Suboptimal Control Design for Eye Movement System

Takafumi Oki * **Bijoy. K. Ghosh** **

* The Texas Tech Universitythe Dept. of Mathematics and Statistics at , Lubbock, Texas, USA(e-mail: takafumi.oki@ttu.edu).
** The Texas Tech Universitythe Dept. of Mathematics and Statistics at , Lubbock, Texas, USA (e-mail: bijoy.ghosh@ttu.edu)

Abstract: The Eye Movement System is represented as Lagrange Dynamical System on the **List** Manifold. We constructed Suboptimal controller which minimized quadratic type cost functional without solving Hamilton Jacobi Bellman Equation based on Inverse Optimal Design and compared designed controller with an local optimal controller derived from HJB equation.

1. INTRODUCTION

In Ghosh and Wijayasinghe [2012], Wijayasinghe et al. [to be published], an eye moment system is described as the Language Dynamical System on the **List** Manifold S^2 and its optimal control problem have been studied intensively from the perspective of feed-forward control. Indeed, the finite horizontal optimal control of the eye movement is solved. On the other hand, a construction of optimal feedback control law has been almost never done. The difficulty of optimal feedback is caused to solve Hamilton Jacobi Bellman PDE. In practice, one has no effective method to solve it numerically and analytically for high dimensional optimal control problems. Recently, designing base approach for the optimal control problem, it's called Inverse Optimal Control Problem, has been established (See Sepulchre et al. [1996]) . A main advantage of the inverse optimal approach is that one can construct an optimal input $u(x)$ without solving HJB equation and it can minimized a corresponding meaningful cost functional which is made from a designing with an arbitrariness. A main drawback is that it is hard to find a Control Lyapunov function from which $u(x)$ is made. In Lagrange dynamical system, it has a useful property which is called an Passivity (See Lozano et al. [2000]). Passivity significantly plays important rules on the stability analysis and construction of stabilizing controllers. Moreover, a Total Energy function which characterizes the Passivity of the System is a candidate of the control Lyapunov function. In our research, we showed that an feedback law which is obtained form Total Energy base control Lyapunov function(See Ordaz-Oliver et al. [2009]) is stabilizing input and minimized a cost index with specific type of nonlinear penalty cost function. Finally, we compared our designed optimal control law with an optimal control law coming from HJB equation numerically.

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2. BASIC NOTATIONS

2.1 Stability of the Dynamical System

This section gives some basic concepts of Stability of the Dynamical System. We consider a time independent nonlinear affine dynamical system defined on $\Omega \in R^n$.

$$\dot{x} = f(x) + g(x)u \tag{1}$$

with *equilibrium* $f(x^*) = 0$ in Ω where $x \in R^n, f : R^n \to R^n, g : R^n \to R^{n\times m}$ are *Lipschitz continuous* on Ω, and $u : R^n \to R^m$ is an input function. Basic stability of equilibrium point x^* of the system is following.

Definition 2.1. an equilibrium x^* of the system is called *Lyapunov Stable* if for any $\epsilon > 0$, there exists $delta > 0$ such that if $\|x(t_0) - x^*\| < \delta$, then $\|x(t) - x^*\| < \epsilon$ for all $t \geq t_0$.

Definition 2.2. an equilibrium x^* of the system is called *Asymptotically Stable* if x^* is stable and $\|x(t) - x^*\| \to 0$ as $t \to \infty$.

3. DYNAMICAL SYSTEM ON THE **LIST**

3.1 Lagrangian Control System

We construct an eye movement system on the **List** manifold S^2(See Wijayasinghe et al. [to be published], Ghosh and Wijayasinghe [2012], Polpitiya et al. [2007]) from the perspective of *Lagrangian Control System.* Let $q = [\theta, \phi]^T \in [-\pi,\ \pi] \times (0,\ \pi)$ be a generalized coordinate on S^2 then *Riemann Metric* on the **LIST** S^2 is given by

$$M(q) = \begin{bmatrix} \sin^2\dfrac{\phi}{2} & 0 \\ 0 & \dfrac{1}{4} \end{bmatrix} \tag{2}$$

From the Riemann Metric, we obtained a Kinematic Energy KE as

$$\begin{aligned} KE\,(q,\dot{q}) &= \frac{1}{2}\dot{q}^T M(q)\dot{q} \\ &= \frac{1}{2}(\dot{\theta})^2 \sin^2\frac{\phi}{2} + \frac{1}{8}(\dot{\phi})^2 \end{aligned} \tag{3}$$

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Next, we introduce an Potential Function $U(q) \in C^2 : q \to R$. We only assume that, for given $q^* \in S^2$, the Potential satisfies conditions

$$U(q^*) = 0 \text{ and } \frac{\partial U}{\partial q}(q) = 0 \text{ iff } q = q^*, \tag{4}$$

and there exists a constant $L_q > 0$ such that

$$L_q \|q\|^2 \leq U(q) \tag{5}$$

By combining above equations, we define *Lagrangian* as

$$L(q, \dot{q}) = KE(q, \dot{q}) - U(q) \tag{6}$$

Let $u = [u_\theta \ u_\phi]^T$ be an input function, then by Euler Lagrangian Formula, we have Lagrange dynamical system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = u. \tag{7}$$

Eq (7) can be rewritten as the following formula.

Definition 3.1. (Simple Mechanical System). Lagrange equation (6) can be written as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \frac{\partial U}{\partial q} = u \tag{8}$$

or, equivalently

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ M(q)^{-1}\left(-C(q, \dot{q})\dot{q} - \frac{\partial U}{\partial q} + u\right) \end{bmatrix} \tag{9}$$

where $C_{ij}(q, \dot{q}) = \sum_{k=1}^{2} \Gamma_{ijk}\dot{q}_k$ and Γ_{ijk} are called Christoffel's symbols associated with Riemann Metric $M(q)$ defined by

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{kj}}{\partial q_i}\right) \tag{10}$$

In (8), if set $U(q) \equiv 0$ and $u \equiv 0$, then the remaining equation

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} = 0 \tag{11}$$

represents *Geodesic Curves* of the **LIST** S^2

$$\begin{cases} \ddot{\theta} = -\theta \ \dot{\phi} \cot \dfrac{\phi}{2} \\ \ddot{\phi} = (\dot{\theta})^2 \sin \phi \end{cases} \tag{12}$$

In general, a *Geodesic Curve* on a manifold $\mathcal{M}$ has no equilibrium point on $\mathcal{M}$. By adding, however, a *Potential* function, destruct a structure of *Geodesic*.

Definition 3.2. (Eye Movement System). Define $x := [x_1 \ x_2 \ x_3 \ x_4]^T = [\theta \ \dot{\theta} \ \phi \ \dot{\phi}]^T$ be a state variable. Then *Lagrange Dynamical System* (7) is rewritten as a nonlinear affine form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ &= \begin{bmatrix} x_2 \\ -x_2 x_4 \cot\left(\frac{x_3}{2}\right) - \csc^2\left(\frac{x_3}{2}\right) \frac{\partial U}{\partial x_1}(x_1, x_3) \\ x_4 \\ x_2^2 \sin(x_3) - 4 \frac{\partial U}{\partial x_3}(x_1, x_3) \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ \csc^2\left(\frac{x_3}{2}\right) & 0 \\ 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned} \tag{13}$$

defined on $\Omega = [-\pi, \ \pi] \times [-\pi, \ \pi] \times (0, \ \pi) \times [-\pi, \ \pi]$. In practice, we restrict our attention on a close sub domain of Ω, such as $\tilde{\Omega} = [-\pi, \ \pi] \times [-\pi, \ \pi] \times [\delta, \ \pi - \delta] \times [-\pi, \ \pi]$ for $\delta > 0$, in order to avoid a singularity of the variable $x_3 = \phi$.

3.2 Stability of Equilibrium

The system (13) has an equilibrium point $x^* \in \Omega$ which gives $f(x^*) = 0$. By (8), x^* is determined by $\frac{\partial U}{\partial q}(q^*) = 0$ i.e

$$x^* = [\ \theta^* \ 0 \ \phi^* \ 0 \ ]^T \tag{14}$$

where $\theta^* \in [-\pi, \ \pi]$ and $\phi^* \in (0, \pi)$. Next, we are interested in the *Stability of Equilibrium* of (13) with a Potential U. If a choice of U imposes an asymptotic stability on an equilibrium x^*, our control problem is achieved without an external input u. By *Lagrange Dirichlet Th* (See Lozano et al. [2000]), however, the equilibrium is either locally Lyapunov stable or unstable. In fact, define the Total Energy as

$$E(q, \dot{q}) = KE\,(q, \dot{q}) + U\,(q) \tag{15}$$

and takes its time derivative along with the system trajectories (7) with the external input u

$$\begin{aligned} &\frac{d}{dt}E\left(q(t), q(\dot{t})\right) \\ &= \frac{d}{dt}\left\{\dot{q}(t)^T \frac{\partial L}{\partial \dot{q}} - L(q(t), \dot{q}(t))\right\} \\ &= \ddot{q}^T \frac{\partial L}{\partial \dot{q}} + \dot{q}^T \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \dot{q}^T \frac{\partial L}{\partial q} - \ddot{q}^T \frac{\partial L}{\partial \dot{q}} \\ &= \dot{q}^T u \end{aligned} \tag{16}$$

As $u \equiv 0$, the above equation implies that $\frac{d}{dt}E(x(t)) = 0$. It shows that a level surface of the total energy $E(q, \dot{q}) = E(x(t))$ is invariant under the dynamics of (7) without the external input i.e

$$E(x(t)) = E(x_0) + \int_{t_0}^{t} \dot{E}\,(x(\tau))\,d\tau = E(x_0) \tag{17}$$

Moreover, if a Hessian of $U(q)$ satisfies

$$\frac{\partial^2 U}{\partial q^2}(q^*) > 0 \tag{18}$$

then q^* is a strict local minimum of $U(q)$. So, the convexity of the local minimum implies that

$$E(x(t, q^*)) \cap E(x(t, \bar{q})) = \phi \ \text{ for all t if } q^* \neq \bar{q} \tag{19}$$

The above equation is same as the definition local Lyapunov stability of q^*.

3.3 Potential Function

In the previous subsection, we see that a choice of potential function is independent of the asymptotic stability of an equilibrium of (7). We used an quotation based potential function introduced in Wijayasinghe et al. [to be published].

Definition 3.3. (Quotation Based Potential). Define the potential as follow

$$U_q(\theta, \phi) = \gamma(1 - \langle q(\theta^*, \phi^*), q(\theta, \phi)\rangle) \tag{20}$$

where $\gamma > 0$ is the magnitude of the Potential and $q : [-\pi, \ \pi] \times (0, \ \frac{\pi}{2}) \to S^2 \subset R^3$ is called a *quotation* mapping defined by

$$q = [\cos\left(\frac{\phi}{2}\right), \ \sin\left(\frac{\phi}{2}\right)\cos(\theta), \ \sin\left(\frac{\phi}{2}\right)\sin(\theta)\,]^T \tag{21}$$

By the construction U_q its zeros is given by $U_q(\theta^*, \phi^*) = 0$.

The gradient and Hessian matrix of U_q is given by

$$\frac{\partial}{\partial\theta}U_q = \gamma \sin\left(\frac{\phi^*}{2}\right)\sin\left(\frac{\phi}{2}\right)\sin(\theta-\theta^*). \quad (22)$$

$$\begin{aligned}&\frac{\partial}{\partial\theta}U_q\\ &= \frac{\gamma}{4}\{\sin\left(\frac{\phi+\phi^*}{2}\right)(1-\cos(\theta-\theta^*))\\ &+\sin\left(\frac{\phi-\phi^*}{2}\right)(1+\cos(\theta-\theta^*))\}\end{aligned} \quad (23)$$

$$\frac{\partial^2 U_q}{\partial q^2}(\theta,\phi) = \begin{bmatrix}\sin^2\frac{\phi}{2} & 0\\ 0 & \frac{1}{4}\end{bmatrix} \quad (24)$$

4. OPTIMAL CONTROL DESIGN

From previous section, we know that an asymptotic stability of an equilibrium q^* of (7) or (13) with no external input u cannot be achieved whatever a potential U is. Hence, we need to add an additional feedback control law to (7).

4.1 Optimal Control

Consider the dynamical system has a form (1) and we can assume that an equilibrium of the system is $x^* = 0$ w.r.t by the shift $x \to x - x^*$. We consider the Optimal Control problem which is to find an stabilizing input u^* of (1) minimizing the cost index

$$J(x(0), u) = \int_0^\infty l(x(t)) + u^T R u \; dt \quad (25)$$

where the state penalty function $l : R^n \to R$ is positive definite, monotonically increasing, and continuously differentiable and R is positive define matrix $R = diag\{r_i\} \in R^{m\times m}$. The optimal stabilizing solution u^* is given by the following.

Theorem 1. (Hamilton Jacobi Bellman Equation). The solution of the optimal control problem of the system (1) with the cost index $J(x_0, u)$ is given by the positive definite solution of the Hamilton Jacobi Bellman Equation

$$u^*(x) = -\frac{1}{2}R^{-1}g^T(x)V_x(x) \quad (26)$$

where $V(x)$ is the solution of HJB

$$\text{HJB}(V, x) = V_x^T f - \frac{1}{4}V_x^T g R^{-1} g^T V_x + l = 0 \quad (27)$$

with boundary condition $V(0) = 0$.

4.2 Inverse Optimal Design

In direct approach, we need to solve HJB equation analytically or numerically, but this is not feasible task. Alternatively, we used the Inverse Optimal Design to construct a suboptimal stabilizing control law. In the inverse optimal approach, first, a stabilizing feedback u is given, then to be shown that a cost index

$$J = \int_0^\infty l(x) + u^T R(x) u \, d\tau \quad (28)$$

for some peculiar functions $l(x) \geq 0$ and $R(x) > 0$ is minimized by u.

Definition 4.1. (Inverse Optimal Design). Let a function $V(x) : R^n \to R$ be a positive semidefinite and define the control $u^*(x) := -\frac{1}{2}R^{-1}(x)gV_x$. Then, u^* solves the inverse optimal control of the system (1) if it satisfies

(1) it achieved (global) asymptotic stability of $x = 0$
(2) $\dot{V}|_{u=\frac{1}{2}u^*} = V_x^T f - \frac{1}{4}V_x^T g R^{-1}(x) g^T V_x \leq 0$.

Also, set $\tilde{l}(x) := -V_x^T f + \frac{1}{2}V_x^T g u^* \geq 0$ then V is the solution of HJB equation

$$\tilde{l} + V_x^T f - \frac{1}{4}V_x^T g R^{-1}(x) g^T V_x = 0 \quad (29)$$

We show the example of The Inverse Optimal Design. Consider the system (13) with a potential $U(x_1, x_3)$. Let $R(x) = I$ and set $V(x) = E(x)$ i.e consider the Total Energy as a candidate of Lyapunov function. Then, by calculating,

$$\begin{aligned}V_x^T f &= x_2 U_{x_1} - \frac{1}{2}x_2^2 x_4 \text{sin}(x_3) - x_2 U_{x_1} + \frac{1}{4}x_2^2 x_4 \text{sin}(x_3)\\ &+ x_4 U_{x_3} + \frac{1}{4}x_2^2 x_4 \text{sin}(x_3) - x_4 U_{x_3}\\ &= 0\end{aligned} \quad (30)$$

and

$$\begin{aligned}\frac{1}{4}V_x^T g R(x)^{-1} g^T V_x &= \frac{1}{4}\left(\csc^4\left(\frac{x_3}{2}\right)V_{x_2}^2 + 16V_{x_4}^2\right)\\ &= \frac{1}{4}\left(\csc^4\left(\frac{x_3}{2}\right)KE_{x_2}^2 + 16KE_{x_4}^2\right)\\ &= \frac{1}{4}\left(x_2^2 + x_4^2\right)\end{aligned} \quad (31)$$

In the above equation (31), define $\tilde{l}(x) = \frac{1}{4}\left(x_2^2 + x_4^2\right)$ then $V(x) = E(x)$ is the solution of HJB. Next, we need to show the stability of $x^* = 0$ under the input $u^*(x) = -\frac{1}{2}g^T V_x$. To see this, we prove that V is a Lyapunov function. Taking the derivative along with the system trajectories under the input u,

$$\begin{aligned}\frac{d}{dt}V(x(t)) &= V_x^T f - \frac{1}{4}V_x^T g R^{-1} g^T V_x\\ &= -\frac{1}{4}\left(x_2^2 + x_4^2\right) \leq 0\end{aligned} \quad (32)$$

Let $\dot{V}^{-1}(0) := \{x \in R^4 \,|\, x_2 = x_4 = 0\}$. The only invariant set of the system in $\dot{V}^{-1}(0)$ is the equilibrium $x = 0$. Hence, by the LaSalle Invariance Principle, $x = 0$ is asymptotically stable. This consideration implies that the Total Energy $E(x)$ provide the optimal stabilizing input which minimize the cost index with the standard type cost penalty function $l(x) = \frac{1}{4}\left(x_2^2 + x_4^2\right)$ and $R = I$ for any choice of the Potential $U(x_1, x_3)$.

5. CONSTRUCTION OF THE SUBOPTIMAL CONTROL

In general, we have no systematic approach to design the inverse optimal control for given a nonlinear system, but, in the Lagrange system, one can find it by the existence of the Total Energy. In this section, we construct a suboptimal control based on the Total Energy base approach.

5.1 Control Lyapunov Function

Consider Lagrange System (7) whose kinetic matrix satisfies the non-degenerate condition

$$L_k \|\dot{q}\|^2 \le \dot{q}^T M(q) \dot{q} \tag{33}$$

where L_k is a positive constant. Consider the candidate of the control Lyapunov function has form

$$V(x) = \frac{1}{2} r_E\, E^2(x) + \frac{1}{2} x^T A x \tag{34}$$

where $r_E > 0$ and A is a positive symmetric constant matrix(See Ordaz-Oliver et al. [2009]). Define a notations

$$X := [x_1\, x_3]^T \text{ and } A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Then, the designed input function is constructed as follow

$$\begin{aligned} u^* =& -\frac{1}{2} R^{-1}\, M^{-1}(X) \frac{\partial V}{\partial \dot{X}} \\ =& -\frac{1}{2} R^{-1} \left\{ r_E\, E(x) \dot{X} + M^{-1} \left(A_{12} X + A_{22} \dot{X} \right) \right\}. \end{aligned} \tag{35}$$

Our inverse optimal design is that for given an equilibrium point $x^* = (z_1, 0, z_3, 0)^T$, specifics matrices components of A such that the close loop system $\dot{x} = f + g u^*$ is asymptotically stable and a corresponding state penalty should be meaningful. First, we shows that u^* a stabilizing input.

Proposition 2. If $r_E > 0$, and A_{12} and A_{22} are positive diagonalize matrix, then (35) are (locally) stabilizing input of (7).

Proof 5.1. First, consider an input $u = A_{12} X + A_{22} \dot{X} = A_{12} q + A_{22} \dot{q}$. Define a modified Total Energy as

$$\tilde{E} = E(q, \dot{q}) + \frac{1}{2} q^T A_{12}\, q \tag{36}$$

Then $\tilde{E}$ has the same local minimum as E. Consider the derivative of the $\tilde{E}$. By the Euler Lagrange formula (7) and the relation (16),

$$\begin{aligned} \frac{d}{dt} E &= \dot{q}^T \left(-A_{12} q - A_{22} \dot{q} \right) \\ \frac{d}{dt} \left(\frac{1}{2} q^T A_{12} q \right) &= \dot{q}^T A_{12} q \end{aligned}$$

It shows the negativity semidefinite of the derivative

$$\frac{d}{dt} \tilde{E} = -\dot{q}^T A_{22} \dot{q} \le 0 \tag{37}$$

So, by the LaSalle Invariance Principle, $x = 0$ is locally asymptotically stable. Construct a new input as

$$u = r_E\, E(q, \dot{q})\, M(q) \dot{q} + A_{12} q + A_{22} \dot{q} \tag{38}$$

If set $r_E = 0$, then the new input is the previous one. Since E and M is positive definite, u can keep an passivity of the system

$$E(x(t)) - E(x(0)) \le -\int_0^t \dot{q}^T \left(A_{22} + E(q, \dot{q})\, M(q) \right) \dot{q}\, dt. \tag{39}$$

So, it is also stabilizing input of the system. Substitute u into (9) then we have the result.

5.2 Construction State Penalty Function

We construct a state penalty function $\tilde{l}$ from (34). By (29),

$$\begin{aligned} G(x) =& \frac{1}{4} V_x^T g R^{-1} g^T V_x \\ =& u^T R^{-1} u \\ =& \frac{1}{4} R^{-1} \left\| M^{-1} A_{12} X + \left(r_E\, E(x)\, I + M^{-1} A_{22} \right) \dot{X} \right\|^2 \end{aligned} \tag{40}$$

and

$$\begin{aligned} P(x) =& V_x^T\, f \\ =& r_E\, E\, E_x^T\, f + \frac{1}{2} \frac{d}{dt} \left(x^T A x \right) \\ =& [X^T \dot{X}^T]\, A\, [\dot{X}^T \ddot{X}^T]. \end{aligned} \tag{41}$$

Then

$$\tilde{l}(x) = G(x) - P(x). \tag{42}$$

The positivity of $\tilde{l}$ guarantee a stability of the origin $x = 0$. We have following proposition.

Proposition 3. There exist some positive symmetric matrix A in (34) such that $\tilde{l}(x) = G(x) - P(x)$ is positive semidefinite.

Proof 5.2. $G(x)$ is clearly positive semidefinite because of its construction. Let $\alpha = \min \sigma(A)$. By (5) and (33),

$$0 \le \max\{L_k, L_p\} \|x\| \le E(x). \tag{43}$$

By the passive inequality (39), $E(x(t))$ monotonically decrease under the input u^* defined by (35). If set $2\,\alpha \le \max\{L_k, L_p\}$,then

$$\frac{1}{2} x^T(t) A x(t) \le E(x(t)) \tag{44}$$

and it decreases monotonically i.e $P(x) = \frac{1}{2} \frac{d}{dt} x^T A x \le 0$. Hence, $\tilde{l} \ge 0$.

By prop 3, $\tilde{l}(x)$ is a meaningful cost penalty function because its positive semidefinite and it fulfills with the requirement of the inverse optimal design.
Next, we show a example of $\tilde{l}$ for the system (13). For given an equilibrium point $x^* = (z_1, 0, z_3, 0)^T$, consider the shift operation $x_1 \to x_1 + z_1$ and $x_1 \to x_3 + z_3$. Then the equilibrium moves to $x^* = 0$ and the kinetic matrix is

$$M(x) = \begin{bmatrix} \sin^2 \left(\dfrac{x_3 + z_3}{2} \right) & 0 \\ 0 & \dfrac{1}{4} \end{bmatrix}. \tag{45}$$

Set the sub-matrix of A as follow

$$\begin{aligned} A_{11} =& diag\{k_1,\ k_2\} \\ A_{12} = A_{21} =&\ diag\{\sin^2 \left(\frac{z_3}{2} \right),\ \frac{1}{4}\} \\ A_{22} =& diag\{k_3 \sin^2 \left(\frac{z_3}{2} \right),\ \frac{k_4}{4}\} \end{aligned} \tag{46}$$

where k_i are positive constants such that A is positive definite. One can see that $G(x)$ is nonlinear quadratic cost function for X and $\dot{X}$, but it is difficult to describe a shape of $P(x)$ because it includes the term $\ddot{X} = \ddot{q}$. Consider a shape of $\tilde{l}(x)$ around $x = 0$. Since $P(x) = \mathcal{O}(\|x\|^2)$, $P(x) \approx 0$ as x is closed to the origin. Since $\lim_{x \to 0} E(x) = 0$ and

$$\lim_{x_3 \to 0} \sin^2(\frac{z_3}{2}) \csc^2\left(\frac{x_3 + z_3}{2}\right) = 1,$$
$$G(x) \approx diag\{1, 1, k_3, k_4\}. \quad (47)$$

It implies that the suboptimal control u^* close to the optimal control which is derived from HJB equation as the state x is in near equilibrium enough.

6. NUMERAL SIMULATION

7. CONCLUSION

It will be provided in the final version.

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