§12.1

① $\int_0^2\int_0^1 (x^2+xy+y^2)\,dy\,dx$

$= \int_0^2 \left[x^2 + \frac{xy^2}{2} + \frac{y^3}{3}\right]_{y=0}^{y=1} dx$

$= \int_0^2 x^2 + \frac{x}{2} + \frac{1}{3}\,dx$

$= \left[\frac{x^3}{3} + \frac{x^2}{4} + \frac{x}{3}\right]_0^2$

$= \frac{8}{3} + \frac{4}{4} + \frac{2}{3}$

$= \frac{32+12+8}{12}$

$= \frac{\cancel{52}\,13}{\cancel{12}\,3}$

$= \frac{13}{3}$

⑤ $\int_3^4\int_1^2 \frac{x}{x-y}\,dy\,dx$

$= \int_3^4 \left[x \ln|x-y|\right]_1^2 dx$

$= \int_3^4 x\ln|x-2| - x\ln|x-1|\,dx$

$= \int_3^4 (x-2)\ln|x-2| + 2\ln|x-2|$

$\quad - (x-1)\ln|x-1| - \ln|x-1|\,dx$

$= \Big[\frac{(x-2)^2}{2}\left(\ln|x-2| - \frac{1}{2}\right) - \frac{(x-1)^2}{2}\left(\ln|x-1| - \frac{1}{2}\right)$

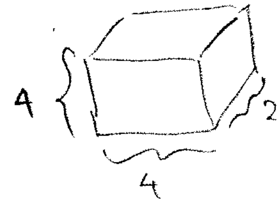
$\quad 2\left((x-2)\ln|x-2| - (x-2)\right)$

$\quad - \left((x-1)\ln|x-1| - (x-1)\right)\Big]_3^4$

(from page A-31)

⑦ $\iint_R 4\,dA$; $R: 0 \le x \le 2;\ 0 \le y \le 4$

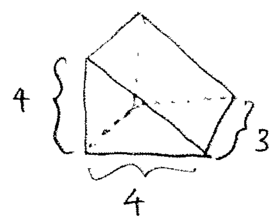
Note that it is a rectangular block



so, volume $= 4 \times 4 \times 2$

$= 32$

⑨ $\iint 4-y\,dA$; $R: 0 \le x \le 3;\ 0 \le y \le 4$



volume $= \frac{1}{2}\cdot 4\cdot 4\cdot 3$

$= 24$

⑬ $\iint_R x^2 y\,dA$; $R: 1 \le x \le 2;\ 0 \le y \le 1$

$= \int_0^1\int_1^2 x^2 y\,dx\,dy$

$= \int_0^1 \left[\frac{x^3}{3}y\right]_1^2 dy = \int_0^1 \frac{8y}{3} - \frac{y}{3}\,dy$

$= \int_0^1 \frac{7}{3}y\,dy = \left[\frac{7}{6}y^2\right]_0^1$

$= \frac{7}{6}$

⑮ $\iint_R 2xe^y\,dA$; $R: -1 \le x \le 0;\ 0 \le y \le \ln 2$

$= \int_0^{\ln 2}\int_{-1}^0 2xe^y\,dx\,dy$

$= \int_0^{\ln 2} \left[x^2 e^y\right]_{-1}^0 dy = \int_0^{\ln 2} -e^y\,dy$

$= \left[-e^y\right]_0^{\ln 2}$

$= -e^{\ln 2} + e^0$

$= -2 + 1$

$= -1$

(17) $\iint_R \frac{2xy}{x^2+1}\, dA \quad R: 0 \le x \le 1;\ 1 \le y \le 3$

$$= \int_1^3 \int_0^1 \frac{2xy}{x^2+1}\, dx\, dy$$

$$= \int_1^3 \Big[y \ln|1+x^2| \Big]_0^1 dy$$

$$= \int_1^3 y \ln 2\, dy$$

$$= \left[\frac{y^2}{2} \ln 2 \right]_1^3$$

$$= \left[\frac{9}{2} - \frac{1}{2} \right] \ln 2$$

$$= \frac{8}{2} \ln 2$$

$$= 4 \ln 2$$

---

(21) $\int_0^2 \int_0^1 2x + 3y\, dx\, dy$

$$= \int_0^2 [x + 3yx]_0^1 dy$$

$$= \int_0^2 1 + 3y\, dy$$

$$= \left[y + \frac{3y^2}{2} \right]_0^2$$

$$= 2 + \frac{(3)\cancel{(4)}^{2}}{\cancel{2}}$$

$$= 8$$

---

(27) $\int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\, dx\, dy$

$$= \int_0^1 \Big[y\sqrt{x^2+y^2+1} \Big]_0^1 dy$$

$$\frac{d}{dx}\sqrt{a+x^2} = \frac{2x}{2\sqrt{a+x^2}} = \frac{x}{\sqrt{a+x^2}}$$

$$= \int_0^1 y\sqrt{2+y^2} - y\sqrt{1+y^2}\, dy$$

$$= \frac{1}{3}\int_0^1 3y\sqrt{2+y^2} - 3y\sqrt{1+y^2}\, dy$$

$$\frac{d}{dy}(a+y^2)^{3/2} = \frac{3}{2}(a+y^2)^{1/2} \cdot 2y = 3y\sqrt{(a+y^2)}$$

$$= \frac{1}{3}\left((2+y^2)^{3/2} - (1+y^2)^{3/2} \right)$$

$$= \frac{1}{3}\left[3^{3/2} - 2^{3/2} - 2^{3/2} + 1^{3/2} \right]$$

$$= \sqrt{3} - \frac{4\sqrt{2}}{3} + \frac{1}{3}$$

§12.2

(3) $$\int_0^4 \int_0^{4-x} xy\,dy\,dx$$

$$= \int_0^4 \left[\frac{xy^2}{2}\right]_0^{4-x} dx$$

$$= \int_0^4 \frac{x(4-x)^2}{2}\,dx$$

$$= \int_0^4 \frac{x(16-8x+x^2)}{2}\,dx$$

$$= \int_0^4 8x - 4x^2 + \frac{x^3}{2}\,dx$$

$$= \left[4x^2 - \frac{4x^3}{3} + \frac{x^4}{8}\right]_0^4$$

$$= (4)(16) - \frac{(4)(64)}{3} + \frac{256}{8}^{32}$$

$$= 32\left(2 - \frac{8}{3} + 1\right)$$

$$= 32\left(\frac{9-8}{3}\right) = \frac{(32)(1)}{3} = \frac{32}{3}$$

---

(7) $$\int_0^{2\sqrt{2}} \int_{y^2/4}^{\sqrt{12-y^2}} dx\,dy$$

$$= \int_0^{2\sqrt{2}} \sqrt{12-y^2} - \frac{y^2}{4}\,dy$$

$$\int \sqrt{12-y^2}\,dy = \sqrt{12}\int \sqrt{1-\left(\frac{y}{\sqrt{12}}\right)^2}\,dy$$

Let $\frac{y}{\sqrt{12}} = \sin\theta \Rightarrow dy = \sqrt{12}\cos\theta\,d\theta$

$$\Rightarrow \sqrt{12}\int \sqrt{1-\sin^2\theta}\,\sqrt{12}\cos\theta\,d\theta$$

$$= 12\int \cos^2\theta\,d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

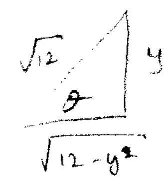
$\Rightarrow \cos^2\theta = \frac{\cos 2\theta + 1}{2}$

$$= 6\int \cos 2\theta + 1\,d\theta$$

$$= \frac{\sin 2\theta}{3} + \theta + c$$

So, $\int \sqrt{12-y^2}\,dy$:

$\theta = \sin^{-1}\left(\frac{y}{\sqrt{12}}\right)$



$\sin 2\theta = 2\sin\theta\cos\theta$

$$= 2\cdot\frac{y}{\sqrt{12}}\cdot\frac{\sqrt{12-y^2}}{\sqrt{12}}$$

$\sqrt{12} = 2\sqrt{3}$

$$\sin 2\theta = \frac{y\sqrt{12-y^2}}{6}$$

$$\int \sqrt{12-y^2}\,dy = \frac{y\sqrt{12-y^2}}{18} + \sin^{-1}\left(\frac{y}{\sqrt{12}}\right) + c$$

So, $$\int_0^{2\sqrt{2}} \sqrt{12-y^2}\,dy = \frac{2\sqrt{2}\cdot\sqrt{12-8}}{18_9} + \sin^{-1}\left(\frac{2\sqrt{2}}{2\sqrt{3}}\right) - 0$$

$$= \frac{2\cdot\sqrt{2}}{9} + \sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$$

and $$\int_0^{2\sqrt{2}} \frac{y^2}{4}\,dy = \frac{y^3}{12}\Big|_0^{2\sqrt{2}} = \frac{8^4\cdot 2\cdot\sqrt{2}}{12_3} = \frac{4\sqrt{2}}{3}$$

So, $$\int_0^{2\sqrt{2}} \sqrt{12-y^2} - \frac{y^2}{4}\,dy = \frac{2\sqrt{3}}{9} + \sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right) - \frac{4\sqrt{2}}{3}$$

---

(11) $$\int_0^1\int_0^x x^2 + 2y^2\,dy\,dx$$

$$= \int_0^1 \left[x^2y + \frac{2}{3}y^3\right]_0^x dx$$

$$= \int_0^1 x^3 + \frac{2}{3}x^3 - (0+0)\,dx$$

$$= \int_0^1 \frac{5}{3}x^3\,dx$$

$$= \frac{5x^4}{12}\Big|_0^1$$

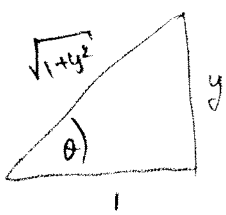
$$= \frac{5}{12}$$

(19) $\displaystyle\int_0^1 \int_{\tan^{-1} y}^{\pi/4} \sec x \, dx \, dy$

$\boxed{\text{PA-29}}$ Recall $\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$

$$= \int_0^1 \Big[\ln|\sec x + \tan x|\Big]_{\tan^{-1} y}^{\pi/4} dy$$

$$= \int_0^1 \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln\left|\sec(\tan^{-1} y) + \tan(\tan^{-1} y)\right| dy$$



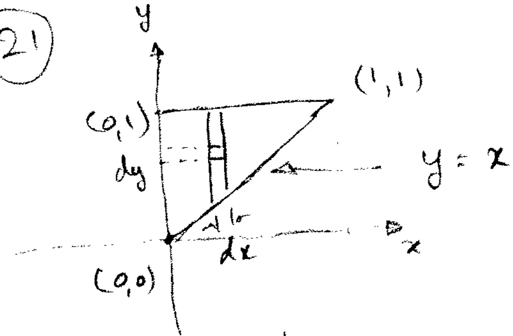
$\tan\theta = y \Rightarrow \tan^{-1} y = \theta$

$\sec\theta = \sqrt{1+y^2}$

$$= \int_0^1 \ln|\sqrt{2}+1| - \ln\left|\sqrt{1+y^2} + y\right| dy$$

$\vdots$

(21)



$$\iint_0 x+y \, dA = \int_0^1 \int_x^1 (x+y) \, dy \, dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2}\right]_x^1 dx$$

$$= \int_0^1 \left[x + \frac{1}{2} - \left(x^2 + \frac{x^2}{2}\right)\right] dx$$

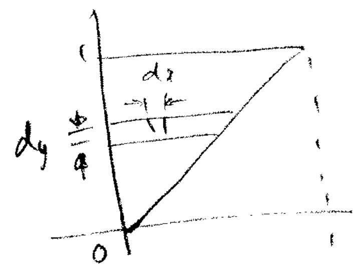
$$= \int_0^1 -\frac{3x^2}{2} + x + \frac{1}{2} \, dx$$

$$= \left[-\frac{x^3}{2} + \frac{x^2}{2} + \frac{x}{2}\right]_0^1$$

$$= -\frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$= \frac{1}{2}$$

$\boxed{\text{OR}}$



$$\int_0^1 \int_0^y (x+y) \, dx \, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} + xy\right]_0^y dy$$

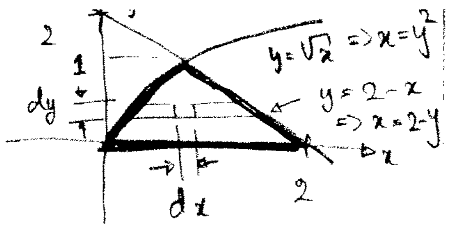
$$= \int_0^1 \left[\frac{y^2}{2} + y^2\right] - [0+0] \, dy$$

$$= \int_0^1 \frac{3y^2}{2} \, dy$$

$$= \left[\frac{y^3}{2}\right]_0^1 = \frac{1}{2}$$

This is a little easier

23 $\iint_D y\,dA$



Solve $\sqrt{x}=2-x$ to find the point of intersection:

$x=(2-x)^2 = 4-4x+x^2$

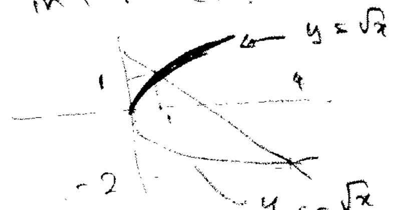
$\Rightarrow x^2-5x+4=0$

$(x-4)(x-1)=0$

$\Rightarrow x=4$ or $x=1$

Clearly the solution should be $x=1 \Rightarrow \begin{cases} y=\sqrt{x} \Rightarrow y=1 \\ y=2-x \Rightarrow y=1 \end{cases}$ ✓

* $x=4$ is where $y=-\sqrt{x}$ and $y=2-x$ intersect:



So, $\iint_D y\,dA = \int_0^1 \int_{y^2}^{2-y} y\,dx\,dy$

$= \int_0^1 [yx]_{x=y^2}^{x=2-y} dy$

$= \int_0^1 y(2-y) - y(y^2)\,dy$

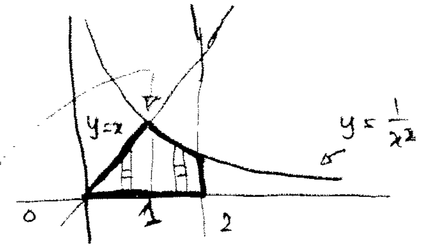
$= \int_0^1 2y - y^2 - y^3\,dy$

$= \left[y^2 - \frac{y^3}{3} - \frac{y^4}{4}\right]_0^1$

$= 1 - \frac{1}{3} - \frac{1}{4} = \frac{12-3-4}{12} = \frac{5}{12}$

25 $\iint_D 2x\,dA$

$y=x=\frac{1}{x^2} \Rightarrow x^3=1 \Rightarrow x=1$



$\int_0^1 \int_0^x 2x\,dy\,dx + \int_1^2 \int_0^{1/x^2} 2x\,dy\,dx$

$= \int_0^1 [2xy]_{y=0}^{x} dx + \int_1^2 [2xy]_0^{1/x^2} dx$

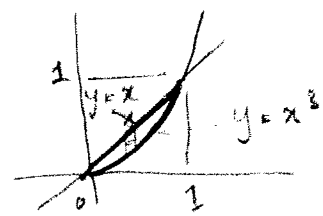
$= \int_0^1 2x^2\,dx + \int_1^2 \frac{2}{x}\,dx$

$= \left[\frac{2x^3}{3}\right]_0^1 + \left[2\ln|x|\right]_1^2$

$= \frac{2}{3} + 2\ln 2$

#27 gives a good reason why you may need to switch the order

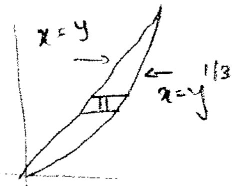
27 $\iint_D 12x^2 e^{y^2}\,dA$



$\int_0^1 \int_{x^3}^{x} 12x^2 e^{y^2}\,dy\,dx$

We cannot integrate this way!

So, change the order.



$\int_0^1 \int_y^{y^{1/3}} 12x^2 e^{y^2}\,dx\,dy$

$= \int_0^1 \left[4x^3 e^{y^2}\right]_y^{1/3} dy = \int_0^1 4ye^{y^2} - 4y^3 e^{y^2}\,dy$

$\Rightarrow 2\int_0^1 2ye^{y^2}\,dy = 2\left[e^{y^2}\right]_0^1 = 2(e-1)$

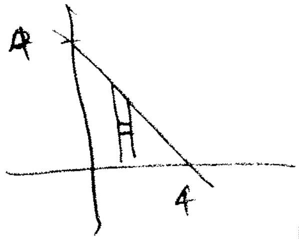
$-2\int_0^1 2y^3 e^{y^2}\,dy \Rightarrow dv = 2ye^{y^2}dy \Rightarrow v=e^{y^2}$

$u=y^2 \Rightarrow du=2y$

$\left[y^2 e^{y^2}\right]_0^1 - \int_0^1 2ye^{y^2}\,dy = \left[y^2 e^{y^2}\right]_0^1 - \left[e^{y^2}\right]_0^1$

...

(33) $\int_0^4\int_0^{4-x} xy\,dy\,dx$



$= \int_0^4 \left[\frac{xy^2}{2}\right]_0^{4-x} dx$

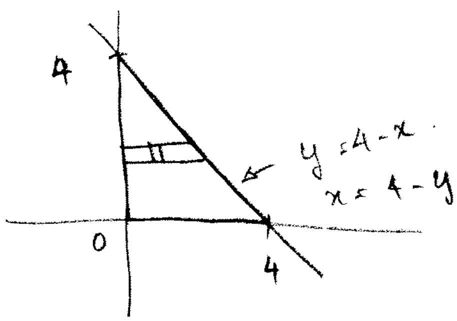
$= \int_0^4 \frac{x(4-x)^2}{2} - 0\,dx$

$= \int_0^4 \frac{x\,(16-8x+x^2)}{2}\,dx$

$= \int_0^4 8x - 4x^2 + \frac{x^3}{2}\,dx$

$= \left[4x - \frac{4}{3}x^3 + \frac{x^4}{8}\right]_0^4$

$= \frac{32}{3}$ (as in #3 of 12.2)



$\int_0^4\int_0^{4-y} xy\,dx\,dy$

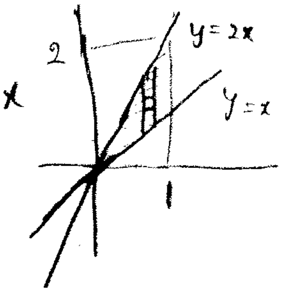
$= \int_0^4 \left[\frac{x^2y}{2}\right]_0^{4-y} dy$

$= \int_0^4 \frac{(4-y)^2y}{2}\,dy$

⋮

(Similar to above)

(35) $\int_0^1\int_x^{2x} e^{y-x}\,dy\,dx$

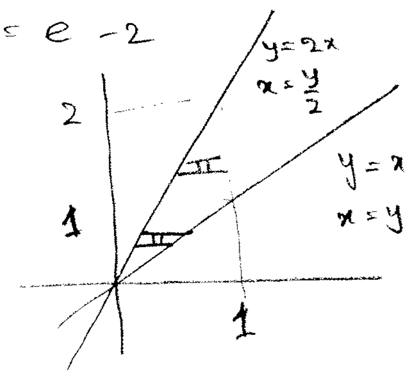


$= \int_0^1 \left[e^{y-x}\right]_x^{2x} dx$

$= \int_0^1 e^{\overbrace{2x-x}^{=x}} - e^{\overbrace{x-x}^{=0}}\,dx = \int_0^1 e^x - 1\,dx$

$= \left[e^x - x\right]_0^1 = (e^1 - 1) - (\underbrace{e^0}_{=1} - 0)$

$= e - 2$



$\int_0^1\int_{y/2}^{y} e^{y-x}\,dx\,dy + \int_1^2\int_{y/2}^{1} e^{y-x}\,dx\,dy$

$= \int_0^1 \left[-e^{y-x}\right]_{y/2}^{y} dy + \int_1^2 \left[-e^{y-x}\right]_{y/2}^{1} dy$

$= \int_0^1 -e^{y-y} + e^{y-y/2}\,dy + \int_1^2 -e^{y-1} + e^{y-\frac{y}{2}}\,dy$

$= \int_0^1 -1 + e^{y/2}\,dy + \int_1^2 -e^{y-1} + e^{y/2}\,dy$

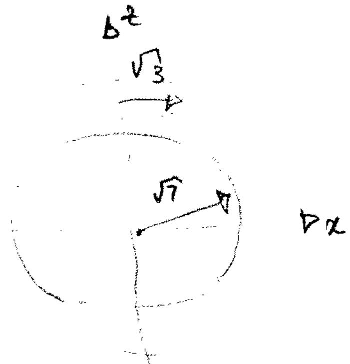
$= \left[-y + 2e^{y/2}\right]_0^1 + \left[-e^{y-1} + 2e^{y/2}\right]_1^2$

$= (-1 + 2e^{1/2}) - (0 + 2e^0)$

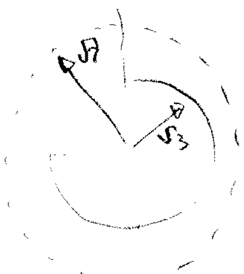
$+ (-e^1 + 2e^1) - (-e^0 + 2e^{1/2})$

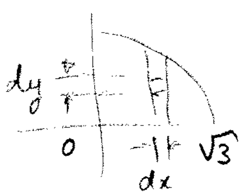
$= e - 2$

(53) Volume of the solid between cylinder $x^2+y^2=3$ & sphere $x^2+y^2+z^2=7$



Let's use the fact that the solid is symmetric about the coordinate axis i.e. about the origin.





$$V = 8 \int_0^{\sqrt{3}} \int_0^{\sqrt{3-y^2}} \sqrt{7-x^2-y^2}\, dy\, dx$$

§12.5

(5)

$$\int_1^2\int_0^1\int_{-1}^2 8x^2yz^3\,dx\,dy\,dz$$

$$= \int_1^2\int_0^1\left[\frac{8}{3}x^3yz^3\right]_{-1}^2 dy\,dz$$

$$= \int_1^2\int_0^1 \left(\frac{8}{3}yz^3\right.\ \underbrace{(2^3-(-1)^3)}_{9}\,dy\,dz$$

$$= \int_1^2\int_0^1 24\,yz^3\,dy\,dz$$

$$= \int_1^2\left[\frac{24}{2}y^2z^3\right]_0^1 dz$$

$$= \int_1^2 12z^3(1^2-0^2)\,dz$$

$$= 12\int_1^2 z^3\,dz$$

$$= \overset{3}{\cancel{12}}\left[\frac{z^4}{4}\right]_1^2 = 3(2^4-1^4) = 5(16-1)$$

$$= 45$$

(15)

$$\iiint_D x^2y+y^2z\,dV \ ;\quad \begin{matrix}1\le x\le 3\\ -1\le y\le 1\\ 2\le z\le 4\end{matrix}$$

$$= \int_2^4\int_{-1}^1\int_1^3 x^2y+y^2z\,dx\,dy\,dz$$

$$= \int_2^4\int_{-1}^1\left[\frac{x^3y}{3}+xy^2z\right]_1^3 dy\,dz$$

$$= \int_2^4\int_{-1}^1 \frac{27}{3}y+3y^2z-\frac{y}{3}-y^2z\ \ dy\,dz$$

$$= \int_2^4\int_{-1}^1 \frac{26y}{3}+2y^2z\ \ dy\,dz$$

$$= \int_2^4\left[\frac{13y^2}{3}+\frac{2y^3z}{3}\right]_{-1}^1 dy\,dz$$

$$= \int_2^4 \frac{13}{3}+\frac{2z}{3}-\left(\frac{13}{3}-\frac{2z}{3}\right)dz$$

$$= \int_2^4 \frac{4z}{3}\,dz$$

$$= \left.\frac{2z^2}{3}\right|_2^4 = \frac{2(16)}{3}-\frac{2(4)}{3} = \frac{32-8}{3} = \frac{24}{3}$$

$$= 8$$

(13)

$$\int_1^4\int_{-1}^{2z}\int_0^{\sqrt{3}x} \frac{x-y}{x^2+y^2}\,dy\,dx\,dz$$

$$= \int_1^4\int_{-1}^{2z}\int_0^{\sqrt{3}x} \frac{x}{(x^2+y^2)}-\frac{2y}{2(x^2+y^2)}\,dy\,dx\,dz$$

$$= \int_1^4\int_{-1}^{2z}\left[\tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\ln|x^2+y^2|\right]_0^{\sqrt{3}x} dx\,dz$$

$$= \int_1^4\int_{-1}^{2z}\Bigg[\left(\overbrace{\tan^{-1}\left(\frac{\sqrt{3}x}{x}\right)}^{\tan^{-1}(\sqrt{3})=\frac{\pi}{3}}-\frac{1}{2}\ln|x^2+3x^2|\right)$$

$$-\left(\cancel{\tan^{-1}0}^{\,0}-\frac{1}{2}\ln(x^2)\right)\Bigg]dx\,dz$$

$$= \int_1^4\int_{-1}^{2z}\left[\frac{\pi}{3}-\frac{1}{2}\ln(4x^2)+\frac{1}{2}\ln x^2\right]dx\,dz$$

$$= \int_1^4\int_{-1}^{2z}\left[\frac{\pi}{3}+\frac{1}{2}\ln\left(\frac{4x^2}{x^2}\right)\right)dx\,dz$$

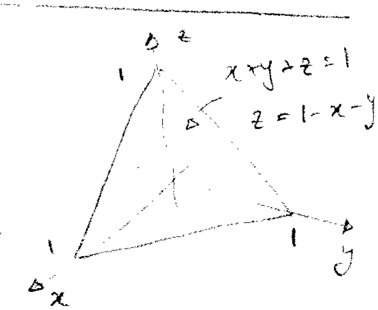
$$= \int_1^4\int_{-1}^{2z}\left(\frac{\pi}{3}+\ln 2\right)dx\,dz$$

$$= \int_1^4\left(\frac{\pi}{3}+\ln 8\right)[2z+1]\,dz$$

$$= \left(\frac{\pi}{3}+\ln 2\right)\underbrace{\left(z^2+z\right)\Big|_1^4}_{16+4-2=18} = 6\pi+18\ln 2$$

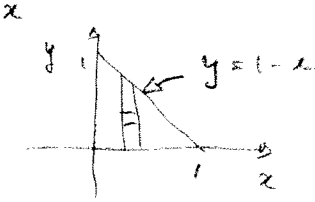
(17)

$$\iiint_D xyz\,dV$$



$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}\left[\frac{xyz^2}{2}\right]_0^{1-x-y} dy\,dx$$



$$= \int_0^1\int_0^{1-x}\frac{xy(1-x-y)^2}{2}\,dy\,dx$$

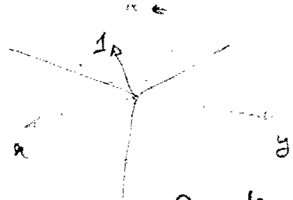
$$= \int_0^1\int_0^{1-x}\frac{xy(1+x^2+y^2-2x-2y+2xy)}{2}\,dy\,dx$$

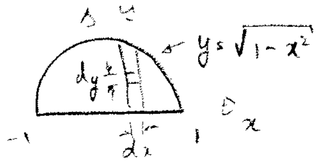
$$= \int_0^1\int_0^{1-x}\frac{xy}{2}+\frac{x^3y}{2}+\frac{xy^3}{2}-\frac{2x^2y}{\cancel{2}}-\frac{2xy^2}{\cancel{2}}+\frac{2x^2y^2}{\cancel{2}}\,dy\,dx$$

$$= \int_0^1\left[\frac{xy^2}{4}+\frac{x^3y^2}{4}+\frac{xy^4}{8}-\frac{x^2y^2}{2}-\frac{xy^3}{3}+\frac{x^2y^3}{3}\right]_0^{1-x} dx$$

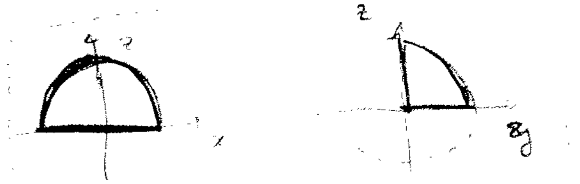
⋮

(19) $\iiint_D xyz\, dV$





quarter of a sphere



for visualization.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx$$

$$= \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \left[xy\frac{z^2}{2}\right]_{0}^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \frac{xy(1-x^2-y^2)}{2}\, dy\, dx$$

$$= \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \frac{xy}{2} - \frac{x^3y}{2} - \frac{xy^3}{2}\, dy\, dx$$

$$= \int_{-1}^{1} \left[\frac{xy^2}{4} - \frac{x^3y^2}{4} - \frac{xy^4}{8}\right]_{0}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \frac{x(1-x^2)}{4} - \frac{x^3(1-x^2)}{4} - \frac{x(1-x^2)^2}{8}\, dx$$

$$\int_{-1}^{1} \frac{x}{4} - \frac{x^3}{4} - \frac{x^3}{4} - \frac{x^5}{4} - \frac{x}{8} + \frac{x^3}{4} - \frac{x^5}{8}\, dx$$

$$= 0$$

(21) $\iiint_D e^z\, dV \quad \begin{cases} 0 \le x \le 1 \\ 0 \le y \le x \\ 0 \le z \le x+y \end{cases}$

$$= \int_0^1\int_0^x\int_0^{x+y} e^z\, dz\, dy\, dx$$

$$= \int_0^1\int_0^x \left[e^z\right]_0^{x+y} dy\, dx$$

$$= \int_0^1\int_0^x e^{x+y} - e^0\, dy\, dx$$

$$= \int_0^1\int_0^x e^{x+y} - 1\, dy\, dx$$

$$= \int_0^1 \left[e^{x+y} - y\right]_0^x dx$$

$$= \int_0^1 \left[e^{2x} - x - e^0 + 0\right] dx$$

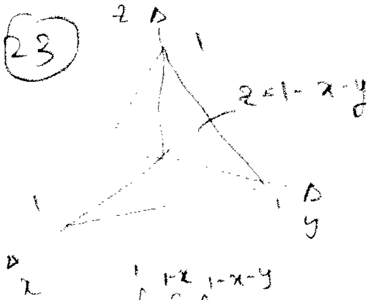
$$= \int_0^1 \left[e^{2x} - x - 1\right] dx$$

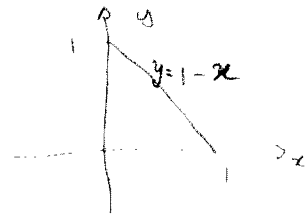
$$= \left[\frac{e^{2x}}{2} - \frac{x^2}{2} - x\right]_0^1$$

$$= \frac{e^2}{2} - \frac{1}{2} - 1 - \left(\frac{e^0}{2} - 0 - 0\right)$$

$$= \frac{e^2}{2} - 2$$

(23)





$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} 1 - x - y \, dy\, dx$$

$$= \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \int_0^1 \left[1 - x - x(1-x) - \frac{(1-x)^2}{2} \right] dx$$

$$= \int_0^1 1 - x - x + x^2 - \frac{1}{2} + x - \frac{x^2}{2}$$

$$= \int_0^1 \frac{1}{2} - x + \frac{x^2}{2} \, dx$$

$$= \int_0^1 \frac{1}{2}(x-1)^2 \, dx$$

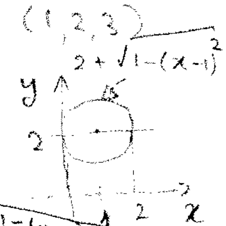
$$= \left. \frac{[x-1]^3}{6} \right|_0^1 = \frac{1}{6}$$

---

(25) $(x-1)^2 + (y-2)^2 + (z-3)^2 = 1$

Sphere of radius 1 center $(1,2,3)$

Ans: $\frac{4}{3}\pi r^3$; $r = 1 \Rightarrow \frac{4}{3}\pi$



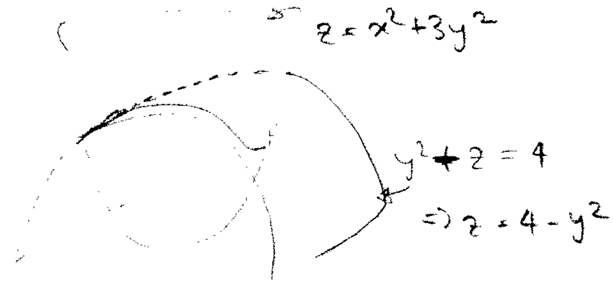
Using integral:

Volume $V = 4 \int_0^2 \int_2^{2+\sqrt{1-(x-1)^2}} \int_3^{3+\sqrt{1-(x-1)^2-(y-2)^2}} dz\, dy\, dx$

$\vdots$

Continue with the rest of the calculation. ☺

(27) ** MAJOR EXAMPLE

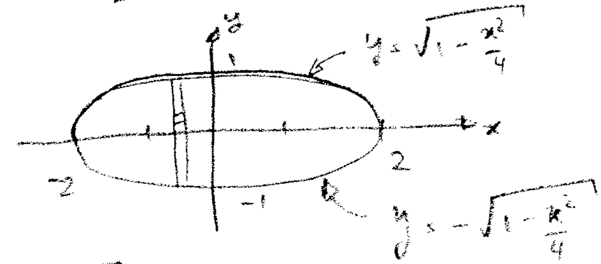


Curve of intersection projected on to the xy plane

set $x^2 + 3y^2 = 4 - y^2$

$\Rightarrow x^2 + 4y^2 = 4$

$\Rightarrow \frac{x^2}{2^2} + y^2 = 1 \Rightarrow y = \pm\sqrt{1 - \frac{x^2}{4}}$



$$\int_{-2}^{2} \int_{-\sqrt{1-\frac{x^2}{4}}}^{\sqrt{1-\frac{x^2}{4}}} \int_{x^2+3y^2}^{4-y^2} dz\, dy\, dx$$

By symmetry $4 \int_0^2 \int_0^{\sqrt{1-x^2/4}} \int_{x^2+3y^2}^{4-y^2} dz\, dy\, dx$

$$= \int_{-2}^{2} \int_{-\sqrt{1-\frac{x^2}{4}}}^{\sqrt{1-\frac{x^2}{4}}} 4 - y^2 - x^2 - 3y^2 \, dy\, dz$$

$$= \int_{-2}^{2} \int_{-\sqrt{1-\frac{x^2}{4}}}^{\sqrt{1-\frac{x^2}{4}}} 4 - x^2 - 4y^2 \, dy\, dz$$

$$= \int_{-2}^{2} \left[4y - x^2 y - \frac{4}{3} y^3 \right]_{-\sqrt{1-\frac{x^2}{4}}}^{\sqrt{1-\frac{x^2}{4}}} dz$$

$$= \int_{-2}^{2} 8\sqrt{1-\frac{x^2}{4}} - 2x^2\sqrt{1-\frac{x^2}{4}} - \frac{8}{3}\left(1 - \frac{x^2}{4}\right)^{3/2} dz$$

$\vdots$

(29) Above $z = 6 - x^2 - y^2$

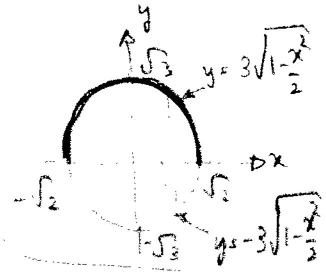
Below $z = 2x^2 + y^2$

Projection of the curve of intersection

$6 - x^2 - y^2 = 2x^2 + y^2$

$6 = 3x^2 + 2y^2$

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{3} = 1$



$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-3\sqrt{1-\frac{x^2}{2}}}^{3\sqrt{1-\frac{x^2}{2}}} \int_{2x^2+y^2}^{6-x^2-y^2} dz\,dy\,dx$$

$$= 2\int_0^{\sqrt{2}} \int_{-3\sqrt{1-\frac{x^2}{2}}}^{3\sqrt{1-\frac{x^2}{2}}} \int_{2x^2+y^2}^{6-x^2-y^2} dz\,dy\,dx$$

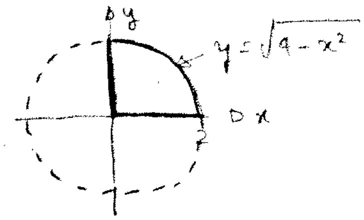
(By Symmetry about $\cancel{yz}$ plane)

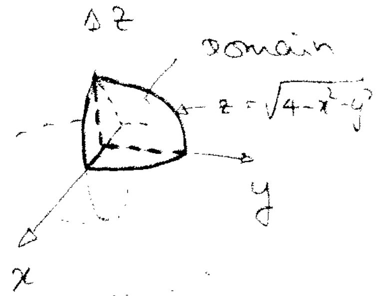
$$= 4\int_0^{\sqrt{2}} \int_0^{3\sqrt{1-\frac{x^2}{2}}} \int_{2x^2+y^2}^{6-x^2-y^2} dz\,dy\,dx$$

(By symmetry about xz plane

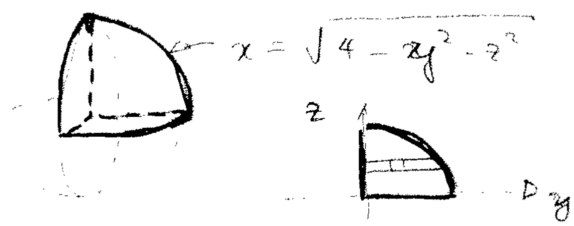
(35) $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$

to $dx\,dy\,dz$





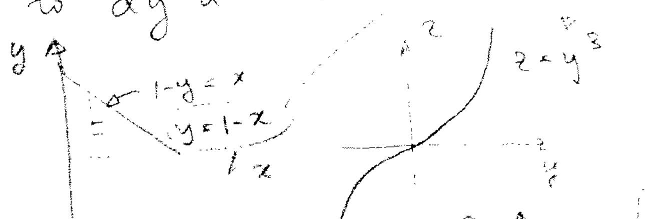
original (given) order

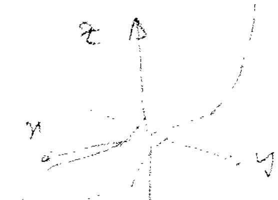


$$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{\sqrt{4-y^2-z^2}} dx\,dy\,dz$$

(37) $\displaystyle\int_0^1 \int_0^{1-y} \int_0^{y^3} f(x,y,z)\,dz\,dy\,dx$

to $dy\,dx\,dz$





$z = (1-x)^3$

$1 - x = z^{1/3}$

$1 - z^{1/3} = x$

$y = 0 \Rightarrow z = 0$
$x = 0$

$$\int_0^1 \int_0^{1-z^{1/3}} \int_0^{1-x} f(x,y,z)\,dy\,dx\,dz$$

$$= \int_0^1 \int_{z^{1/3}}^1 \int_0^{1-x} f(x,y,z)\,dy\,dx\,dz$$

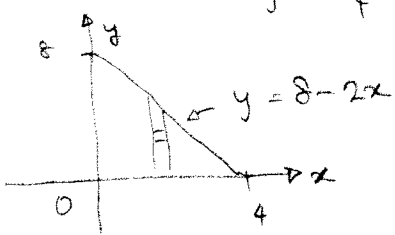
§12.4

① $x \geq 0, y \geq 0, z \geq 0,$

$2x + y + 4z = 8$

$\Rightarrow z = 2 - \frac{y}{4} - \frac{x}{2} = f(x,y)$

$f_x = -\frac{1}{2}, \quad f_y = -\frac{1}{4}$



$\sqrt{f_x^2 + f_y^2 + 1}$
$= \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{4}\right)^2 + 1}$
$= \sqrt{\frac{1}{4} + \frac{1}{16} + 1}$
$= \sqrt{\frac{4+1+16}{16}}$

$$\text{Area} = \int_0^4 \int_0^{8-2x} \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{4}\right)^2 + 1}\, dy\, dx$$

$$= \int_0^4 \int_0^{8-2x} \sqrt{\frac{1}{4} + \frac{1}{16} + 1}\, dy\, dx$$

$$= \int_0^4 \int_0^{8-2x} \frac{\sqrt{21}}{4}\, dy\, dx$$

$$= \int_0^4 \frac{\sqrt{21}}{4} (8 - 2x)\, dx$$

$$= \frac{\sqrt{21}}{4} \left(8x - x^2\right)_0^4$$

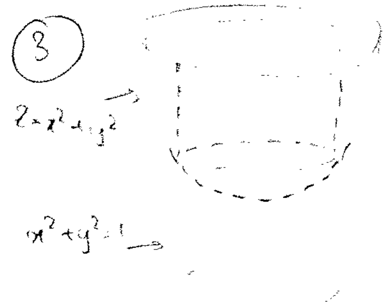
$$= \frac{\sqrt{21}}{4} (32 - 16)$$

$$= \frac{\sqrt{21}}{\cancel{4}} (\cancel{16}^{4})$$

$$= 4\sqrt{21}$$

---

③



Note that these two intersect when $z = 1$.

$f(x,y) = x^2 + y^2$

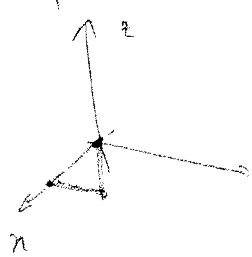
$f_x = 2x, \quad f_y = 2y$

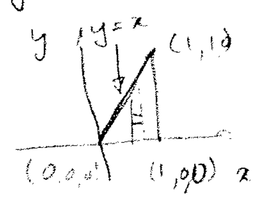
$$\text{Area} = 4 \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{4x^2 + 4y^2 + 1}\, dy\, dx$$

---

⑤ $3x + 6y + 2z = 12$

Note that by substitution,
All three points are below the plane $3x + 6y + 2z = 12$





$f(x,y) = z = 6 - \frac{3}{2}x - 3y$

$f_x = -\frac{3}{2}; \quad f_y = -3$

$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{9}{4} + 9 + 1} = \sqrt{\frac{9 + 36 + 4}{4}} = \frac{\sqrt{49}}{\sqrt{4}}$

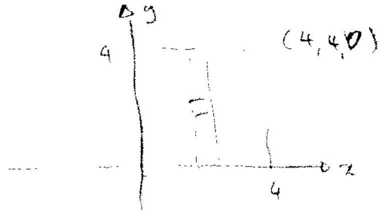
$= \frac{7}{2}$

$$\int_0^1 \int_0^x \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx$$

$$= \int_0^1 \int_0^x \frac{7}{2}\, dy\, dx = \int_0^1 \frac{7}{2} x\, dx$$

$$= \frac{7}{4}$$

(9) $z = x^2$



$f(x,y) = x^2$

$f_x = 2x$

$f_y = 0$

$$\int_0^4\int_0^4 \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx$$

$$= \int_0^4 \int_0^4 \sqrt{4x^2+1}\, dy\, dx$$

$$= \int_0^4 4\sqrt{4x^2+1}\, dx$$

$$= \int_0^4 8\sqrt{x^2 + \tfrac{1}{2^2}}\, dx$$

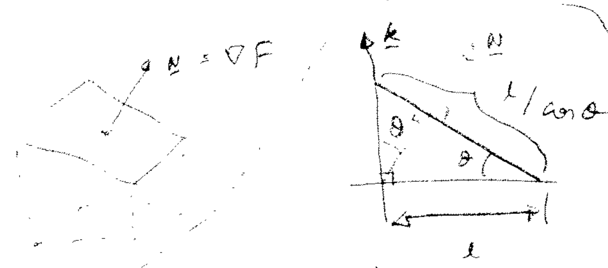
$$= 8\left[\frac{x\sqrt{x^2+\frac{1}{4}}}{2} + \frac{1}{8}\ln\left(x + \sqrt{x^2+\tfrac{1}{4}}\right)\right]_0^4$$

$$= 8\left[\frac{4\sqrt{4^2+\frac{1}{4}}}{2} + \frac{1}{8}\ln\left|4 + \sqrt{4^2+\tfrac{1}{4}}\right| - 0 - \frac{1}{8}\ln\frac{1}{2}\right]$$

$$= 8\sqrt{65} + \ln\left(8 + \sqrt{65}\right)$$

(47) Surface given by $F(x,y,z) = 0$

Then, the normal to the surface is $\nabla F = F_x \underline{i} + F_y \underline{j} + F_z \underline{k}$



$$\cos\theta = \frac{\underline{N}\cdot\underline{k}}{\|\underline{N}\|\,\|\underline{k}\|}$$

But $\|\underline{k}\| = 1$

$$\therefore \quad \cos\theta = \frac{\underline{N}\cdot\underline{k}}{\|\underline{N}\|}$$

$$\therefore \quad \frac{l}{\cos\theta} = l\,\frac{\|\underline{N}\|}{|\underline{N}\cdot\underline{k}|}$$

Same happens with area

$$\|\underline{N}\| = \sqrt{F_x^2 + F_y^2 + F_z^2} = \|\nabla F\|$$

$$\underline{N}\cdot\underline{k} = F_z$$

So, Area becomes

$$\iint_D \frac{\|\nabla F\|}{|\nabla F\cdot \underline{k}|}\, dA$$

$$= \iint_D \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|}\, dA$$

This is just a sketch of the proof but the idea is correct.

In (48), projection is along $\underline{j}$ instead of $\underline{k}$.