MATH 1550: PRECALCULUS

Fall 2010, Section 10

Review for the Final / Test 04 (Take home)

Instructions

- Answer all problems.
- You may make use of any resource, but the final answers SHOULD BE IN YOUR OWN WORDS.
- Please write the answers on a Blue book.
- Answer the problems in the order given on the test.
- Show **all necessary** work to earn full credit.
- Please write clearly.
- Turn in your work on or before Monday, December 06 2010. Late work WILL NOT be accepted.

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1. Consider the rational function $f(x) = \dfrac{3x^2 + 15x}{x^2 - x - 12}$
   (a) Rewrite the given rational function with the numerator and the denominator factored.
   (b) Find the vertical asymptotes
   (c) Find the $x-$intercepts and $y-$intercepts
   (d) Find the horizontal asymptotes
   (e) Find interval(s) where f is positive
   (f) Find intervals where f is negative.
   (g) Use the information you found above to sketch the graph.

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2. Consider the straight line $2y - x + 4 = 0$
   (a) Rewrite the equation of this line in the slope-intercept form
   (b) Find the slope, x-intercept, and the y-intercept
   (c) Find the equation of the line parallel to this line and goes through the point $(6, -3)$
   (d) Find the equation of the line perpendicular to this line and goes through the point $(3, 6)$

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3. Use the discriminant to find the value of k such that the quadratic equation $2x^2 + 6x + k = 0$ will have exactly one solution.

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4. Consider the function $f(x) = 8(x-2)^3 + 4$
   (a) Sketch the graph of this function *[HINT: Use the knowledge of x^3]*
   (b) Find the inverse of this function
   (c) Sketch the graph of the inverse function *[HINT: Use the knowledge of $\sqrt[3]{x}$]*
   (d) Use composition to verify that the inverse you found is correct.

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5. Find the function domain of the function $f(x) = \sqrt{\dfrac{2x+6}{x^2-4}}$

6. In 1981, the population of Lubbock County was 213 350 and in 2001, it was 246 400. (source: U.S. Census Bureau, via `www.google.com`)
   (a) Assuming the Population growth of Lubbock is linear, and find the equation that relates the population, p, to the time, t, (in years).
   (b) Use this to predict what the population in Lubbock County would have been in 2008.
   (c) Given that the actual population in Lubbock County was 264 418, compute the error in the predicted value found above.

7. Consider the quadratic function $2x^2 + 4x + 2y^2 - 8y + 3 = 0$:
   (a) What conic section can this be?
   (b) Using the method of completing the squares, rewrite this in the standard form for the conic you assumed.
   (c) Find the key points of this conic (i.e. center, radius if it is a circle; center, foci and the lengths of major and minor axes if it is an ellipse; vertex and axis if it is a parabola; vertices, foci and center if it is a hyperbola)
   (d) Sketch the conic given by this equation.

8. Repeat problem 7 for the two following quadratic functions:
   (a) $9x^2 + 25y^2 = 225$
   (b) $x^2 + 2x + y^2 + 26 = 0$

9. Find all the roots of the cubic equation $x^3 + 5x^2 + 6x + 2 = 0$.

10. Solve the inequality: $x^3 + 5x^2 + 6x \geq 0$.

11. Solve the equation $\log_6(x+2) + \log_6(x-1) = 1$. Be sure to check the solutions you obtain.

12. Solve the equation $6^{(x+2)} = 9^{(x-1)}$. Round the solution to two decimal places.

13. Solve the equation $8^{(x+1)} = 2^{(x-3)^2}$.

14. Solve the system of equations: $\begin{cases} 3x - 4y + 2z &= 7 \\ 5x + 3y + 3z &= 2 \\ 2x + 7y + 9z &= -5 \end{cases}$

15. Find the coordinates of the points of intersection between the circle $x^2 + y^2 = 1$ and the parabola $y^2 = 4x$.

16. Find partial fraction decomposition of the following:

(a) $\dfrac{1}{x(x-2)(x+3)}$ (b) $\dfrac{3x+2}{(x+1)(x^2+1)}$ (c) $\dfrac{x^2+x+1}{(x+1)^2(x^2+1)}$

17. At the start of a microbiology experiment on a sample of a particular bacterium, there were 3 200 bacteria in the sample. After 72, it was observed that the bacteria population has doubled.

(a) Assuming that the bacteria growth is exponential, (i.e. $\mathcal{N}(t) = \mathcal{N}_0\, e^{kt}$, in usual notation) find the rate constant k.

(b) How long will it take for the bacteria population to reach a population of 144 000?

(c) After the bacteria population reaches 200 000, the researchers administer an antibiotic to the sample. It is known that this antibiotic reduces the bacterial population exponentially according to the formula $\mathcal{M}(t) = \mathcal{M}_0\, e^{-103t}$, where t is the time measured in hours, starting from the moment when the antibiotic is administered, $\mathcal{M}_0$ is the population at the beginning of administering the antibiotic.

For how long should the antibiotic should be administered before the population of the bacteria to reach 1 000?

18. Consider the trigonometric function given by $y = 2\sin(\pi x + \frac{\pi}{4})$

(a) Find its (fundamental) period

(b) Find its Phase shift

(c) Find its amplitude

(d) Find its vertical shift

(e) Sketch its graph for a few cycles

19. Find all the solutions of the trigonometric equation $2\cos^2(x) + 5\cos(x) - 3 = 0$.

20. Give the EXACT VALUES of the following trigonometric expressions. No credit will be given for calculator approximations.

(a) $\cos\left(\dfrac{11\pi}{4}\right)$ (b) $\tan(22.5°)$ (c) $\sin(-270°)$ (d) $\cos(75°)$

(e) $\cos(105°)\cos(15°)$ (f) $\sec\left(\dfrac{\pi}{8}\right)$

21. Give the EXACT VALUES of the following trigonometric expressions. No credit will be given for calculator approximations.

(a) $\cos\left(\cos^{-1}\left(\dfrac{1}{4}\right)\right)$ (b) $\sin\left(\cos^{-1}\left(\dfrac{-1}{4}\right)\right)$ (c) $\cos(\tan^{-1}(\sqrt{3}))$

(d) $\cos^{-1}(\cos(75°))$ (e) $\sin^{-1}(\sin(150°))$

22. An angle θ is in the third quadrant and its tangent is $\frac{8}{15}$, find the EXACT VALUES of the other 5 trigonometric functions of θ. No credit will be given for calculator approximations.

23. Given that θ is in the second quadrant and $\sin(\theta) = \frac{4}{5}$. Compute the EXACT VALUES of the following (No credit will be given for calculator approximations):

(a) $\cos(2\theta)$ (b) $\tan(2\theta)$ (c) $\cos\left(\frac{\theta}{2}\right)$ *[HINT: Which quadrants are θ, 2θ and $\theta/2$ are in?]*

24. Sally and Tammy lives 6 miles apart. Robert's lives 8 miles away from Sally's house and 4 miles away from Tammy's house. If all three houses are on the same plane, find the angles between each of the houses.

25. The building I stand in front of, has a telecommunications tower on its roof top. When I am 100 feet away from the building, the roof top of the building is seen at an elevation of 32°, and the top of the telecommunications tower is seen at an elevation of 43°. Find the height of the telecommunications tower (from the roof top).

26. Two of the angles of a triangle are 55° and 44°. The side opposite the 44° angle is 8 inches long. Find the lengths of the other two sides of the triangle.

27. Two sides of a triangle are 10 and 15 inches long. The angle between them (i.e. the included angle) is 59°. Find the length of the other side and the other two angles.

28. Prove the following identity: $\sin\theta\,(\tan\theta + \cot\theta) = \sec\theta$

29. Prove the following identity: $\sec^2\theta + \csc^2\theta = \sec^2\theta\csc^2\theta$

30. Prove the following identity: $\sin\theta = \csc\theta - \cos\theta\cot\theta$

31. Prove the following identity: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. *[HINT: Note that $\sin(3\theta) = \sin(2\theta + \theta)$]*

(1) $f(x) = \dfrac{3x^2+15x}{x^2-x-12}$

(a) $f(x) = \dfrac{3x(x+5)}{(x-4)(x+3)}$

(b) Vertical asymptotes: $x=4$, $x=-3$

(c) x-intercepts: $x=0$, $x=-5$

y-intercept: 0

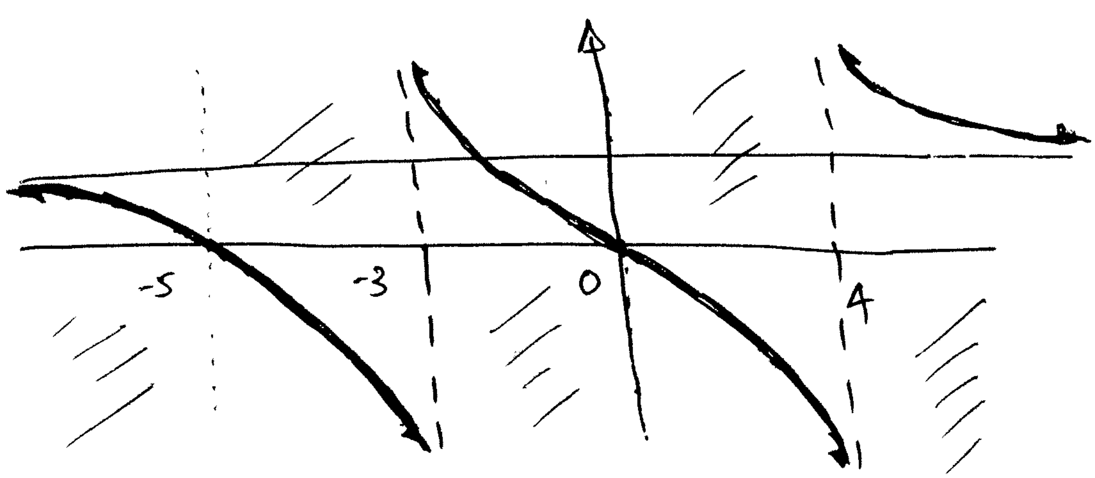
(d) Horizontal asymptote: $y=3$

(e), (f)

$\frac{n\cdot n}{n\cdot n}=p$ $\quad\frac{n\cdot p}{n\cdot n}=n$ $\quad\frac{n\cdot p}{n\cdot p}=p$ $\quad\frac{p\cdot p}{n\cdot p}=n$ $\quad\frac{p\cdot p}{p\cdot p}=p$

Positive $\;-5\;$ neg. $\;-3\;$ pos. $\;0\;$ neg. $\;4\;$ positive.

(g)



(2) $2y - x + 4 = 0$

(a) $2y = x - 4$

Hence slope-intercept form: $\boxed{y = \frac{1}{2}x - 2}$

(b) Slope: $\frac{1}{2}$

y-intercept: 2

(c) parallel lines' slope $= \frac{1}{2}$

$\therefore$ Equation of the line parallel ang passing through $(6, -3)$:

$y = \frac{1}{2}(x - 6) - 3$ (from $y = m(x - x_0) + y_0$)

$\therefore y = \frac{1}{2}x - 3 - 3$

$\boxed{y = \frac{1}{2}x - 6}$

(d) Perpendicular line's slope $= -\frac{1}{\left(\frac{1}{2}\right)} = -2$

$\therefore$ Equation of the line perpendicular and passing through $(3, 6)$:

$y = -2(x - 3) + 6 = -2x + 6 + 6$

$\therefore \boxed{y = -2x + 12}$

(3) $2x^2 + 6x + k = 0$

If $b^2 - 4ac > 0$: 2 distinct real roots

$b^2 - 4ac = 0$: 1 real root

$b^2 - 4ac < 0$: No real roots

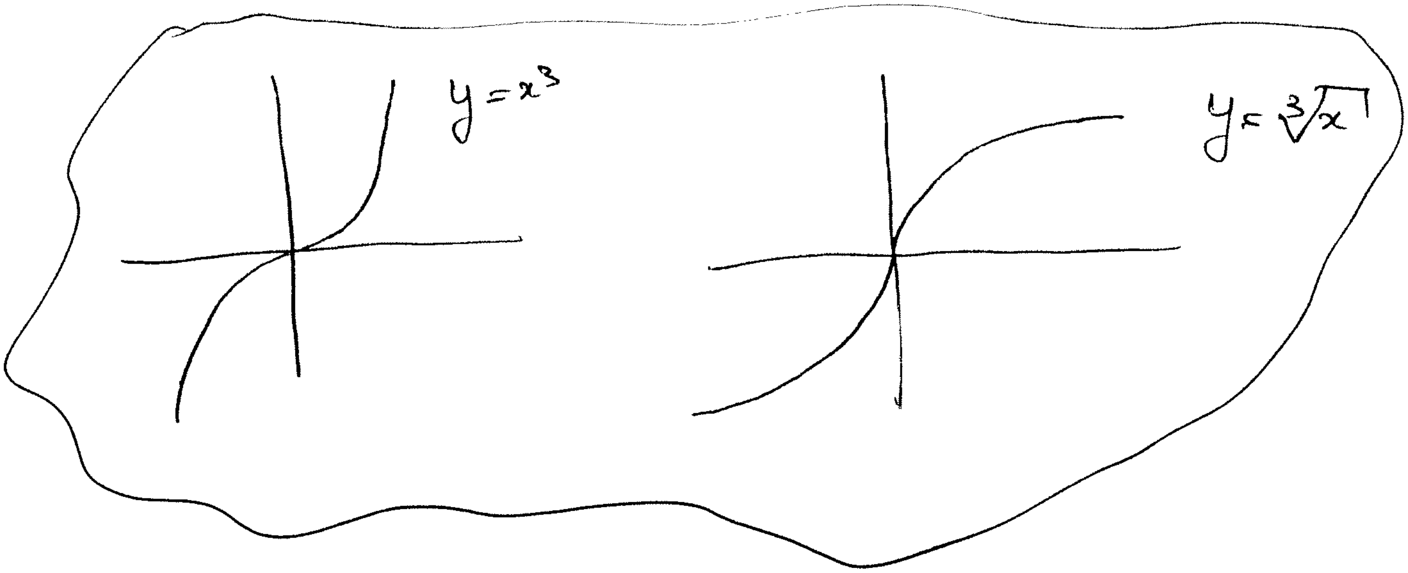
$6^2 - (4)(2)(k) = 0$

$36 - 8k = 0$

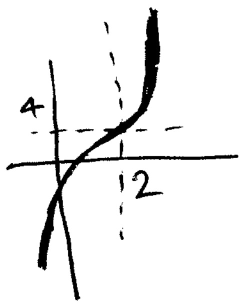
$\therefore \frac{-8k}{-8} = \frac{-36}{-8}$

$\boxed{k = \frac{9}{2}}$

④ $f(x) = 8(x-2)^3 + 4$



(a)



(b) Set $x = 8(y-2)^3 + 4$

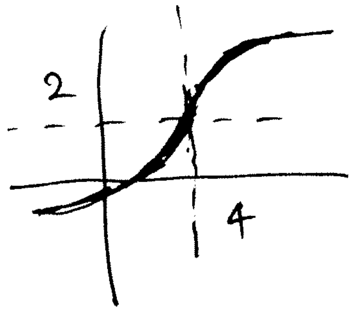
so, $\frac{x-4}{8} = \frac{8(y-2)^3}{8}$

$\frac{x-4}{8} = (y-2)^3$

$\therefore \sqrt[3]{\frac{x-4}{8}} = y-2$

$\therefore \boxed{y = \sqrt[3]{\frac{x-4}{8}} + 2}$

④ (c)



(d) $f\left(f^{-1}(x)\right)$

$$= 8\left(\left(\sqrt[3]{\frac{x-4}{8}}+2\right)-2\right)^3+4$$

$$= 8\left(\sqrt[3]{\frac{x-4}{8}}+\cancel{2}-\cancel{2}\right)^3+4$$

$$= 8\left(\sqrt[3]{\frac{x-4}{8}}\right)^3+4$$

$$= \cancel{8}\left(\frac{x-4}{\cancel{8}}\right)+4$$

$$= x-\cancel{4}+\cancel{4}$$

$f\left(f^{-1}(x)\right) = x$ ✓

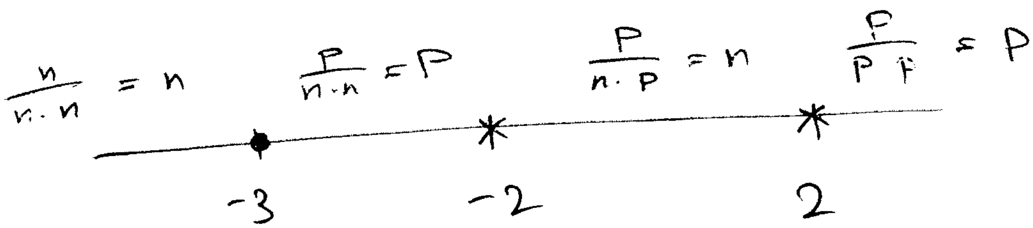
⑤ Domain of $\sqrt{\dfrac{2x+6}{x^2-4}}$

Since this is an even root we need to have,

$$\frac{2x+6}{x^2-4} \geq 0$$

factoring,

$$\frac{2(x+3)}{(x-2)(x+2)} \geq 0$$

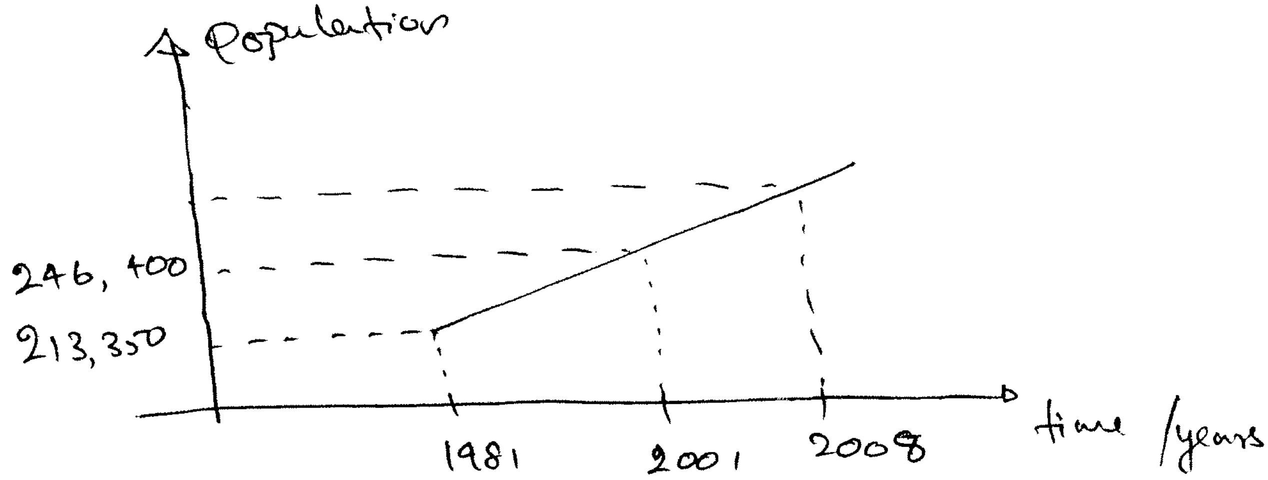


$$[-3, -2) \cup (2, \infty)$$

$\therefore$ Domain is

$$[-3, -2) \cup (2, \infty).$$

(6)



(a) Slope: $\dfrac{246{,}400 - 213{,}350}{2001 - 1981}$

$= \dfrac{33050}{20}$

$= 1652.5$

$\therefore$ Equation: {from $y = m(x - x_0) + y_0$}

$y = 1652.5(x - 1981) + 213350$

$y = 1652.5x - 3273602.5 + 213350$

$\boxed{y = 1652.5x - 3060252.5}$

(b) $y = 1652.5(2008 - 1981) + 213350$

$= (1652.5)(27) + 213350$

$= 257967.5$

$y \approx 257968$ people.

(c) Error = Actual − Predicted

$= 264418 - 257968$

Error $= 6450$

7. $2x^2+4x+2y^2-8y+3=0$

(a) This <u>can</u> be a circle
(x^2 and y^2 has the <u>same</u> Coefficients)

(b) $2(x^2+2x)+2(y^2-4y)+3=0$

$$2\left(x^2+2x+\left(\tfrac{2}{2}\right)^2-\left(\tfrac{2}{2}\right)^2\right)+2\left(y^2-4y+\left(\tfrac{-4}{2}\right)^2+\left(\tfrac{-4}{2}\right)^2\right)+3=$$

$$2(\underbrace{x^2+2x+1}_{=(x+1)^2}-1)+2(\underbrace{y^2-4y+4}_{(y-2)^2}-4)+3=0$$

$$2(x+1)^2-2+2(y-2)^2-8+3=0$$

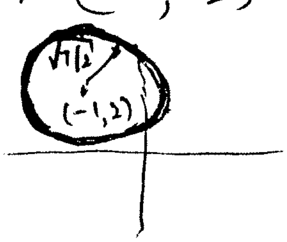
$$2(x+1)^2+2(y-2)^2-7=0$$

$$\therefore \frac{2(x+1)^2+2(y-2)^2}{2}=\frac{7}{2}$$

$$(x+1)^2+(y-2)^2=\frac{7}{2}$$

$$(x+1)^2+(y-2)^2=\left(\sqrt{\tfrac{7}{2}}\right)^2$$

(c)
$\therefore$ Center $=(-1,2)$, radius $=\sqrt{\frac{7}{2}}$

(d)


(8) (a) $9x^2+25y^2=225$

(a) This can be an ellips (x^2 and y^2 has different Coefficients but of the same sign).

(b) $$\frac{9x^2+25y^2}{225}=\frac{225}{225}$$

$$\frac{9x^2}{225}+\frac{25y^2}{225}=1$$

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

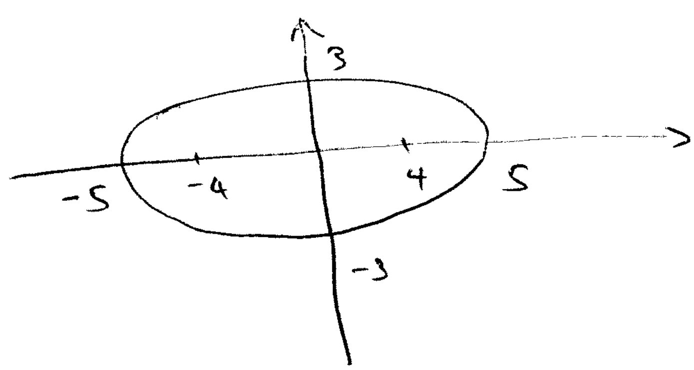
$$\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$$

(c) Center = (0, 0)

major axis Length = $2\times5=10$

minor axis Length = $2\times3=6$

(d)



~~focus~~

foci $=(\pm\sqrt{5^2-3^2},0)$

$=(\pm\sqrt{25-9},0)$

$=(\pm\sqrt{16},0)$

foci $=(\pm4,0)$

8 (b) $x^2+2x+y^2+26=0$

(a) This can be a circle

(x^2 and y^2 has same Coefficients)

(b) $x^2+2x+\left(\frac{2}{2}\right)^2-\left(\frac{2}{2}\right)^2+y^2+26=0$

$\underbrace{x^2+2x+1}-1+y^2+26=0$

$(x+1)^2+y^2+25=0$

~~Bm~~ $(x+1)^2+y^2=-25$

(c), (d) This is not a circle because, the r^2 term cannot be negative. !

(a) $x^3+5x^2+6x+2=0$

Use the rational root test:

If $\frac{p}{q}$ is a root,

p is a factor of the constant term

q is a factor of the leading term

$2 = 2\times 1 = (-2)\times(-1) \Rightarrow p = \pm 1, \pm 2$

$1 = 1\times 1 = (-1)\times(-1) \Rightarrow q = \pm 1$

$\therefore$ Possible choices for $\frac{p}{q} = \pm 1, \pm 2$.

Try -1 first:

$(-1)^3 + 5(-1)^2 + 6(-1) + 2 \overset{?}{=} 0$

$-1+5-6+2=0$ ✓

$\therefore$ $x=-1$ is a solution and

$(x+1)$ is a factor.

$$\begin{array}{r|rrrr} & 1 & 5 & 6 & 2 \\ -1 & & -1 & -4 & -2 \\ \hline & 1 & 4 & 2 & 0 \end{array}$$

Synthetic division.

$\underbrace{1 \quad 4 \quad 2}_{x^2+4x+2}$

$\therefore \quad x^3 + 5x^2 + 6x + 2 = (x+1)(x^2 + 4x + 2) = 0$

Now consider $x^2 + 4x + 2 = 0$.

$$\therefore \quad x = \frac{-4 \pm \sqrt{4^2 - (4)(1)(2)}}{(2)(1)}$$

$$= \frac{-4 \pm \sqrt{16 - 8}}{2}$$

$$= \frac{-4 \pm \sqrt{8}}{2}$$

$$= \frac{-4 \pm \sqrt{(4)(2)}}{2}$$

$$= \frac{-4 \pm 2\sqrt{2}}{2}$$

$$x = -2 \pm \sqrt{2}$$

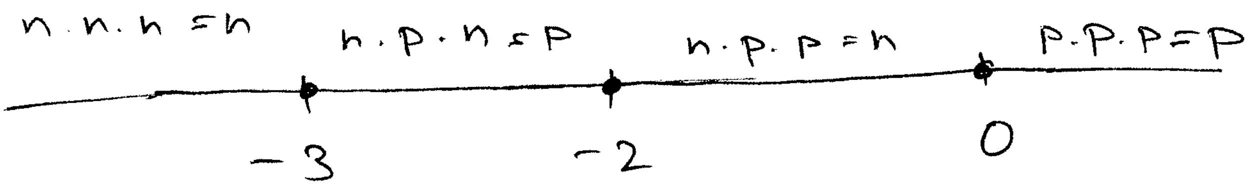
$\therefore$ All roots:

$x = -1, \quad -2 + \sqrt{2}, \quad -2 - \sqrt{2}$

(10) $x^3 + 5x^2 + 6x \geq 0$

$x(x^2+5x+6) \geq 0$

$x(x+3)(x+2) \geq 0$



$[-3, -2] \cup [0, \infty)$.

(11) $\log_6 (x+2) + \log_6 (x-1) = 1$ —— (*)

$$\log_6 [(x+2)(x-1)] = \log_6 6$$

$\therefore$ $(x+2)(x-1) = 6$

$$x^2 + x - 2 = 6$$

$$x^2 + x - 8 = 0$$

$$\therefore \quad x = \frac{-1 \pm \sqrt{1-(4)(1)(-8)}}{2}$$

$$= \frac{-1 \pm \sqrt{1+32}}{2}$$

$$x = \frac{-1 \pm \sqrt{33}}{2}$$

$\therefore$ $x \approx 2.3723, \ -3.\cancel{372}7,$

So if we plug in the negative number to the original equation:

IT DOE NOT WORK

$$\log_6(-3.3727+2) + \log_6(-3.3727-1) \neq 1.$$

but $\log_6(2.3723+2) + \log(2.3723-1) = \underline{1}$

(12) $6^{(x+2)} = 9^{(x-1)}$

Take logarithms of both sides:

$$\log_{10} 6^{(x+2)} = \log_{10} 9^{(x-1)}$$

$$\therefore (x+2)\log_{10} 6 = (x-1)\log_{10} 9$$

$$x\log_{10} 6 + 2\log_{10} 6 = x\log_{10} 9 - \log_{10} 9$$

$$x\log_{10} 6 - x\log_{10} 9 = -\log_{10} 9 - 2\log_{10} 6$$

$$x(\log_{10} 6 - \log_{10} 9) = -\log_{10} 9 - 2\log_{10} 6$$

$$\therefore x = \frac{-\log_{10} 9 - 2\log_{10} 6}{\log_{10} 6 - \log_{10} 9}$$

$$x = \frac{\log_{10} 9 + 2\log_{10} 6}{\log_{10} 9 - \log_{10} 6}$$

$$x \approx 14.2571$$

$$\approx 14.26$$

(13) $8^{(x+1)} = 2^{(x-3)^2}$

Note that $8 = 2^3$

$\therefore \left(2^3\right)^{(x+1)} = 2^{(x-3)^2}$

$\therefore 2^{3(x+1)} = 2^{(x-3)^2}$

$\therefore 3(x+1) = (x-3)^2$

$3x+3 = x^2-6x+9$

$0 = x^2-9x+6$

$$x = \frac{9 \pm \sqrt{9^2-(4)(1)(6)}}{(2)(1)}$$

$$= \frac{9 \pm \sqrt{81-24}}{2}$$

$$x = \frac{9 \pm \sqrt{57}}{2}$$

$\therefore x \approx 8.2749, \ 0.7251$

(14)

$$3x - 4y + 2z = 7 \quad \text{—— (1)}$$
$$5x + 3y + 3z = 2 \quad \text{—— (2)}$$
$$2x + 7y + 9z = -5 \quad \text{—— (3)}$$

$(1) \times 5 \Rightarrow \quad 15x - 20y + 10z = 35$

$(2) \times 3 \Rightarrow \quad 15x + 9y + 9z = 6 \qquad (-)$

$$-29y + z = 29 \quad \text{—— (4)}$$

$(1) \times 2 \Rightarrow \quad 6x - 8y + 4z = 14$

$(3) \times 3 \Rightarrow \quad 6x + 21y + 18z = -15 \qquad (-)$

$$-29y - 14z = 29 \quad \text{—— (5)}$$

~~From~~

$(4) \Rightarrow \quad -29y + z = 29$

$(5) \Rightarrow \quad -29y - 14z = 29 \qquad (-)$

$$15z = 0$$

$$\therefore \boxed{z = 0}$$

From 4

$$-29y + 0 = 29.$$

$$\therefore \frac{-29y}{-29} = \frac{29}{-29}$$

$$\therefore \boxed{y = -1}$$

From 1,

$$3x - 4(-1) + 2(0) = 7.$$

$$3x + 4 = 7.$$

$$-4 \qquad -4$$

$$\frac{\cancel{3}x}{\cancel{3}} = \frac{\cancel{3}}{\cancel{3}}$$

$$\boxed{x = 1}$$

$$\therefore \boxed{\begin{aligned} x &= 1 \\ y &= -1 \\ z &= 0 \end{aligned}}$$

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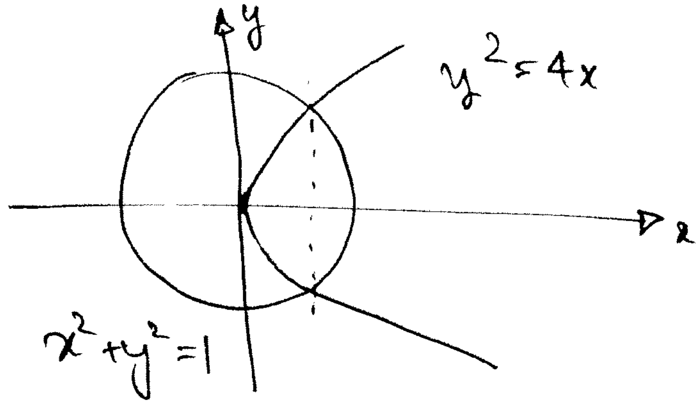


$$y^2 = 4x \quad \text{— (1)}$$

$$x^2 + y^2 = 1 \quad \text{— (2)}$$

Substitute y^2 in (2)

$$x^2 + (4x) = 1$$

$$x^2 + 4x - 1 = 0$$

$$\therefore \quad x = \frac{-4 \pm \sqrt{(4)^2 - (4)(1)(-1)}}{(2)(1)}$$

$$= \frac{-4 \pm \sqrt{16 + 4}}{2}$$

$$= \frac{-4 \pm \sqrt{20}}{2}$$

$$\sqrt{20} = \sqrt{5 \times 4} = \sqrt{5} \times \sqrt{4} = 2\sqrt{5}$$

$$= \frac{-4 \pm 2\sqrt{5}}{2}$$

$$x = -2 + \sqrt{5}$$

$$\therefore \quad x \approx 0.2361, \ -4.2361$$

From ①, note that,

$y^2 = 4x$,

So, x cannot be negative,

$\therefore x = -2 + \sqrt{5}$ (i.e. 0.2361, ...)

is the only valid solution for x.

Then, $y = \pm\sqrt{4x}$

$= \pm 2\sqrt{x}$

$y = \pm 2\sqrt{-2+\sqrt{5}}$

$\therefore y = \pm 0.9717$

$\therefore$ Points of intersection are,

$\left(-2+\sqrt{5},\ 2\sqrt{-2+\sqrt{5}}\right)$, $\left(-2+\sqrt{5},\ -2\sqrt{-2+\sqrt{5}}\right)$.

16 (a) $\dfrac{1}{x(x-2)(x+3)} = \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+3}$

$= \dfrac{A(x-2)(x+3)+Bx(x+3)+Cx(x-2)}{x(x+2)(x+3)}$

$\dfrac{1}{x(x-2)(x+3)} = \dfrac{A(x^2+x-6)+B(x^2+3x)+C(x^2-2x}{x(x+2)(x+3)}$

Compare coefficients:

$x^2 \Rightarrow \quad A+B+C = 0$ —— (1)

$x \Rightarrow \quad A+3B-2C = 0$ —— (2)

Const. $\Rightarrow \quad -6A = 1$ —— (3)

From (3), $\boxed{A = \dfrac{-1}{6}}$

(2) $\Rightarrow \quad A+3B-2C = 0$

(1) $\Rightarrow \quad A+B+C = 0 \qquad (-)$

$2B-3C = 0$

$\therefore \quad 2B = 3C$

i.e $\quad B = \frac{3}{2}C$. —— (4)

∴ ~~From~~ (4)

from ①,

$$\frac{-1}{6} + \frac{3}{2}C + C = 0.$$

$$\frac{3}{2}C + C = \frac{1}{6}.$$

$$C\left(\frac{3}{2} + 1\right) = \frac{1}{6}$$

$$C\left(\frac{3+2}{2}\right) = \frac{1}{6}$$

$$C\left(\frac{5}{2}\right) = \frac{1}{6}.$$

$$\therefore \quad C = \frac{1}{\cancel{6}_3} \times \frac{\cancel{2}}{5}$$

$$\boxed{C = \frac{1}{15}}$$

$\therefore$ from ④, $$B = \frac{\cancel{3}}{2} \cdot \frac{1}{\cancel{15}_5}$$

$$\therefore \boxed{B = \frac{1}{10}}$$

$$\therefore \boxed{\frac{1}{x(x-2)(x+3)} = \frac{-1}{6x} + \frac{1}{10(x-2)} + \frac{1}{15(x+3)}}$$

(b) $$\frac{3x+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1) + (Bx+C)(x+1)}{(x+1)(x^2+1)}$$

$$= \frac{Ax^2 + A + Bx^2 + Bx + Cx + C}{(x^{\cancel{2}}+1)(x^2+1)}$$

$$\frac{3x+2}{(x+1)(x^2+1)} = \frac{(A+B)x^2 + (B+C)x + (A+C)}{(x+1)(x^2+1)}$$

Compare Coefficients:

Const. $\Rightarrow$ $A + C = 2$ —— (1)

x $\Rightarrow$ $B + C = 3$ —— (2)

x^2 $\Rightarrow$ $A + B = 0$ —— (3)

from (3) $\Rightarrow$ $A = -B$ —— (4)

$\therefore$ From (1) $\Rightarrow$ $-B + C = 2$ —— (5)

(2) $\Rightarrow$ $B + C = 3$ (+)

$$2C = 5$$

$\therefore$ $\boxed{C = \frac{5}{2}}$

On the other hand,

$$② \Rightarrow \quad B + C = 3$$

$$⑤ \Rightarrow \quad -B + C = 2 \qquad (-)$$

$$2B = 1$$

$$\therefore \boxed{B = \frac{1}{2}}.$$

$\therefore$ from ④, $\boxed{A = -\frac{1}{2}}$

$$\therefore \frac{3x+2}{(x+1)(x^2+1)} = \frac{(-1/2)}{x+1} + \frac{(1/2)x + (5/2)}{x^2+1}.$$

$$\boxed{\frac{3x+2}{(x+1)(x^2+1)} = \frac{-1}{2(x+1)} + \frac{x+5}{2(x^2+1)}}$$

(C)

$$\frac{x^2+x+1}{(x+1)^2(x^2+1)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+1)}$$

$$= \frac{A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2}{(x+1)^2(x^2+1)}$$

$$= \frac{A(x^3+x^2+x+1) + B(x^2+1) + (Cx+D)(x^2+2x+1)}{(x+1)^2(x^2+1)}$$

$$= \cancel{A(x^3+x^2+x+1) + B(x^2+1) + (Cx^3+Dx^2+2Cx^2+2Dx}$$

$$= \frac{A(x^3+x^2+x+1) + B(x^2+1) + (Cx^3+2Cx^2+Cx+Dx^2+2Dx+D)}{(x+1)^2(x^2+1)}$$

Compare coefficients:

Const. $\Rightarrow$ $A + B + D = 1$ — (1)

$x \Rightarrow A + C + 2D = 1$ — (2)

$x^2 \Rightarrow A + B + 2C + D = 1$ — (3)

$x^3 \Rightarrow A + C = 0.$ — (4)

Then,

$$\begin{array}{lrl} 1 \Rightarrow & A + B + D & = 1 \\ 2 \Rightarrow & A + C + 2D & = 1 \quad (-) \\ \hline & B - C - D & = 0 \quad — 5 \end{array}$$

$$\begin{array}{lrl} 3 \Rightarrow & A + B + 2C + D & = 1 \\ 4 \Rightarrow & A + B + D & = 1 \quad (-) \\ \hline & 2C & = 0 \end{array}$$

$$\therefore \boxed{C = 0}$$

$\therefore$ 4 $\Rightarrow$ $A + C = 0$

$\therefore A + 0 = 0$

$\therefore \boxed{A = 0}$.

$$\begin{array}{lrl} \therefore \text{ from } 1 \Rightarrow & B + D & = 1 \\ \text{from } 5 \Rightarrow & B - D & = 0 \quad (+) \\ \hline & 2B & = 1 \end{array}$$

$$\therefore \boxed{B = \frac{1}{2}}$$

From 5 again, $\frac{1}{2} - 0 - D = 0$.

$$\therefore \boxed{D = \frac{1}{2}}$$

$$\therefore \frac{x^2+x+1}{(x+1)^2(x^2+1)} = \frac{0}{(x+1)} + \frac{(1/2)}{(x+1)^2} + \frac{0x+(1/2)}{x^2+1}$$

$$\therefore \boxed{\frac{x^2+x+1}{(x+1)^2(x^2+1)} = \frac{1}{2(x+1)^2} + \frac{1}{2(x^2+1)}}$$

(17) (a) $N(t) = N_0 e^{kt}$

$$\frac{2 \times 3200}{3200} = \frac{3200 e^{k \cdot 72}}{3200}$$

$$2 = e^{72k}.$$

$$\ln 2 = \ln\left(e^{72k}\right)$$

$$\frac{\ln 2}{72} = \frac{\cancel{72}k}{\cancel{72}}$$

$$\therefore \quad k = \frac{\ln 2}{72}$$

$k \approx 0.009627$ per hour

In scientific notation, $k \approx 9.627 \times 10^{-3}$ per hour

(b) $$\frac{144{,}000}{3200} = \frac{3200 e^{0.009627t}}{3200}$$

$$45 = e^{0.009627t}$$

$$\ln 45 = \ln\left(e^{0.009627t}\right)$$

$$\ln 45 = 0.009627t$$

about 16 days

$$\therefore \quad t = \frac{\ln 45}{0.009627} \approx 395.42 \text{ hours}$$

(c) $M(t) = M_0 e^{-103t}$

$$\frac{1000}{200000} = \frac{\cancel{200{,}000}}{\cancel{200{,}000}} e^{-103t}$$

$$\frac{1}{200} = e^{-103t}$$

$$\ln\left(\frac{1}{200}\right) = \ln\left(e^{-103t}\right).$$

$$\ln\left(\frac{1}{200}\right) = -103t$$

$$\therefore \quad t = \frac{1}{(-103)} \ln\left(\frac{1}{200}\right)$$

$$t = 0.0514 \text{ hours.}$$

about 3 minutes

(18) $y = 2\sin\left(\pi x + \frac{\pi}{4}\right)$.

(a) Period $= \frac{2\pi}{\pi} = 2$

(b) Phase shift $= -\left(\frac{\pi}{4}\right)/\pi$

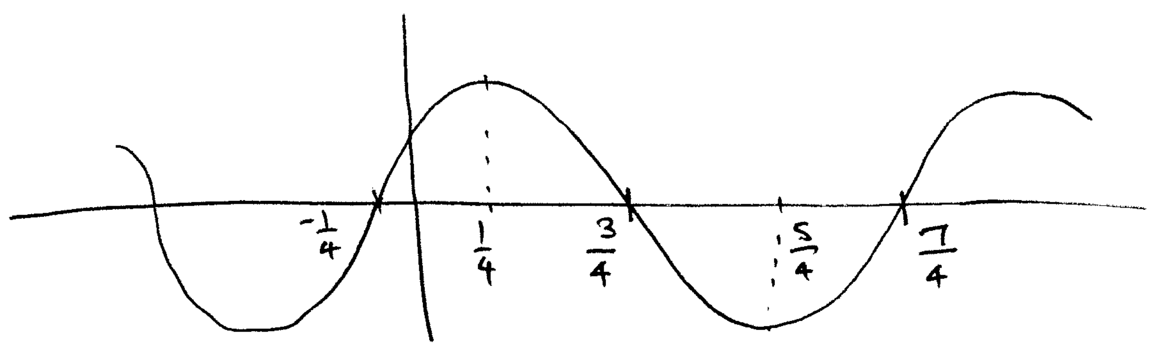
$= -\frac{1}{4}$

(c) Amplitude $= 2$

(d) Vertical shift $= 0$

Compare to
$y = A \sin(Bx - C) +$
Period $= \frac{2\pi}{B}$
Phase shift $= \frac{C}{B}$
Amplitude $= A$
Vertical shift $= D$

(e)



(19) $2\cos^2 x + 5\cos x - 3 = 0$

Let $y = \cos x$.

Then, $2y^2 + 5y - 3 = 0$

$$y = \frac{-5 \pm \sqrt{5^2 - (4)(2)(-3)}}{(2)(2)}$$

$$= \frac{-5 \pm \sqrt{25 + 24}}{4}$$

$$y = \frac{-5 \pm \sqrt{49}}{4}$$

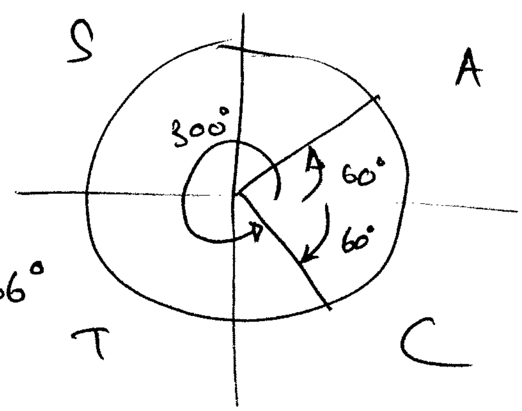
$$y = \frac{-5 \pm 7}{4}$$

i.e. $y = \frac{-5-7}{4} = \frac{-12}{4} = -3$

or $y = \frac{-5+7}{4} = \frac{2}{4} = \frac{1}{2}$

But $y = \cos x$, and $-1 \le \cos x \le 1$, so, $y = -3$ is not a valid solution.

$\therefore$ $y = \cos x = \frac{1}{2}$



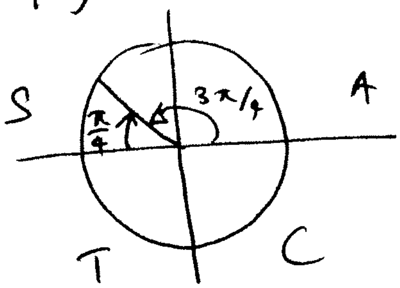
The primary angles which gives $\cos x = \frac{1}{2}$ are $x = 60° + 36°$

$\therefore$ $x = 60° + 360° k$

or $x = 300° + 360° k$ for any <u>integer</u> k.

(20) (a) $\cos\left(\frac{11\pi}{4}\right) = \cos\left(2\pi + \frac{3\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$

$\therefore \cos\left(\frac{11\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right)$



$$\boxed{\cos\left(\frac{11\pi}{4}\right) = -\frac{\sqrt{2}}{2}}$$

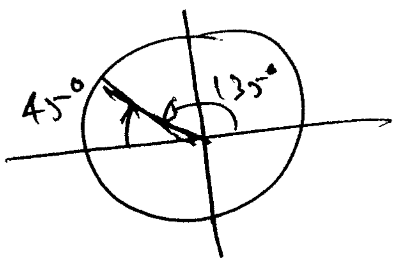
Or, $\frac{11\pi}{4} \times \frac{180^\circ}{\pi} = 495^\circ = 360^\circ + 135^\circ$

Then, $\cos\left(\frac{11\pi}{4}\right) = \cos(495^\circ) = \cos(360^\circ + 135^\circ)$

$= \cos(135^\circ)$

$= -\cos(45^\circ)$

$= -\frac{\sqrt{2}}{2}$



(b) $\tan(22.5^\circ) = \tan\left(\frac{45^\circ}{2}\right)$

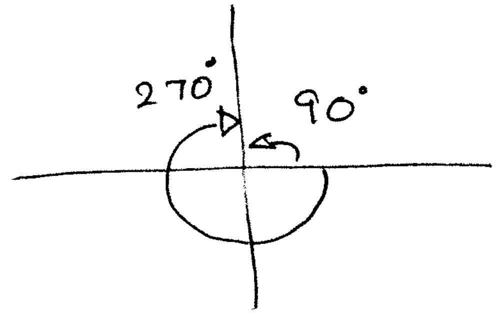
$= \frac{\sin 45^\circ}{1 + \cos 45^\circ}$ (Half angle rule).

$= \frac{\sqrt{2}/2}{1 + \sqrt{2}/2}$

$$\boxed{\tan(22.5^\circ) = \frac{\sqrt{2}}{2 + \sqrt{2}}}$$

(c) $\sin(-270^\circ) = \sin 90^\circ$

$$\boxed{\sin(-270^\circ) = 1}$$



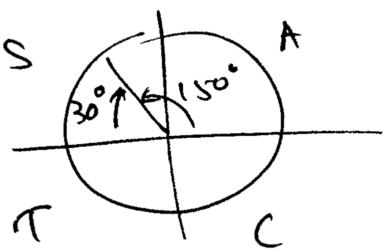
(d) $\cos 75^\circ = \cos(30^\circ + 45^\circ)$

$= \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ$

$\cancel{= \cos} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$

$$\boxed{\cos 75^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}}$$

or $\cos 75^\circ = \cos \frac{150^\circ}{2} = \pm\sqrt{\frac{1 + \cos 150^\circ}{2}}$



~~th~~ But we need the (+) Only because 75° is in First Quadrant.

$\cos 150^\circ = -\cos 30^\circ$
$= -\frac{\sqrt{3}}{2}$

$\therefore \cos 75^\circ = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}}$

$= \sqrt{\frac{2 - \sqrt{3}}{4}}$

$\cos 75^\circ = \frac{\sqrt{2 - \sqrt{3}}}{4}$

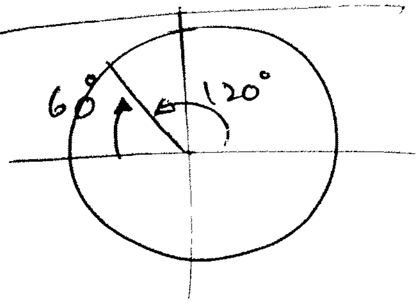
Showing that $\frac{\sqrt{6} - \sqrt{2}}{4} = \frac{\sqrt{2 - \sqrt{3}}}{4}$ is a good problem! see if you can do it.

(e) $\cos 105^\circ \cos 15^\circ = \frac{1}{2}\left(\cos(105^\circ - 15^\circ) + \cos(105^\circ + 15^\circ)\right)$

$= \frac{1}{2}\left(\cos 90^\circ + \cos(120^\circ)\right)$

$= \frac{1}{2}\left(-\frac{1}{2}\right)$

$\boxed{= -\frac{1}{4}}$



$\therefore \cos 120^\circ = -\cos 60^\circ$

$= -\frac{1}{2}$

$\boxed{\cos 90^\circ = 0}$

(f) $\sec\left(\frac{\pi}{8}\right) = \sec(22.5^\circ)$

$$= \frac{1}{\cos 22.5^\circ}$$

$$= \frac{1}{\cos\left(\frac{45^\circ}{2}\right)}$$

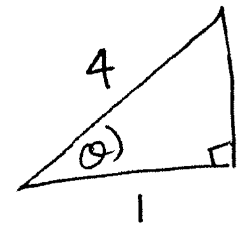
$$= \frac{1}{\sqrt{\frac{1 + \cos 45^\circ}{2}}}$$

$$= \frac{1}{\sqrt{\frac{1 + \sqrt{2}/2}{2}}}$$

$$= \frac{1}{\sqrt{\frac{2 + \sqrt{2}}{4}}} = \frac{1}{\left(\frac{\sqrt{2 + \sqrt{2}}}{2}\right)}$$

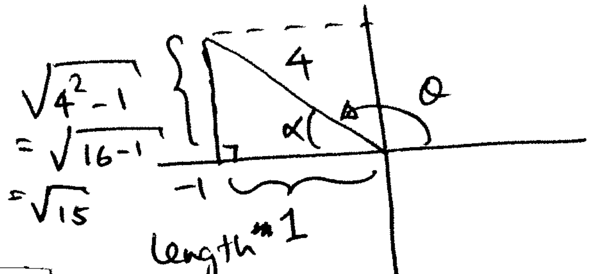
$$\therefore \boxed{\sec \frac{\pi}{8} = \frac{2}{\sqrt{2 + \sqrt{2}}}}$$

(21) (a) $\cos\left(\cos^{-1}\left(\frac{1}{4}\right)\right) = \frac{1}{4}$



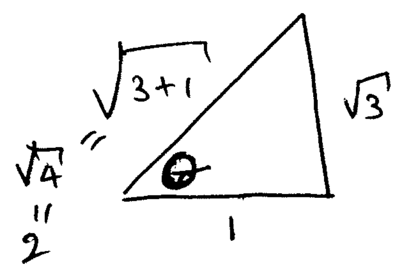
(b) $\sin\left(\cos^{-1}\left(\frac{-1}{4}\right)\right)$

$= \frac{\sqrt{15}}{4}$



$$\boxed{\therefore \sin\left(\cos^{-1}\left(\frac{-1}{4}\right)\right) = \frac{\sqrt{15}}{4}}$$

(c) $\boxed{\cos\left(\tan^{-1}(\sqrt{3})\right) = \frac{1}{2}}$



(d) $\cos^{-1}(\cos 75^\circ) = 75^\circ$

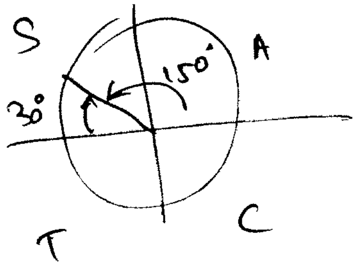
(Since 75° is in the 1^{st} quadrant)

(e) $\sin^{-1}(\sin(150^\circ)) \neq 150^\circ$

$\sin^{-1}(\sin(150^\circ)) = \sin^{-1}(\sin 30^\circ)$

$= \sin^{-1}\left(\frac{1}{2}\right)$

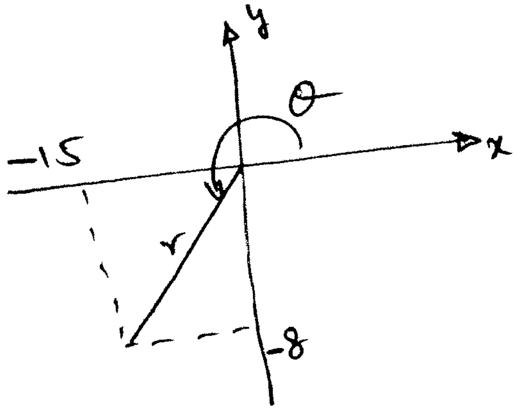
$$\boxed{= 30^\circ}$$



This is a very important point.

The range of $\sin^{-1}$ function is $[-90^\circ, 90^\circ]$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ radians.

(22)



Recall that

$\tan\theta = \dfrac{y}{x}$

$\therefore r = \sqrt{15^2 + 8^2}$

$= \sqrt{225 + 64}$

$= \sqrt{289}$

$r = 17$

$\therefore \sin\theta = \dfrac{y}{r} = \dfrac{-8}{17}$

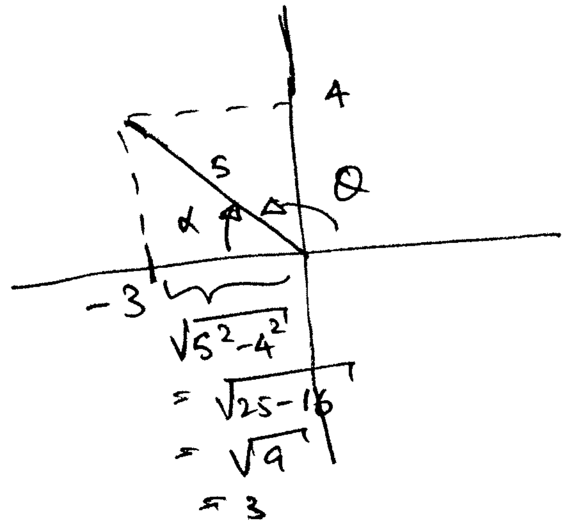
$\cos\theta = \dfrac{x}{r} = \dfrac{-15}{17}$

$\cot\theta = \dfrac{x}{y} = \dfrac{-15}{-8} = \dfrac{15}{8}.$

$\sec\theta = \dfrac{1}{\cos\theta} = -\dfrac{17}{15}$

$\csc\theta = \dfrac{1}{\sin\theta} = -\dfrac{17}{8}$

(23) $\sin\theta = \frac{4}{5}$



Note that

$\alpha = \sin^{-1}\left(\frac{4}{5}\right) \approx 53°$

So, $\theta \approx 127°$

So, $2\theta \approx 254°$, which is in the 3rd quadrant,

and $\frac{\theta}{2} \approx 63.5°$, which is in the 1st quadrant (To do this checking you <u>NEED the calculator</u>)

Then, Also note that, Since

$\sin\theta = \frac{4}{5}$, and θ in the 2nd quadrant,

$\cos\theta = \frac{-3}{5}$, and $\tan\theta = -\frac{4}{3}$

(a)

$\therefore \cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\cdot\left(\frac{4}{5}\right)^2 = 1 - 2\cdot\left(\frac{16}{25}\right)$

$= 1 - \frac{32}{25} = \frac{25-32}{25} = \frac{-7}{25}$

$$\boxed{\cos 2\theta = \frac{-7}{25}}$$

(b) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$= \dfrac{2\left(\frac{-4}{3}\right)}{1-\left(\frac{-4}{3}\right)^2}$

$= \dfrac{\left(\frac{-8}{3}\right)}{\left(1-\frac{16}{9}\right)}$

$= \dfrac{\left(\frac{-8}{3}\right)}{\left(\frac{9-16}{9}\right)}$

$= \dfrac{\left(\frac{-8}{3}\right)}{\left(\frac{-7}{9}\right)}$

$= \dfrac{-8}{\cancel{3}} \cdot \left(-\dfrac{\overset{3}{\cancel{9}}}{7}\right)$

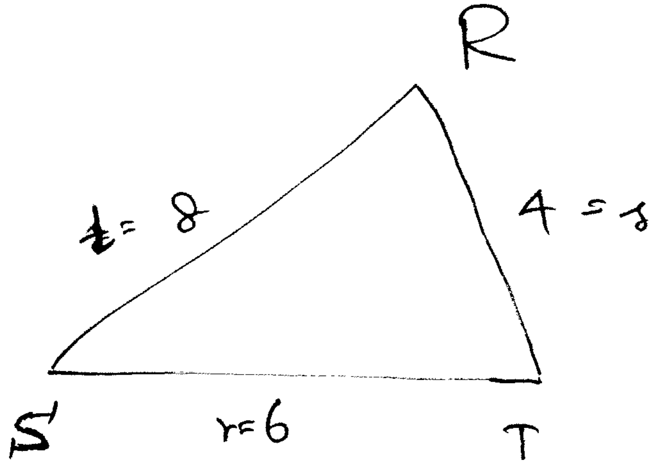
$\boxed{\tan 2\theta = \dfrac{24}{7}}$

(c) $\cos\left(\frac{\theta}{2}\right) = \sqrt{\dfrac{1+\cos\theta}{2}} = \sqrt{\dfrac{1+\left(\frac{-3}{5}\right)}{2}} = \sqrt{\dfrac{\left(\frac{5-3}{5}\right)}{2}}$

(Since $\frac{\theta}{2}$ is in the 1st quadrant)

$\therefore \cos(\theta/2) = \sqrt{\dfrac{\left(\frac{2}{5}\right)}{2}} = \sqrt{\dfrac{1}{5}}$ i.e. $\boxed{\cos\left(\frac{\theta}{2}\right) = \dfrac{1}{\sqrt{5}}}$

24



$r^2 = s^2 + t^2 - 2st\cos R$

$6^2 = 4^2 + 8^2 - (2)(4)(8)\cos R$

$36 = 16 + 64 - 64\cos R$

$36 = 80 - 64\cos R$

$-80 \qquad -80$

$-44 = -64\cos R$

$\therefore \cos R = \frac{44}{64}$

$\therefore R \approx 46^\circ.\cancel{6}$

Similarly,

$s^2 = r^2 + t^2 - 2rt\cos S$

$16 = 36 + 64 - (2)(6)(8)\cos S$

$16 = 100 - 96\cos S$

$-100 \qquad -100$

$-84 = -96\cos S$

$\cos S = \frac{84}{96} \Rightarrow S \approx 29^\circ$

Use Law of Cosines

$t^2 = s^2 + r^2 - 2sr\cos T$

$64 = 16 + 36 - (2)(4)(6)\cos T$

$64 = 52 - 48\cos T$

$12 = -48\cos T$

$\therefore \cos T = -\frac{12}{48}$

$\therefore T \approx 105^\circ$

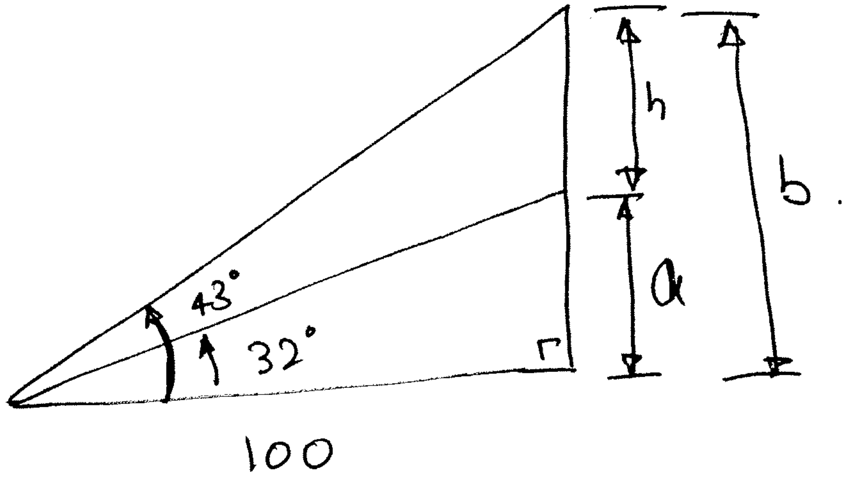
Check

$R + S + T$

$= 46 + 29 + 105$

$= 180^\circ$

25



$a = 100 \tan 32° \approx 62.5'$

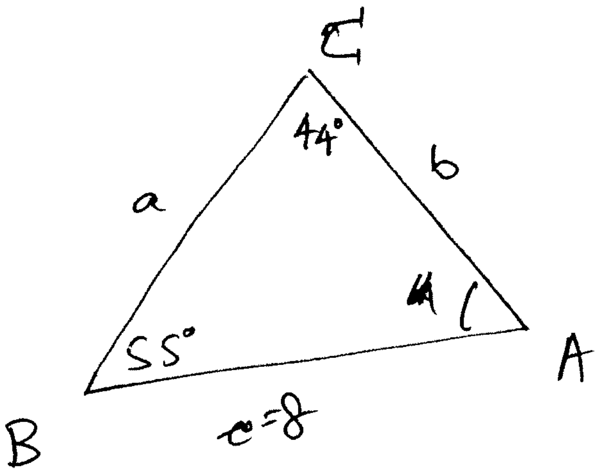
$b = 100 \tan 43° \approx 93.3'$

$\therefore h = b - a$

$= 62.5 - 93.3$

$\boxed{h \approx 30.8'}$

(26)



$A = 180^\circ - (55^\circ + 44^\circ)$

$= 180^\circ - 99^\circ$

$\boxed{A = 81^\circ}$

Using law of Sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin C}{c} = \frac{\sin 44^\circ}{8}$$

$$\therefore \quad \frac{\sin A}{a} = \frac{\sin 81^\circ}{a} = \frac{\sin C}{c}$$

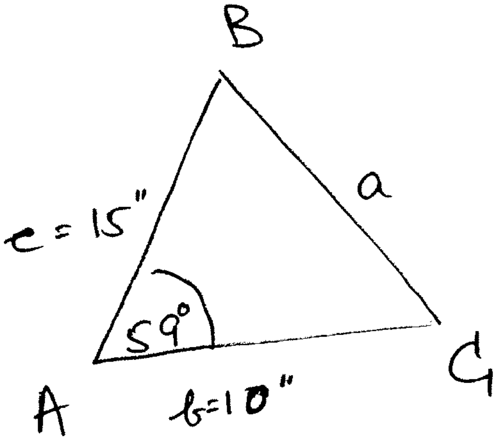
$$\frac{\sin 81^\circ}{a} = \frac{\sin 44^\circ}{8}.$$

$$\therefore \boxed{a = \frac{8 \sin 81^\circ}{\sin 44^\circ} \approx 11.37''}$$

Similarly $\dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

$$\frac{\sin 55^\circ}{b} = \frac{\sin 44^\circ}{8} \Rightarrow \boxed{b = \frac{8 \sin 55^\circ}{\sin 44^\circ} \approx 9.43''}$$

(27)



~~[illegible]~~

Using law of Cosines,

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$= 10^2 + 15^2 - (2)(10)(15)\cos 59°$$

$$= 100 + 225 + 300\cos 59$$

$$a^2 \approx 479.51$$

$$\therefore a \approx 21.9''$$

Use the law of Sines to find the angles B and C

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin 59}{21.9} = \frac{\sin B}{10} = \frac{\sin C}{15}$$

$$\therefore \sin B = \frac{10 \times \sin 59}{21.9} \approx 0.3194 \Rightarrow \boxed{B \approx 23°}$$

$$\therefore C = 180° - 59° - 23° = 98° \Rightarrow \boxed{C \approx 98°}$$

(28) $\sin\theta(\tan\theta + \cot\theta) = \sec\theta$

L.H.S.

$\sin\theta\,(\tan\theta + \cot\theta)$

$= \sin\theta \left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \right)$ (ratio identi

$= \sin\theta \left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} \right)$ (common denom.)

$= \cancel{\sin\theta} \left(\frac{1}{\cancel{\sin\theta}\cos\theta} \right)$ (Pythagorean $\sin^2\theta + \cos^2\theta = 1$)

$= \frac{1}{\cos\theta}$

$= \sec\theta = $ R.H.S.

(24) $\sec^2 + \csc^2\theta = \sec^2\theta \csc^2\theta$

L.H.S.

$$\sec^2 + \csc^2\theta = \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}$$ (Reciprocal identities

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta \ \sin^2\theta}$$ (Common denom.

$$= \frac{1}{\cos^2\theta \sin^2\theta}$$ (Pythagorean $\sin^2\theta + \cos^2\theta = 1$

$$= \frac{1}{\cos^2\theta} \cdot \frac{1}{\sin^2\theta}$$

$$= \sec^2\theta \cdot \csc^2\theta$$ (Reciprocals)

= R.H.S.

✓

(30) $\sin\theta = \csc\theta - \cos\theta\cot\theta$

R.H.S.

$= \csc\theta - \cos\theta\cot\theta$

$= \dfrac{1}{\sin\theta} - \cos\theta \cdot \dfrac{\cos\theta}{\sin\theta}$

$= \dfrac{1}{\sin\theta} - \dfrac{\cos^2\theta}{\sin\theta}$

$= \dfrac{1-\cos^2\theta}{\sin\theta}$

From $\sin^2\theta + \cos^2\theta = 1$
we see that
$\sin^2\theta = 1 - \cos^2\theta$

$= \dfrac{\sin^{2}\theta}{\cancel{\sin\theta}}$

$= \sin\theta = \text{L.H.S.}$

(31) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

From the hint,

$$L.H.S = \sin 3\theta = \sin(2\theta + \theta)$$

$$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$$

$$= (2\sin\theta\cos\theta)\cos\theta + (1 - 2\sin^2\theta)\sin\theta$$

$$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$$

$$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$$

$$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$$

$$= 3\sin\theta - 4\sin^3\theta \qquad = R.H.S.$$

This is called the TRIPLE ANGLE formula for sin.

(Similarly one can show that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta \,)$$