Chapter 5

This chapter introduces **probability** to quantify randomness.

Section 5.1: How Can Probability Quantify Randomness?

The probability of an event is viewed as a numerical measure of the chance that the event will occur.

Ex 1: Tossing a fair coin. Chance of getting head is 50%. (=0.5)

Randomness

The possible outcomes are known, but it is uncertain which will occur for any given observation.

Ex 2: Rolling a die, Drawing cards from a deck

Random Phenomenon

For a random phenomenon,

- Individual outcomes are unpredictable
- With a large number of observations, predictable patterns occur
- The proportion of times that something happens is highly random and variable in the short run, but very predictable in the long run

Ex 3: The outcome of tossing a fair coin once is unpredictable. However, in the long run, when the coin is tossed a large number of times, we can predict that the ratio (# of times heads occurred**:** # of times tails occurred) to be approximately 1:1. (Because none of them is favored over the other as the coin is fair)

Probability of a Random Phenomenon

The probability of a particular outcome in a random phenomenon is the proportion of times that the outcome would occur in a long run of the observations.

Ex 3: The probability of getting heads when tossing a fair coin being ½ means that the proportion of times heads would occur in a long run of the observations is ½.

The probability (change) of rain being 40% for today means, out of a large number of days with similar atmospheric conditions like today, proportion of days that rain occurs is 0.4.

Independent Trials

Different trials of a random phenomenon are **independent** if the outcome of any one trial is not affected by the outcome of any other trial.

Ex 4: Assume you have 4 A's of ♦,♥,♣, and ♠.

- Drawing a card with replacement → Independent trials. (The card you get in the first draw does not affect what you can get in the second draw. I.e. if you get ♥ in the first draw, the change of getting ♥ in the second draw still remains to be ¼)

- Drawing a card without replacement → Not independent. (If you get ♥ in the first draw, the change of getting ♥ in the second draw is 0. So what you got in the first draw has affected the chances of what you can get in the second draw)

Section 5.2 How Can We Find Probabilities

Two terms related to probabilities are introduced here.

- Sample Space
- Event

Sample Space

For a random phenomenon, the **sample space** is the set of all possible outcomes.

Ex 5: Roll a die once

$$Sample\ Space = \{1,2,3,4,5,6\}$$

Tossing two coins

$$Sample\ Space = \{(H,H),(H,T),(T,H),(T,T)\}$$

There was a blind test to identify three types of beer (A, B, C) by tasting them. If the interested random variable is number of correct identifications, list out the sample space. (Three cups contain three different beers and they test all three before they know the result).

$$S = \{0,1,3\}$$

Explanation: It is possible that you do not identify any of the three correctly, or get one of them correctly and the other two incorrect or get all three correct. But you cannot identify two correctly and get the other one wrong. (Identifying two correctly means the third one automatically becomes correct)

Event

An **event** is a subset of the sample space.

Ex 6: Refer to the case of rolling a die (Ex 5)

$$Event\ of\ rolling\ a\ even\ number = \{2,4,6\}$$

$$Event\ of\ rolling\ a\ multiple\ of\ 3 = \{3,6\}$$

Referring to the tossing of two coins

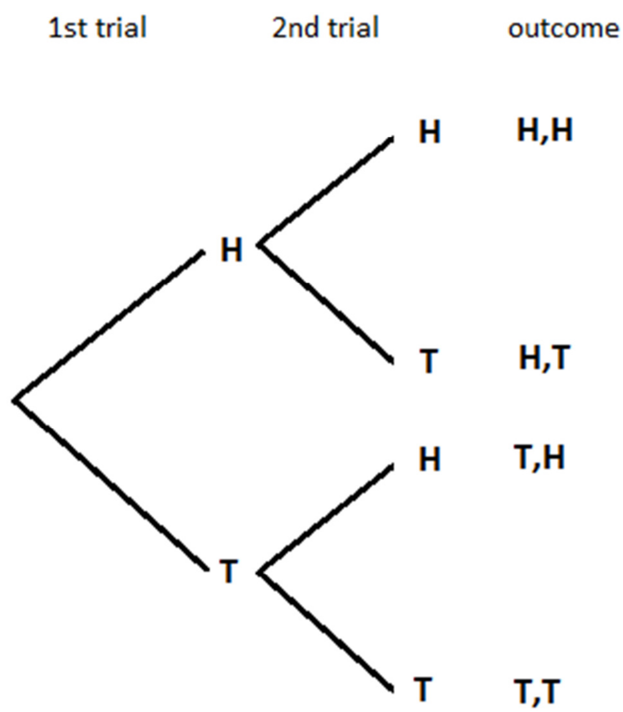
$$Event\ of\ getting\ at\ least\ one\ heads = \{(H,H),(H,T),(T,H)\}$$

$$Event\ of\ getting\ exactly\ one\ head = \{(H,T),(T,H)\}$$

Tree Diagrams

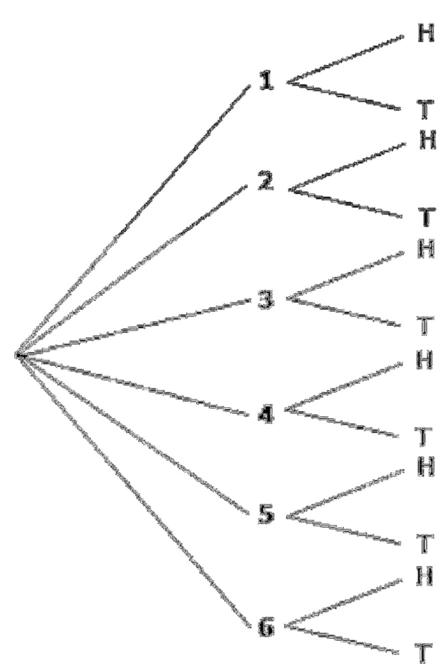
- An ideal way of visualizing sample spaces with a small number of outcomes
- As the number of trials or the number of possible outcomes on each trial increases, the tree diagram becomes impractical

Ex 7: Flipping a coin twice



S = {(H,H), (H,T), (T,H), (T,T)}

Rolling a die and tossing a coin



Probabilities for a Sample Space

The probabilities for the outcomes in a sample space must follow two basic rules

- The probability of each individual outcome is between 0 and 1
- The total of all the individual probabilities equals one

Ex 8: Tossing a fair coin twice

Outcome	**Probability**
(H,H)	¼
(H,T)	¼
(T,H)	¼
(T,T)	¼
Total	**1**

Could the following be a probability distribution?

Tossing a biased coin: Probability of getting heads = 0.65; Probability of getting tails = 0.3;

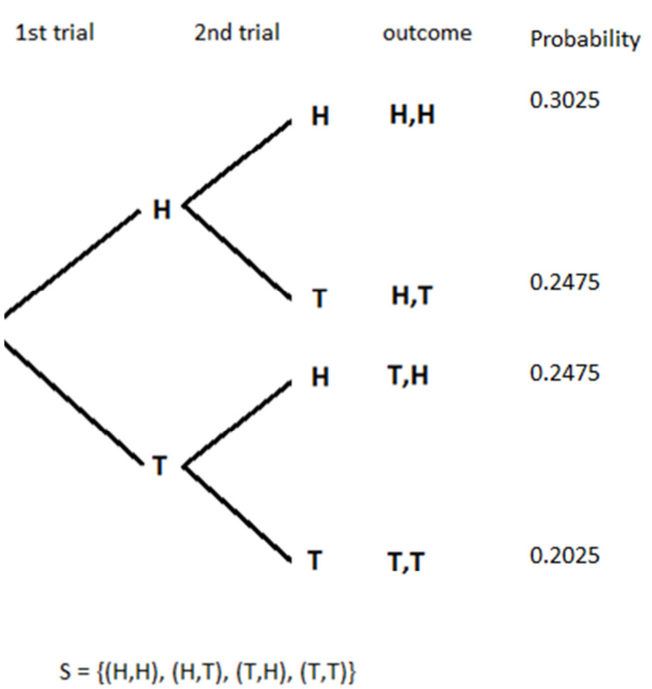
No. Since the probability of each individual outcome do not add up to 1.

Probability of an Event

- The probability of an event A, denoted by $P(A)$, is obtained by add the probabilities of the individual outcomes in the event.
- *When all the possible outcomes are equally likely,*

$$P(A) = \frac{number\ of\ outcomes\ in\ event\ A}{number\ of\ outcomes\ in\ the\ sample\ space}$$

Ex 9: Tossing an unfair coin twice



A – getting only one head $A = \{(H,T),(T,H)\}$

B – getting no tails $B = \{(H,H)\}$

$$P(A) = P\big((H,T)\big) + P\big((T,H)\big) = 0.2475 + 0.2475 = 0.495$$

$$P(B) = P\big((H,H)\big) = 0.3025$$

Example for equally likely outcomes: Rolling a fair die

A – getting a multiple of three

$$S = \{1,2,3,4,5,6\}$$

$$A = \{3,6\}$$

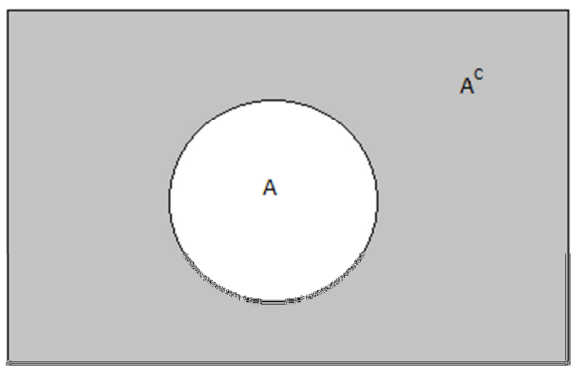
Therefore

$$P(A) = \frac{number\ of\ outcomes\ in\ A}{number\ of\ outcomes\ in\ S} = \frac{2}{6} = \frac{1}{3}$$

Complement of an Event

The **complement** of an event A consists of all the outcomes in the samples space that are not in A. It is denoted by A^C. The probabilities of A and A^C add to 1. So,

$$P(A^C) = 1 - P(A)$$

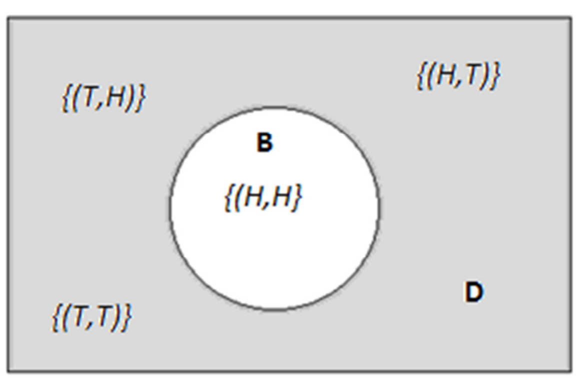


Ex 10: Refer to the Ex 9. What is the probability of getting at least one tails?

D – getting at least one tails = B^C

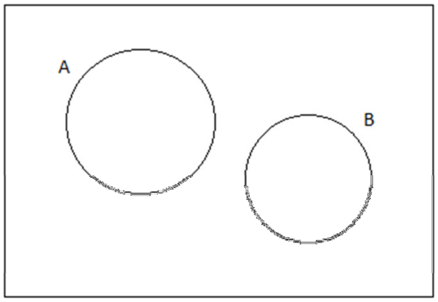
$$P(D) = P(B^C) = 1 - P(B)$$

$$P(D) = 1 - 0.3025 = 0.6975$$

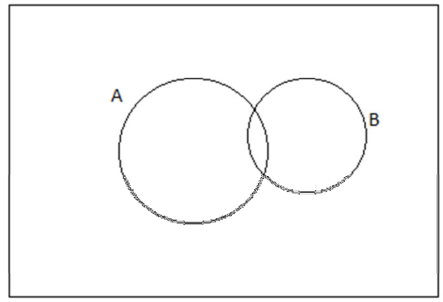


Disjoint Events

Two events, A and B, are **disjoint** if they do not have any common outcomes.



Two disjoint sets



Two sets which are **not** disjoint

Ex 11: Rolling a fair die

A – getting an even number

B – getting an odd number

$$A = \{2,4,6\}$$

$$B = \{1,3,5\}$$

Since A and B do not have any common outcomes, they are disjoint.

C – getting a multiple of 3

D – getting a multiple of 2

$$C = \{3,6\}$$

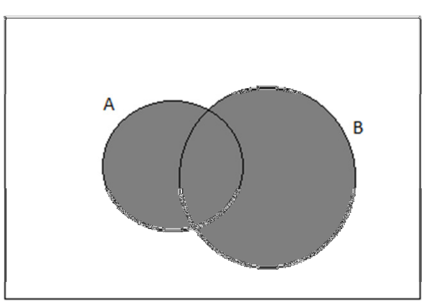
$$D = \{2,4,6\}$$

Since C and D have a common outcome which is 6, they are **not** disjoint.

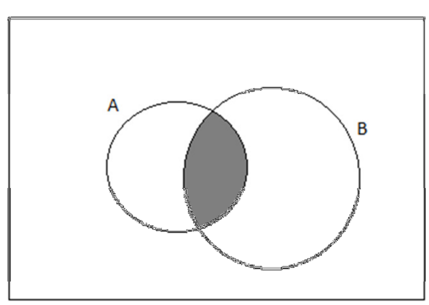
Intersection and Union of Two Events

The **intersection** of A and B consists of outcomes that are in both A *and* B. $(A \cap B)$

The **union** of A and B consists of outcomes that are in A *or* B. In probability, "A or B" denotes that A occurs or B occurs or both occur. $(A \cup B)$



A or B

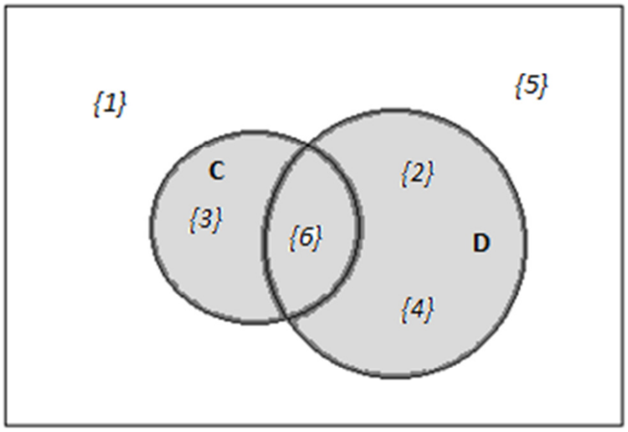


A and B

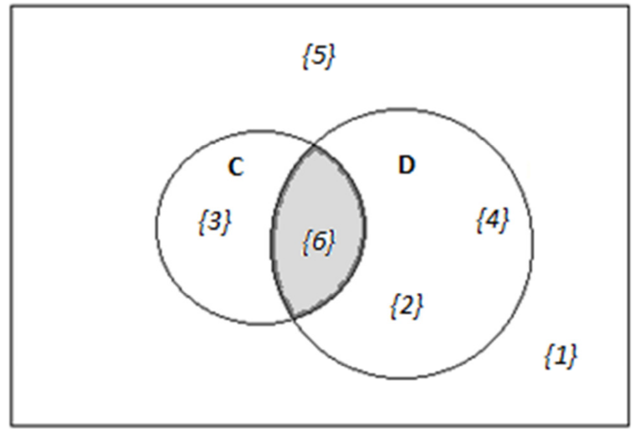
<u>Ex 12:</u> Refer to Ex 11.

C and D = $C \cap D = \{6\}$

C or D = $C \cup D = \{2,3,4,6\}$



C or D



C and D

Consider the case of tossing two coins

A – getting at least one heads

B – getting at least one tails

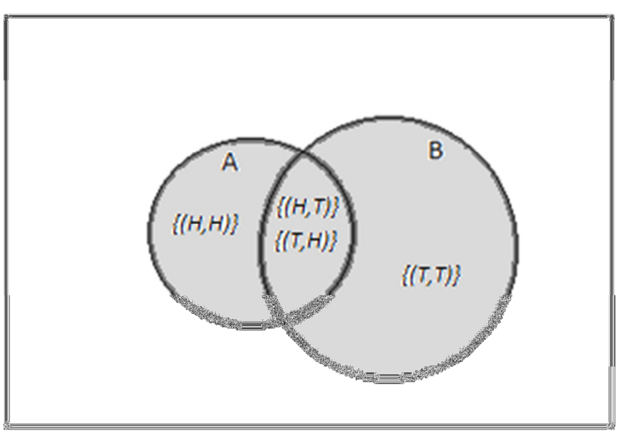
$S = \{(H,H),(H,T),(T,H),(T,T)\}$

$A = \{(H,H),(H,T),(T,H)\}$

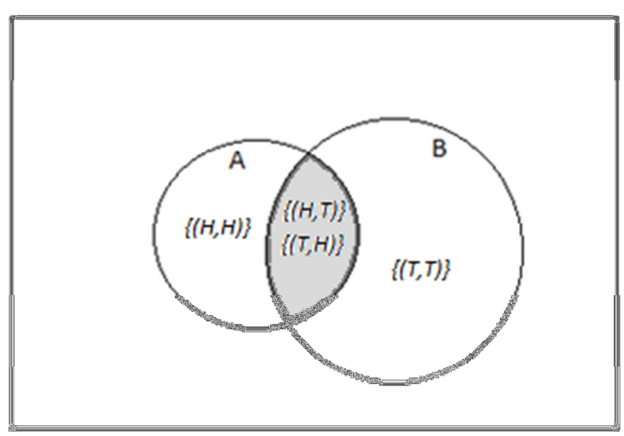
$B = \{(H,T),(T,H),(T,T)\}$

$A\ and\ B = A \cap B = \{(H,T),(T,H)\}$

$A\ or\ B = A \cup B = \{(H,H),(H,T),(T,H),(T,T)\} = S$



A or B

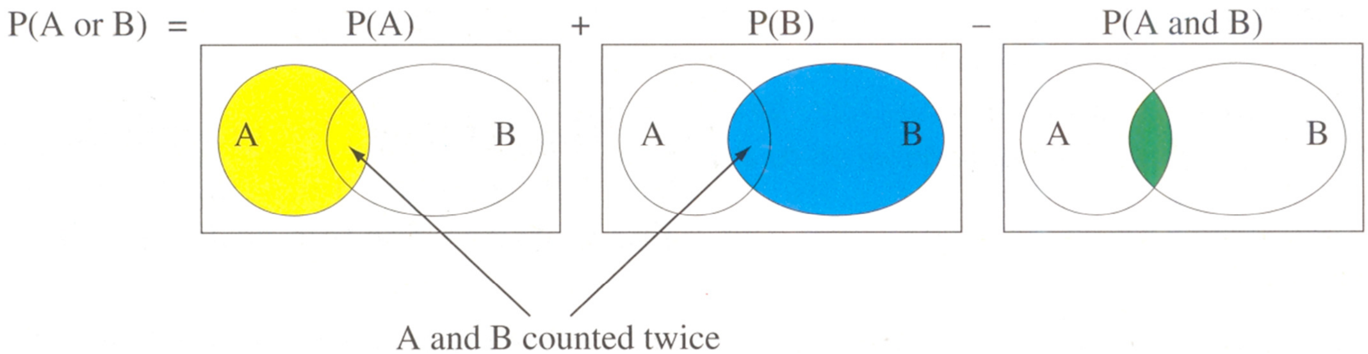


A and B

Addition Rule: Probability of the Union of Two Events

For the **union** of two events, $P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$.

If the events are **disjoint**, then $P(A\ and\ B) = 0$, so $P(A\ or\ B) = P(A) + P(B)$



Ex 13:

Out of a class of 50, the number of students who passed exam 1 is 30, number of students who passed exam 2 is 20, and the number of students who passed both exams are 15. Find the probability of a randomly selected student out of the class to have passed at least one exam.

A – selected student has passed exam 1

B – selected student has passed exam 2

The event of selected student having passed at least one exam $= A\ or\ B$

From the above formula,

$$P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$$

$P(A) = \frac{30}{50};\ P(B) = \frac{20}{50};$

$A\ and\ B$ - is the event of the selected student having passed both exams. Therefore,

$P(A\ and\ B) = \frac{15}{50}$

Therefore,

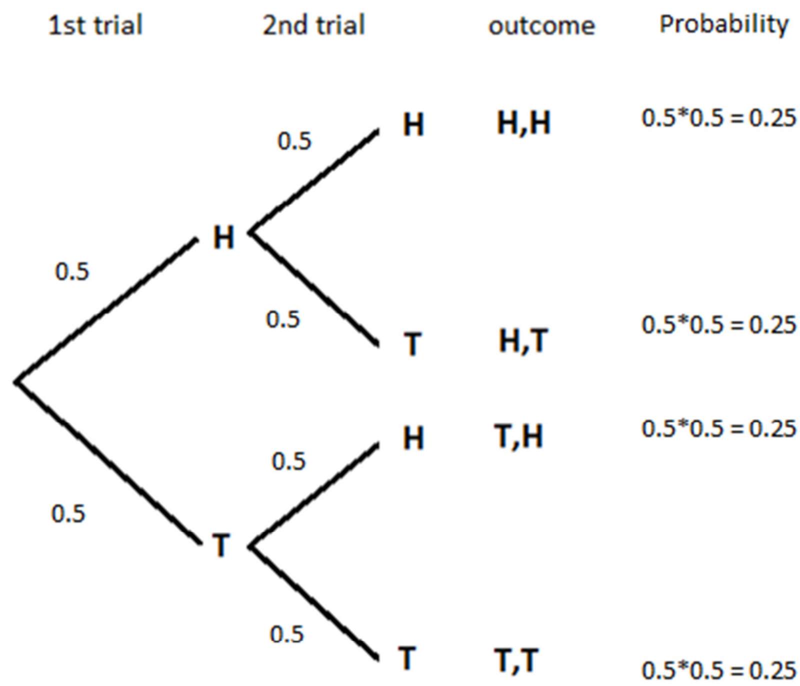
$P(A\ or\ B) = \frac{30}{50} + \frac{20}{50} - \frac{15}{50} = \frac{35}{50} = 0.7$

Multiplication Rule: Probability of the Intersection of Independent Events

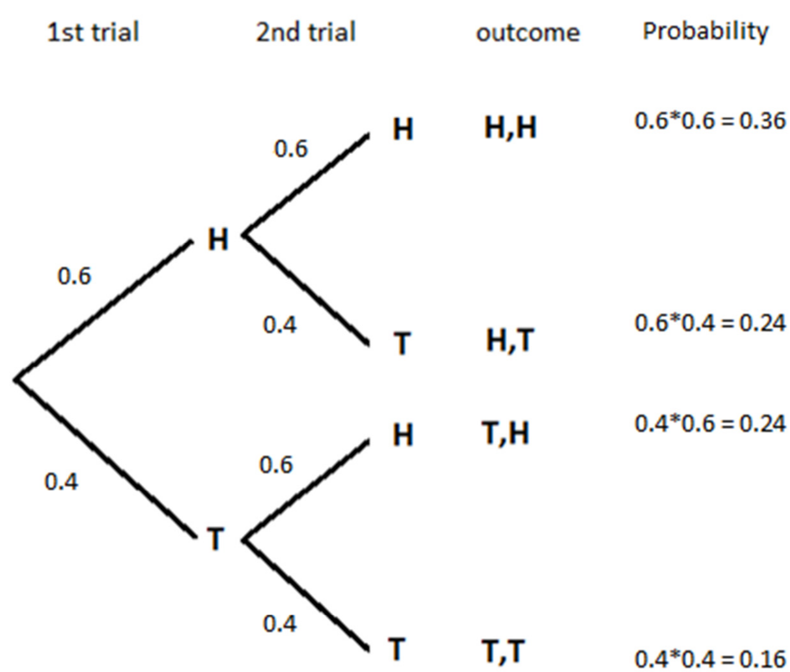
For the **intersection** of two **independent** events, A and B,

$$P(A\ and\ B) = P(A) \times P(B)$$

Ex 14: Tossing a fair coin twice



Tossing a coin with $P(H) = 0.6$ and $P(T) = 0.4$ twice



$P((H,H)) = P(H) \times P(H) = 0.6 \times 0.6 = 0.36$

$P((H,T)) = P(H) \times P(T) = 0.6 \times 0.4 = 0.24$

$P((T,H)) = P(T) \times P(H) = 0.4 \times 0.6 = 0.24$

$P((T,T)) = P(T) \times P(T) = 0.4 \times 0.4 = 0.16$

$P((T,T)) = P(T) \times P(T) = 0.4 \times 0.4 = 0.16$

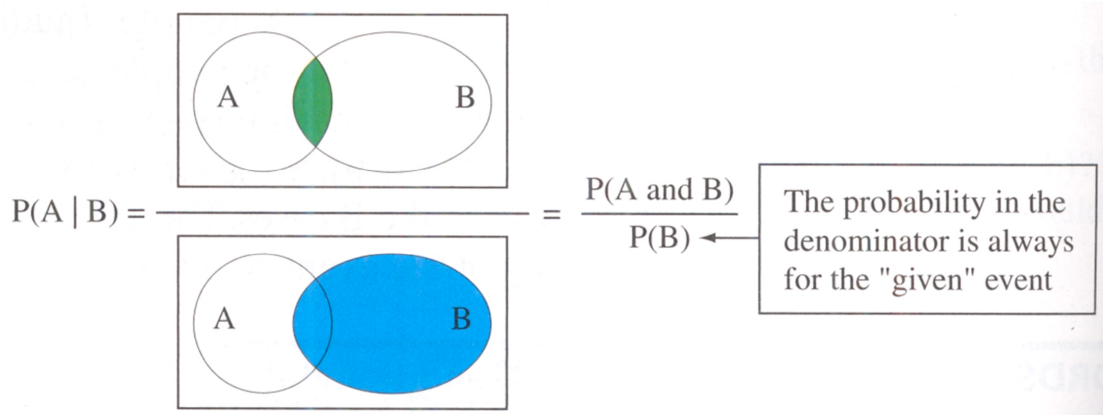
Section 5.3: Conditional Probability

Conditional Probability

For events A and B, the **conditional probability** of event A, given that event B has occurred, is

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

$P(A|B)$ is read as "the probability of event A given event B".



Ex 15:

The graduate director in the department of Mathematics is considering whether the department should offer Time series course during the next semester. So he has contacted 50 students in the department and collects the data on whether they would plan to take the class during the next semester. The collected data are as follows.

	Want to take	**Do not want to take**	
Male	20	5	25
Female	10	15	25
	30	20	50

- Find the probability of a student wanting to take the class
- Find the probability of a student wanting to take the class given that the student is female
- Find the probability of a student not wanting to take the class given that the student is male

Multiplication Rule for Evaluating P(A and B)

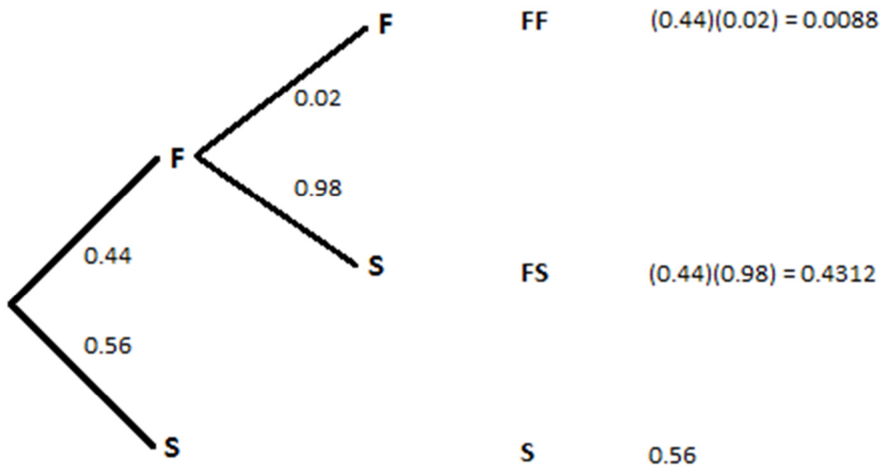
For events A and B, the probability that A and B both occur equals,

$$P(A \text{ and } B) = P(B) \times P(A|B)$$

Applying the conditional probability formula to $P(B|A)$, we also see that,

$$P(A \text{ and } B) = P(A) \times P(B|A)$$

Ex 16: In a tennis match, a player made faults on his first serve 44% of the time and only 2% on his second serves. Find the probability that he makes a double fault.



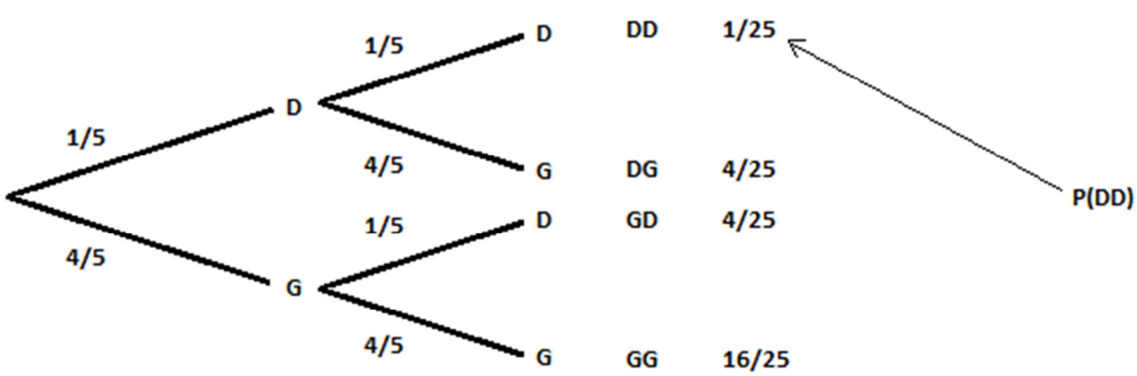
Therefore $P(FF) = 0.0088$

Ex 17:

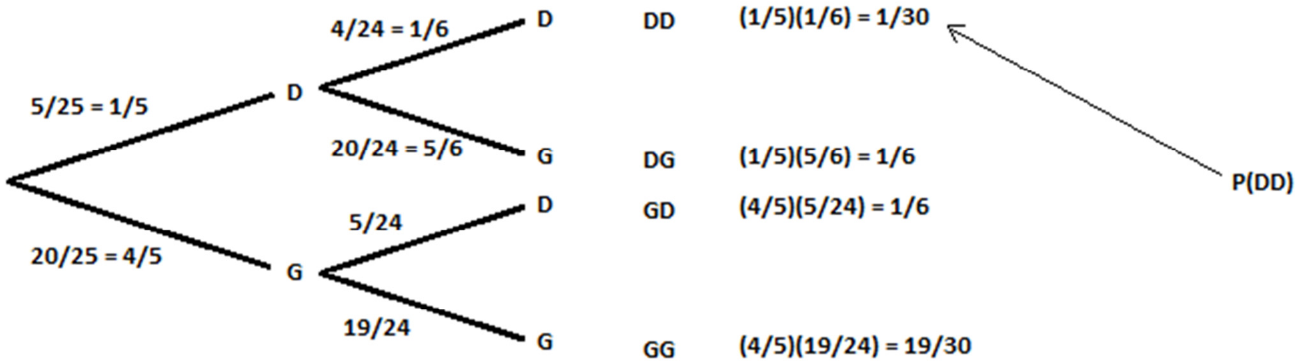
Sampling with or without replacement

There is a container in front of you which contains 25 pens. Out of these 25 pens, 20 of them write well. And 5 of them are defective. You will be asked to select two of them. Find the probability that both of them are defective.

***Case I*:** With replacement



***Case II*:** Without replacement



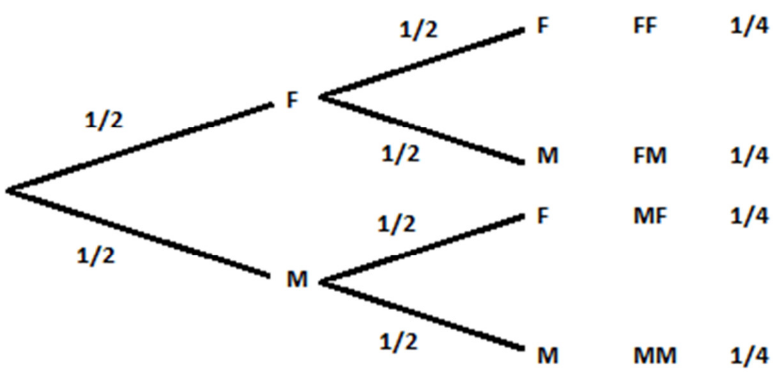
Independent Events, in Terms of Conditional Probability

Events A and B are **independent**, if $P(A|B) = P(A)$, or equivalently, if $P(B|A) = P(B)$. If either of them holds, then the other does too, and also, $P(A\ and\ B) = P(A) \times P(B)$.

Note: You can check for independence of two events A and B by checking one of the following three conditions.

- Is $P(A|B) = P(A)$?
- Is $P(B|A) = P(B)$?
- Is $P(A\ and\ B) = P(A) \times P(B)$?

Ex 18: 5.45: For a family with two children, let A denote {first child is female} and let C denote {both children are female}. Find whether A and C are independent.



Find whether A and C are independent.

$$P(A) = \frac{1}{2}; P(C) = \ P(FF) = \frac{1}{4};$$

$$P(A\ and\ C) = \ P(FF) = \frac{1}{4}; P(A) \times P(C) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

Therefore, $P(A\ and\ C) \neq P(A) \times P(C)$.

Hence, A and C are not independent.

Let B denote {second child is female}. Are A and B independent?

$P(B) = \frac{1}{2};$

$P(A\ and\ B) = P(FF) = \frac{1}{4}; P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Therefore, $P(A\ and\ B) = P(A) \times P(B)$.

Hence A and B are independent.