Lec 1

Regulation by State Feedback

$$\dot{x} = Ax + Bu + Pw \qquad (1)$$

$$y = Cx + Qw \qquad (2)$$

u is the control.

w is the disturbance.

Goal is to have

$$\underset{t \to \infty}{Lt} \|y(t) - y_{ref}(t)\| = 0.$$

— x —

Define

$$\tilde{w} = \begin{pmatrix} w \\ y_{ref} \end{pmatrix}$$

$$\dot{x} = Ax + Bu + \begin{pmatrix} P & 0 \end{pmatrix} \tilde{w}(t). \quad (3)$$

$$e(t) = Cx + \begin{pmatrix} Q & -I \end{pmatrix} \tilde{w}(t) \quad (4)$$

we want

$$\underset{t \to \infty}{Lt} e(t) = 0 \qquad (5)$$

State Feedback

$$u = Kx + Lw \quad (6)$$

Exosystem :-

$$\dot{w} = Sw \quad (7)$$

Output Regulation Problem.

Given $\{A, B, C, P, Q, S\}$ find two matrices K and L such that.

$(S)_{fi}$: $A + BK$ has eigenvalues in $\mathbb{C}^-$

$(R)_{fi}$: For each (x^0, w^0), the solution $(x(t), w(t))$ of

$$\dot{x} = (A + BK)x + (P + BL)w \quad (8)$$

$$\dot{w} = Sw \quad (9)$$

satisfying $(x(0), w(0)) = (x^0, w^0)$ is such that

$$\lim_{t \to \infty} (Cx + Qw) = 0 \quad (10)$$

Hypothesis :

(H 1) The exosystem $\dot{w} = Sw$ is antistable, ie all the eigenvalues of S have non-negative real part.

(H 2) coming later.

Lemma $\boxed{1}$:

Assume (H1). Suppose $\exists$ a feedback

$$u = Kx + Lw \quad (11)$$

for which $(S)_{fi}$ holds.

Then condition $(R)_{fi}$ also holds

iff $\exists$ a matrix Π which solves

the following linear equations:

$$\Pi S = (A + BK)\Pi + (P + BL) \quad (12)$$

$$0 = C\Pi + Q. \quad (13)$$

Proof of Lemma 1:

Equation (12) is a Sylvester eqn.

The spectrum $\sigma(S)$ is in the closed r.h.p.

The spectrum $\sigma(A+BK)$ is in the open l.h.p.

It follows that

$$\sigma(S) \cap \sigma(A+BK) = \underset{\text{empty}}{\underset{\uparrow}{\phi}}.$$

Eqn (12) has a unique solution Π.

Let us now consider a co-ordinate transformation

$$\begin{pmatrix} \tilde{x} \\ w \end{pmatrix} = \begin{pmatrix} x - \Pi w \\ w \end{pmatrix} \qquad (14)$$

We have

$$\begin{aligned}
\dot{\tilde{x}} &= \dot{x} - \Pi \dot{w} \\
&= Ax + Bu + Pw - \Pi S w \\
&= Ax + BKx + BLw + Pw - \Pi S w \\
&= (A+BK)x + (P+BL)w - \Pi S w \\
&= (A+BK)\tilde{x} + \underbrace{(A+BK)\Pi w + (P+BL)w - \Pi S w}_{=0} \\
&= (A+BK)\tilde{x} \qquad (15)
\end{aligned}$$

Moreover

$$e = Cx + Qw \quad (16)$$
$$= C\tilde{x} + (C\Pi + Q)w$$
$$= Ce^{(A+BK)t}\tilde{x}(0) + (C\Pi + Q)e^{St}w(0).$$

Since $A+BK$ has all eigenvalues with negative real parts,

$(R)_{fi}$ is satisfied iff

i.e. $\underset{t\to\infty}{Lt}\ e(t) = 0$

for every $(\tilde{x}(0), w(0))$ iff

$$\underset{t\to\infty}{Lt}\ (C\Pi + Q)e^{St} = 0.$$

and this in turn occurs iff

$$C\Pi + Q = 0.$$

(H 2)

The pair (A, B) is stabilizable.

Theorem 1

Assume (H1) and (H2). Then the problem of output regulation via full information can be solved iff $\exists$ matrices

$$\Pi \quad \& \quad \Gamma$$

which solve

$$\Pi S = A\Pi + B\Gamma + P. \quad (17)$$

$$0 = C\Pi + Q. \quad (18).$$

Proof of Theorem 1

(Necessity)

Assume that the output regulation is solvable using full information. It follows from Lemma 1 that $\exists$ Π that solves (12), (13). Let us define

$$\Gamma = K\Pi + L \quad (19)$$

It follows from (12) that

$$\Pi S = A\Pi + B\Gamma + P \quad (20)$$

Hence (17) & (18) are satisfied.

Sufficiency :-

The proof is constructive.

By hypothesis (H2), the pair (A, B) is stabilizable. Hence $\exists$ matrix K such that $A+BK$ has all eigenvalues in $\mathbb{C}^-$.

Let Π & Γ be such that (17) & (18) are satisfied.

We set

$$u = \Gamma w + K(x - \Pi w) \quad (21)$$

Claim is such that the control law solves the problem of output regulation.

If we define

$$L = \Gamma - K\Pi \quad (22)$$

$$u = Lw + Kx \quad (23)$$

has the desired form.

$(S)_{fi}$ indeed holds by the definition of K.

To prove that $(R)_{fi}$ holds we use Lemma 1 again.

It follows from (17) and (22) that

$$\Pi S = A\Pi + B[L + K\Pi] + P$$
$$= (A + BK)\Pi + (BL + P)$$

This is precisely (12) and it would

follow from ~~[illegible]~~ Lemma 1 that

$(R)_{fi}$ holds.

(QED)

Remark

The system of eq^{n} (8) (9) can be written as

$$\dot{x}_{cl} = A_{cl}\, x_{cl} \quad (24)$$

$$x_{cl} = \begin{pmatrix} x \\ w \end{pmatrix}, \quad A_{cl} = \begin{pmatrix} A+BK & P+BL \\ 0 & S \end{pmatrix} \quad (25)$$

$(n+r) \times (n+r)$ matrix A_{cl} has

n eigenvalues in $\mathbb{C}^-$ (those of $A+BK$)

r eigenvalues in $\overline{\mathbb{C}^+}$ (those of S).

$\mathcal{V}^-$: Invariant subspace of A_{cl} associated with $\mathbb{C}^-$

$\mathcal{V}^+$: Invariant subspace of A_{cl} associated with $\overline{\mathbb{C}^+}$

$\mathcal{V}^-$ is spanned by the columns of

$$M^- = \begin{pmatrix} I_n \\ 0 \end{pmatrix}$$

$\mathcal{V}^+$ is complementary to $\mathcal{V}^-$ in $\mathbb{R}^{n+r}$, will be spanned by columns of M^+

where

$$M^{+} = \begin{pmatrix} X \\ I_r \end{pmatrix}$$

$\because$ $\mathcal{V}^{+}$ is invariant under A_{cl},

it would follow that.

$$\begin{pmatrix} A+BK & P+BL \\ 0 & S \end{pmatrix} \begin{pmatrix} X \\ I_r \end{pmatrix} w = \begin{pmatrix} X \\ I_r \end{pmatrix} \tilde{w}$$

$$\Rightarrow \left[(A+BK)X + (P+BL) \right] w = X \tilde{w}$$

$$S w = \tilde{w}$$

$$\Rightarrow (A+BK)X + (P+BL) = XS .$$

Hence X coincides with the unique solution Π of (12).

Conclusion:

Let Π be the ~~of~~ unique sol$^{\underline{n}}$ of (12). column span of

$$M^{+} = \begin{pmatrix} \Pi \\ I_r \end{pmatrix},$$ is invariant under Acl.

Vectors in $\mathcal{V}^{+}$ are of the form

$$\begin{pmatrix} \Pi \omega \\ \omega \end{pmatrix}$$

$\therefore \mathcal{V}^{+} = \{(X, \omega) : X = \Pi \omega\}$.

Remark:

condition (17) expresses the fact that the subspace $\mathcal{V}^+$

where

$$\mathcal{V}^+ = \{(x, \omega) \in \mathbb{R}^{n+r} : x = \Pi \omega\}$$

is a controlled invariant subspace.

of ~~the~~ system

$$\begin{pmatrix} \dot{x} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} A & P \\ 0 & S \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$

i.e. $\mathcal{V}^+$ is rendered invariant by an appropriate choice of a feedback control, ie our case

$$u = \Gamma \omega.$$

$$\dot{x} = Ax + P\omega + B\Gamma\omega$$

$$= A\Pi\omega + P\omega + B\Gamma\omega$$

$$\dot{x} = (A\Pi + P + B\Gamma)\omega = \Pi S\omega$$

$$\Rightarrow \dot{x} = \Pi\dot{\omega}$$

$$\dot{\omega} = S\omega$$

$$\cancel{x = \Pi\omega \Rightarrow \dot{x} = \Pi\dot{\omega}}$$

Hence $\mathcal{V}^+$ is invariant

— x —

Condition (18) expresses the fact that the controlled invariant subspace is annihilated by the error map.

— x —

Condition (13) expresses the fact that

the error map

$$e = Cx + Qw$$

is zero at each point on $\mathcal{V}^+$.

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