Lec 2

Output Regulation via Error Feedback

Problem:

$\boxed{\text{Output regulation via error feedback}}$

Given A, B, C, P, Q, S.

Find if possible matrices F, G, H

such that

$(S)_{ef}$ The matrix

$$\begin{pmatrix} A & BH \\ GC & F \end{pmatrix} \quad \text{has eigenvalues in } \mathbb{C}^-. \qquad (1)$$

$(R)_{ef}$ For each $(x^\circ, \xi^\circ, w^\circ)$, the solution

$(x(t), \xi(t), w(t))$ of

$$\begin{aligned} \dot{x} &= Ax + BH\xi + Pw \\ \dot{\xi} &= GCx + F\xi + GQw \\ \dot{w} &= Sw \end{aligned} \qquad (2)$$

satisfying $(x(0), \xi(0), w(0)) = (x^\circ, \xi^\circ, w^\circ)$

is such that $\underset{t \to \infty}{Lt} (Cx + Qw) = 0$.

Output Regulation via error feedback:

Lemma 2.1:

Assume (H1), suppose $\exists$ a feedback law

$$\dot{\xi} = F\xi + Ge \qquad (3)$$
$$u = H\xi.$$

for which $(S)_{ef}$ holds. Then condition $(R)_{ef}$ also holds iff $\exists$ matrices Π, Σ such that

$$\Pi S = A\Pi + BH\Sigma + P$$
$$\Sigma S = F\Sigma \qquad (4)$$
$$0 = C\Pi + Q.$$

Proof :-

consider the Sylvester Eqn

$$\begin{pmatrix} \Pi \\ \Sigma \end{pmatrix} S = \begin{pmatrix} A & BH \\ GC & F \end{pmatrix} \begin{pmatrix} \Pi \\ \Sigma \end{pmatrix} + \begin{pmatrix} P \\ GQ \end{pmatrix} \quad (5)$$

Because

$$\sigma(S) \cap \sigma \begin{pmatrix} A & BH \\ GC & F \end{pmatrix} = \phi \quad (6)$$

it follows that the sylvester Eqn has a unique solution

As before the column span of $\begin{bmatrix} \Pi \\ \Sigma \\ I_r \end{bmatrix}$ is the invariant subspace $\mathcal{V}^+$ of (7)

$$\begin{pmatrix} A & BH & P \\ GC & F & GQ \\ 0 & 0 & S \end{pmatrix}$$

This invariant subspace is associated with the eigenvalues in $\overline{\mathbb{C}^+}$.

Let us now consider the co-ordinate transformation:

$$\tilde{x} = x - \Pi w$$
$$\tilde{\xi} = \xi - \Sigma w \qquad (8)$$

It follows that

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{\xi}} \end{pmatrix} = \begin{pmatrix} A & BH \\ GC & F \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\xi} \end{pmatrix} \qquad (9)$$

$$\dot{w} = Sw. \qquad (10)$$
$$e = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\xi} \end{pmatrix} + (C\Pi + Q)w$$

It would follow, using arguments similar to Lemma 1 in Lec 1 that

$$\underset{t \to \infty}{Lt} \; e(t) = 0 \quad \text{for every } (\tilde{x}(0), \tilde{\xi}(0), \omega(0))$$

holds.

$$\text{iff} \quad \underset{t \to \infty}{Lt} \; (C\Pi + Q)\, e^{St} = 0$$

$$\text{iff} \quad C\Pi + Q = 0 \qquad (11)$$

From the sylvester eqs (5) and (11) we have

$$\Pi S = A\Pi + BH\Sigma + P \qquad (12)$$

$$\Sigma S = G(C\Pi + Q) + F\Sigma$$

$$= F\Sigma \qquad (13)$$

& $C\Pi + Q = 0$ (14)

which is identical to (4)

(QED)

Note that 12 & 14 are similar to what we have seen before

12 is Sylvester eqn.

14 is error cancellation.

Q: How do we interpret 13

what does $\Sigma S = F\Sigma$ mean.

Recall $\dim x = n$

$\dim \xi = v$

$\dim \omega = r$

S is a $r \times r$ matrix.

Σ is a $v \times r$ matrix.

F is a $v \times v$ matrix.

Assume that Σ is of full rank.

ie assume r columns of Σ are l.i.

It would imply that $v \geq r$

Thus $F\Sigma = \Sigma S$
would imply that the subspace spanned by columns of Σ is invariant under F.

Moreover:
Restriction of F to this particular invariant subspace is precisely given by S, which characterizes the exosystem.
Thus the compensator infact contains a copy of the exosystem.

Sol^n to the problem of output regulation through error-feedback.

Hypothesis (H 3) The pair (C, A) is detectable.

Hypothesis $(H\,3)_{\text{strong}}$ The pair

$$C^e = \begin{pmatrix} C & Q \end{pmatrix}$$

$$A^e = \begin{pmatrix} A & P \\ 0 & S \end{pmatrix}$$

is detectable.

To see that $(H3)_{strong}$ is stronger than $(H3)$ note that:

$$(H3) \iff \operatorname{rank} \begin{pmatrix} \lambda I - A \\ C \end{pmatrix} = n$$

for every eigenvalue of A in $\mathbb{C}^+$

$$(H3)_{strong} \iff \operatorname{rank} \begin{pmatrix} A - \lambda I & P \\ 0 & S - \lambda I \\ C & Q \end{pmatrix} = n + r$$

for every eigenvalues of A & S in $\mathbb{C}^+$.

When (H3) is not satisfied,

clearly $(H3)_{strong}$ is also not satisfied.

It follows that $(H3)_{strong}$ is stronger

than (H3).

Interesting proposition.

Suppose (H3) holds and $(H3)_{\text{strong}}$ does not hold. Consider the augmented system

$$\dot{x}^e = A^e x^e + B^e u.$$
$$e = C^e x^e$$

where
$$A^e = \begin{pmatrix} A & P \\ 0 & S \end{pmatrix} \qquad x^e = \begin{pmatrix} x \\ w \end{pmatrix}$$

$$C^e = \begin{pmatrix} C & Q \end{pmatrix}$$

$$B^e = \begin{pmatrix} B \\ 0 \end{pmatrix}$$

$\exists$ a transformation

$$\tilde{x}^e = T^e x^e$$

such that in the new co-ordinate.

A^e, B^e, C^e

assumes the form.

$$\left(\begin{array}{c|cc} A & P_1 & 0 \\ \hline 0 & S_{11} & 0 \\ 0 & S_{21} & S_{22} \end{array}\right), \quad \begin{pmatrix} B \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} C & Q_1 & 0 \end{pmatrix}$$

where

$$\begin{pmatrix} A & P_1 \\ 0 & S_{11} \end{pmatrix} \quad \begin{pmatrix} C & Q_1 \end{pmatrix}$$

is detectable i.e.

satisfies (H3) strong.