Optimal Control of Chebyshev Collocation Method and Application to Eye Movement Problem

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***Abstract*— An optimal control problem on rotational mechanical system on $SO(3)$ has been studied for the purpose of controlling human eye movement. Although these problems have been handled earlier, as a two point boundary value problem which in turn is analyzed using pseudo-spectral methods, our approach in this paper is based on solving the associated Hamilton Jacobi Bellman equation. A feedback stabilizing controller is obtained using a combination of 'successive approximation' and 'Chebyshev collocation'.**

I. INTRODUCTION

Successive approximation of the solution of Hamilton Jacobi Bellman Equation for the feed back controller which gives optimal control have been studied by Beard [1] and Saridis [10]. Although solving HJB is a laborious task because of the "Curse of dimensionality", the controller obtained from HJB provided us the feedback law defined on a neighborhood of an equilibrium point of the dynamical system. In [1], a successive approximation is described that is based on solving Generalized Hamilton Jacobi Bellman Equation which is a linear partial differential equation. A new input function is constructed from a derivative of the solution of GHJB. It was proved that in each iteration step, a reconstructed input is also stabilizing control and the solution of GHJB is uniformly convergent to the solution HJB. Our primal concern is to approximate the solution of GHJB while keeping the convergence property and the admissibility of the input function. In [1], [2], Beard and his coworkers established successive approximation algorithm which is compatible with convergence and admissibility using Galerkin Spectral Method. It, however, requires a large number of numerical integration for computing inner products.

In order to reduce the computational cost, we have used Chebyshev Collocation Method (see [3], [6], [9]) based on the polynomial interpolation at Chebyshev Zeros. It is well known [8], that the Chebyshev interpolation in multivariate case has the uniformly convergent property. In this paper, we showed that the successive approximation with Chebyshev Collocation converges uniformly to the solution of HJB.

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II. OPTIMAL CONTROL PROBLEM

A. Optimal Control and Hamilton Jacobi Bellman Equation

Let us consider the infinite horizon optimal control problem of the nonlinear dynamical system in $\Omega \in R^n$

$$\dot{x} = f(x) + g(x)u \tag{1}$$

with equilibrium $f(0) = 0 \in \Omega$ where $x \in R^n, f : R^n \to R^n, g : R^n \to R^{n\times m}$, and $u : R^n \to R^m$ is the input function and the cost index along with the system trajectory

$$J(x(0)) = \int_0^\infty l(x(t)) + ||u(t)||_R^2 \, dt \tag{2}$$

where the state penalty function $l : R^n \to R$ is positive definite, monotonically increasing, and continuous and $||R||_R^2 = u^t Ru$ for positive define matrix $R = diag\{r_i\} \in R^{m\times m}$. Additionally, we assume that all of these functions are continuously differentiable and f, g are Lipschitz continuous on Ω. The optimal control problem is to find a feed back law $u^*(x) = arg_{u\in\mathcal{A}}\min J(x(0))$.

Definition 2.1 (Hamilton Jacobi Bellman Equation): The solution of the optimal control problem of (f, g) with $J(x_0)$ is given by the solution of the Hamilton Jacobi Bellman Equation (HJB):

$$u^*(x) = -\frac{1}{2}R^{-1}g^T(x)\frac{\partial V}{\partial x}(x) \tag{3}$$

where $V(x)$ is the solution of HJB

$$\mathrm{HJB}(\frac{\partial V}{\partial x}, x) = \frac{\partial V}{\partial x}^T f + \frac{\partial V}{\partial x}^T gR^{-1}g^T\frac{\partial V}{\partial x} + l = 0 \tag{4}$$

with boundary condition $V(0) = 0$.

B. Generalized Hamilton Jacobi Bellman Equation

In general, it is difficult to treat HJB equation analytically and numerically, so we use an alternative method to approximate the solution of HJB in a successive manner based on Generalized Hamilton Jacobi Bellman Equation (GHJB). In this section, we introduce important results and terminologies on approximation method from [1].

Definition 2.2 (Admissible Input): An input function $u(x)$ is called admissible input of (f, g) with respect to the state penalty $l(x)$ on Ω if it satisfies following properties

1) $u(x) \in C^1(\Omega)$ and $u(0) = 0$.
2) $u(x)$ stabilize the system (f, g) on Ω.
3) For the solution $\phi(t)$ of (f, g) under the input u, $\int_0^\infty l(\phi(t)) + ||u(\phi(t))||_R^2 \, dt < \infty$.

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We define the set of admissible input on Ω with the state penalty l as $\mathcal{A}_l(\Omega)$.

Definition 2.3 (Generalized HJB Equation): For fixed admissible input $u(x) \in \mathcal{A}_l(\Omega)$, the function $V : R^n \to R$ satisfies GHJB equation if,

$$\text{GHJB}(\frac{\partial V}{\partial x}, u) = \frac{\partial V}{\partial x}(f + gu) + l + ||u||_R^2 = 0 \quad (5)$$

with boundary condition $V(0) = 0$.

Theorem 2.1: (See [1])Let $V(x)$ be the solution of GHJB with $u(x) \in \mathcal{A}_l(\Omega)$, then

$$V(x) = \int_0^\infty l(\phi(\tau)) + ||u(\phi(\tau))||_R^2 \, d\tau \text{ for } \phi(0) = x. \quad (6)$$

and,

1) $V(x)$ exists uniquely on Ω for given $u(x)$.
2) $V(x)$ is continuous differentiable Lyapunov function of the system (f, g) with the input u.
3) $V(x)$ is equal to the cost index $J(x)$ and called $the\ value\ function.$

The expression in (6) includes the system trajectory $\phi(\tau)$, but, in general, this ϕ is unavailable without a solving the system numerically. By explicitly taking derivative of $V(x(t))$ along with the system trajectory $\phi(\tau)$, we can verify that $V(x)$ satisfies GHJB which doesn't depending on the ϕ. Eq (6), however, indicates that to minimize the value function $V(x)$ is to minimize the input $u(x)$ along with GHJB.

Next, we refer to the relation between HJB and GHJB. In the context of a traditional optimal control problem, GHJB is referred to as the Pre Hamiltonian and HJB is called the Hamiltonian. The Hamiltonian is obtained by completing square and minimizing in (5) along with input function $u(x)$. Likewise, we can obtain the following ,by completing the square,

$$\begin{aligned} \text{HJB}(x) &= \text{GHJB}(\frac{\partial V}{\partial x}, -\frac{1}{2}R^{-1}g^T\frac{\partial V}{\partial x}) \\ &= arg_{\hat{u} \in \mathcal{A}_l(\Omega)} \min \text{GHJB}(\frac{\partial V}{\partial x}, \hat{u}). \end{aligned}$$

C. Successive Approximation Algorithm of GHJB

Beard [1], [2] formalized Successive Approximation Algorithm(SAA) based on previous work of Saridis [10].
The following theorem guarantees convergence of the iteration algorithm.

Theorem 2.2 (Improve Cost Index [1], [10]): Assume $u \in \mathcal{A}_l(\Omega)$ and V be the solution of $\text{GHJB}(\frac{\partial V}{\partial x}, u)$. Define $\hat{u} = \min \text{GHJB} = -\frac{1}{2}g^T\frac{\partial V}{\partial x}^T$. Then,

1) $\hat{u} \in \mathcal{A}_l(\Omega)$.
2) Let $\hat{V}$ be the solution of $\text{GHJB}(\frac{\partial \hat{V}}{\partial x}, \hat{u}) = 0$, then $\hat{V}(x) \le V(x)$ on Ω.

III. Collocation Method

Successive Approximation Algorithm provides the way of constricting uniformly convergent approximation by GHJB. It, however, is still problem that how we solve GHJB in iteration steps. We used the Chebyshev Collocation Method (we treat "Collocation" as the synonym of "Polynomial Interpolation"). Beard used the Galerkin Spectral Method based on orthonormal basis in L_2 normed space, but the Galerkin Method need a computation of an inner product by a numerical integral. On contrast, the Collocation Method doesn't require such a computation and are recognized as the approximation of it with high accuracy guaranteed by the Gauss Lobatto quadrature (See [4], [5]). In this section, we introduce the $Chebyshev\ Interpolation$(See [6], [8]) in the multivariate case and uniformly convergent theorem, Then we show the convergence results of SAA by the Chebyshev Interpolation.

A. Chebyshev Polynomial

Definition 3.1 (First kind Chebyshev Polynomial): The $First\ kind\ Chebyshev\ Polynomial$ is defined as follow

$$T_n(x) = cos(n\theta) \quad (7)$$

where $x = arccos(\theta)$. $T_n(x)$ satisfies the three recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \quad (8)$$

The low order polynomials are $T_0(x) = 1$, $T_1(x) = x$,and $T_2(x) = 2x - 1$.

Definition 3.2 (Chebyshev Zeros): the roots of the first kind Chebyshev polynomial $T_{n+1}(x)$ are called as the set of Chebyshev zeros and given by

$$x_k = cos\frac{(k - \frac{1}{2})\pi}{n+1} \text{ for } k = 1, \cdots, n+1. \quad (9)$$

B. Chebyshev Interpolation

we start discussion from an polynomial interpolation of a single variable continuous function of $f(x)$ on $I = [-1,\ 1]$. For given f, the polynomial interpolation of degree n at the $n + 1$ arbitrary distinct grid points $x_1, \cdots, x_{n+1}$ has form

$$p_n(x) = \sum_{i=1}^{n+1} l_i(x)f(x_i) \quad (10)$$

where $l_i(x)$ are $Lagrange\ polynomial$ defined by

$$l_i(x) = \Pi_{k=1,k\neq i}^{n+1}\left(\frac{x - x_k}{x_i - x_k}\right) \quad (11)$$

If we set the $n + 1$ grids point $\{x_i\}$ as roots of $T_{n+1}(x)$ in the above representation, then $Chebyshev\ Interpolation$ is given by

$$l_i(x) = \frac{T_{n+1}(x)}{(x - x_i)T'_{n+1}(x_i)}. \quad (12)$$

By the discrete orthogonality of Chebyshev polynomial $\{T_i(x)\}$ up to n th degree on the set of zeros $\{x_k\}$,

$$\sum_{k=1}^{n+1} T_i(x_k)T_j(x_k) = \begin{cases} 0 & n \neq m \\ n+1 & i = j = 0 \\ \frac{1}{2}(n+1) & 0 < i = j \le n \end{cases} \quad (13)$$

we rewrite the n th degree polynomial p_n as the Chebyshev polynomial expansion

$$p_n(x) = \sum_{k=1}^{n+1}{}' c_i T_i(x). \quad (14)$$

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where $'$ means that the first and last term are divided by 2.

In the multidimensional case, in a similar manner (10) and (11), a multivariate polynomial interpolation of a continuous function f on $I^N = [-1,1]^N$ is given by

$$If = \sum_{i_1=1}^{n_1+1} \cdots \sum_{i_N=1}^{n_N+1} l_{i_1}^{(1)}(x_1) \cdots l_{i_N}^{(N)}(x_N) f(x_1^{i_1}, \cdots, x_N^{i_N}) \tag{15}$$

where the index n_j is the number of interpolation points of variable x_j, and

$$l_i^j(x_j) = \Pi_{k=1,k\neq i}^{n_j+1} \left(\frac{x_j - x_j^{(k)}}{x_j^{(i)} - x_j^{(k)}} \right). \tag{16}$$

In (15), set the interpolation points as the tensor product of zeros of $T_{n+1}(x)$ and apply (13) and (14), then

$$I_n f = \sum_{i_1=1}^{n+1} \cdots \sum_{i_N=1}^{n+1} c_{i_1 \cdots i_N} T_{i_1}(x_i) \cdots T_{i_N}(x_N). \tag{17}$$

The important property of Chebyshev Interpolation If is uniformly convergence property.

Definition 3.3 (Modulars of Continuity): The modulus of continuity of $f(x_i, \cdots, x_N)$ is defined as

$$\omega_i(t) = sup_{|x_i - x_i^*| \leq t} |f(\cdot, x_i, \cdot) - f(\cdot, x_i^* \cdot)|. \tag{18}$$

$\omega_i(t)$ indicates the amplitude of maximum oscillation of f along with the variable x_i on segment of length t.

Theorem 3.1 (Uniformly Convergence [8]): If $f(x_1, x_2, \cdots, x_n)$ satisfies Deni-Lipschitz condition

$$\sum_{j=1}^{N} \omega_j(\delta)(log(\delta))^N \to 0 \quad \text{as} \quad \delta \to 0 \tag{19}$$

where $\omega_j(\delta)$ is the modulus of continuity defined by (18), then the multivariate polynomial interpolation If at the tensor product of Chebyshev zeros converges uniformly to f as $n \to \infty$.

In the assumption Th 3.1, if f is in a class of $H\ddot{o}lder$ function or is $Lipschitz\ Continuous$, then f satisfies necessarily the Deni-Lipschitz condition, but the converse statement is not true.

IV. SSA BY CHEBYSHEV COLLOCATION METHOD

A. Construction

In this section, we formulate the procedure to solve GHJB by Chebyshev Collocation for a four dimensional nonlinear dynamical system with specific input form.

We focuses on the dynamical system defined on compact subset K of Ω which has form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ g_2 & 0 \\ 0 & 0 \\ 0 & g_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{20}$$

and we suppose that f and g are $Lipschitz\ Continuous$ on compact subset K of Ω. The system is multivariate, so we define a basis function ϕ as Multiplication of Chebyshev Polynomial up to N degree by a tensor product technique,

$$\phi_{I_k}(x) = T_{k_1}(x_1) T_{k_2}(x_2) T_{k_3}(x_3) T_{k_4}(x_4) \tag{21}$$

$k_i = 0, 1 \cdots N$. For a simplification of an expression of an index, we used the $multiple\ index$ $I_k : k \in Z_+ \to (k_4, k_3, k_2, k_1) \in Z_+^4$. and each k_i can be computed from k by

$$\begin{aligned} k_i &= \lfloor k \ / \ (N+1)^i \rfloor \quad \text{for } i = 2,3,4 \\ k_1 &= k - \sum_{i=2}^{4} k_i (N+1)^{i-1} \end{aligned} \tag{22}$$

where $\lfloor \cdot \rfloor$ is the $floor\ function$. Conversely, k is computed by

$$k = k_4(N+1)^3 + k_3(N+1)^2 + k_2(N+1) + k_1 \tag{23}$$

we approximate the solution V of GHJB as follow, let $c_{I_k} \in R$ be a coefficient of the basis, then

$$V_N(x) = c_0 + \sum_{k=1}^{(N+1)^4} c_{I_k} \phi_{I_k}(x) \tag{24}$$

For a fixed polynomial degree N, let $\{x_k\}$ for $k = 1, \cdots, N+1$ be the root of $T_{N+1}(x)$, then, in a similar manner to the multi basis ϕ_{I_k}, the multiple zeros is defined as

$$x_{I_k} = (x_{k_1}, x_{k_2}, x_{k_3}, x_{k_4}).$$

According to interpolation formulators which we discussed in the previous section, we determine the coefficients $\{c_{I_k}\}$ to satisfy

$$V(x_{I_k}) = c_0 + \sum_{k=1}^{(N+1)^4} c_{I_k} \phi_{I_k}(x_{I_k}) \text{ with } V(0) = 0 \tag{25}$$

all over $\{x_{I_k}\}$.

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B. Residual Equation

In general, GHJB is not defined on the unit cube $I^4 = [-1,1]^4$, so we consider the case that an interval is given by $\Omega_\tau := [-\tau_1, \tau_1] \times \cdots \times [-\tau_4, \tau_4]$ where $\tau_i > 1$ then introduce the parametrization $X = \tau \circ x$ as

$$X = [X_1, \cdots, X_4]^T = \tau \circ x = [\tau_1 x_1, \cdots, \tau_4 x_4]^T. \quad (26)$$

This parametrization means that we shrink the domain Ω_τ of the independent variable X into the cube I^4 of the independent variable x , then

$$\begin{aligned} V_N(\tau \circ x) &= V_N(X) \\ &= c_0 + \sum_{k=1}^{(N+1)^4} c_{I_k} \phi_{I_k}((\tau^{-1}) \circ X). \quad (27) \end{aligned}$$

We construct the *Residual Equation* for computing the interpolation coefficient c_{I_k} in (25).

Definition 4.1 (The Residual Equation of GHJB): By combing (5),(20),(25), and (27), we have a linear equations for $\{c_{I_k}\}$

$$\text{GHJB}(I_n V, u, x_{I_k}) = \sum_{i=1}^{4} \left(\sum_{j=1}^{N^4} \frac{c_{I_j}}{\tau_i} \frac{\partial \phi_{I_j}}{\partial x_i} \right) \left(f_i + \sum_{j=1}^{m} g_{ij} u_j \right) + l + ||u||_R^2 = 0. \quad (28)$$

Define

$$\begin{aligned} c &= [c_{I_1}, \cdots, c_{I_{(N+1)^4}}]^T \in R^{(N+1)^4-1} \\ b(x_{I_k}, u) &= [-l(\tau \circ x_{I_k}) - ||u(\tau \circ x_{I_k})||_R^2]^T \\ &\in R^{(N+1)^4} \\ A &= [A_{nm}] \in R^{(N+1)^4-1 \times (N+1)^4} \\ A_{nm}(x_{I_n}, u) &= \{ \frac{\partial \phi_{I_m}}{\partial x_1} \frac{\bar{f}_1}{\tau_1} + \frac{\partial \phi_{I_m}}{\partial x_2} \frac{(\bar{f}_2 + \bar{g}_2 \bar{u}_1)}{\tau_2} \\ &+ \frac{\partial \phi_{I_m}}{\partial x_3} \frac{\bar{f}_3}{\tau_3} + \frac{\partial \phi_{I_m}}{\partial x_4} \frac{(\bar{f}_4 + \bar{g}_4 \bar{u}_2)}{\tau_4} \}(x_{I_n}) \end{aligned} \quad (29)$$

where $\bar{f}_i(x) = f_i(\tau \circ x)$, $\bar{g}_i(x) = g_i(\tau \circ x)$, $\bar{u}_i(x) = u_i(\tau \circ x)$, and $\{\cdot\}(x_{I_n})$ means substituting the value x_{I_n} into the all functions inside of $\{\cdot\}$. Then, we have linear equations

$$Ac = b. \quad (30)$$

We are interested in the convergence of SSA of Chebyshev collocation. Let introduce the coefficients $c_i = [c_{i,I_k}]$ depending on iteration times, then

$$I_N^i V = \sum_{k=0}^{(N+1)^4} c_{i,I_k} \phi_{I_k}(x) \quad (31)$$

and we have the following theorem

Theorem 4.1: Let V^* be the solution of HJB. Suppose V^* exist on Ω and for enough large $N > 0$, $Ac_i = b$ is solvable in each iteration step, then

1) $||V^* - I_N^i V||_\infty \to 0$ as $i \to \infty$
2) $||I_N^i V - I_N^j V||_\infty \to 0$ as i and $j \to \infty$

Proof, The theorem follows from [1], [8].

We construct iteration process based on Th 4.1 as follow

1) Find $u_0(x) \in \mathcal{A}_l(I^4)$ and set $u = u_0$, then construct the linear equations (29).
2) Solve a linear problem $\inf||Ac_0 - b|| = 0$
3) Update the next input $u_1 = [u_{1,1} \ u_{1,2}]^T$ as
$$u_{1,i}(x_{I_k}) = -\frac{1}{2}(r_i \tau_i)^{-1} g_i(\tau \circ x_{I_k}) \sum_{k=1}^{(N+1)^4} c_{0,I_k} \frac{\partial \phi_{I_k}(x_{I_k})}{\partial x_i} \quad (32)$$
4) By (28) and (29), construct GHJB$(c_1, u_1) = 0$ and solve $\inf||Ac_1 - b|| = 0$
5) Construct next input u_2 by (32).
6) Continue iteration to
$$||I_N^{i+1} V - I_N^i V||_\infty < \epsilon \quad (33)$$

where $\epsilon > 0$ is error bound.

In the successive approximation, the most exhausting task are to solve the linear equation and construction of it. The former is what the spectral method had in common. The latter, however, in the collocation method, can be mitigated. According to (28) and (29), an element of A_{nm} consist of the value of $\{f_i\}$, $\{g_i\}$, and l on x_{I_k} and the value of ϕ and $\frac{\partial \phi}{\partial x_i}$ which are combination of the value of $T_n(x)$ and $T_n'(x)$ on Chebyshev zeros $\{x_k\}$.

V. SIMULATION

In this section, we apply SAA of Chebyshev collocation to the optimal control of the eye movement problem [12]. The strategy of the eye movement control consists of adding a potential function and a frictional term. The potential function embeds an equilibrium in state space on which the system is defined and we can consider the friction term as a linear feedback control. In this research, we replaced the friction term with the nonlinear feed back control which minimize the cost index (2) around given equilibrium associated with the potential function.

A. State Equation

Definition 5.1 (The Eye Movement Problem [12]): We introduce state variable representation of the system by

$[\theta, \dot{\theta}, \phi, \dot{\phi}]^T = [x_1, x_2, x_3, x_4]^T$. The Potential which assigns the equilibriums α and β into the system is written by

$$\begin{aligned} V(\theta, \phi) &= V(x_1, x_3) \\ &= \frac{C_1}{2}\sin(x_1 - \alpha)^2 + \frac{C_2}{2}\sin(x_3 - \beta)^2 \end{aligned} \quad (34)$$

The dynamical system of the eye movement induced by the potential is

$$\dot{x} = \begin{bmatrix} x_2 \\ -x_2\, x_4\, \cot\left(\frac{x_3}{2}\right) + \frac{C_1}{2}\csc^2\left(\frac{x_3}{2}\right)\sin\left(x_1 - \alpha\right) \\ x_4 \\ \sin(x_3)\; x_2^2 + 2C_2\,\sin(x_3 - \beta) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\csc^2\left(\frac{x_3}{2}\right) & 0 \\ 0 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (35)$$

defined on $\Omega = [-\pi,\ \pi] \times [-\pi,\ \pi] \times (0,\ \pi) \times [-\pi,\ \pi]$ and its equilibrium is $[\alpha, 0, \beta, 0] \in \Omega$.

In (35), x_3 has a singularity which is a simple pole at the boundary $x_3 = 0$ and the singularity hinders a sectoral convergence of a collocation method. Then, we restricted our attention on a neighborhood of the equilibrium which is far from the singularity.

B. Riccati Equation

We consider The Linear Optimal Control Problem associated with the original optimal control problem

Definition 5.2 (Linear Optimal Control Problem): Consider the following linearized problem

$$\begin{cases} \dot{x} = Ax + Bu \\ J(x_0) = \int_0^\infty x^T Q x + ||u||_R^2\, dt \end{cases}$$

where $A = \frac{\partial f}{\partial x}(0)$, $B = g(0)$, and $Q = \frac{\partial^2 l}{\partial x^2}(0)$. A solution of this linearized problem is given by

$$u(x) = -R^{-1}B^T P x \quad (36)$$

where P is the positive symmetric solution of Riccati Equation(Ric) (See [11]).

$$AP + PA - PBR^{-1}B^T P + Q = 0$$

and u can stabilize the original nonlinear system locally. Let V^* be the solution of HJB, then P gives the first approximation of V^* i.e

$$V(x) = \frac{1}{2}x^T P x + O(||x||^3). \quad (37)$$

We used the solution of Ric as the initial input u_0 of SAA. It is used as an alternative to the optimal control of HJB and showed relatively better permanence than another choice of a linear control law. So, we compared the input of our approximation solution with the input of Ric case.

C. Numerical Simulation

We verified the accuracy of SAA by MATLB in the following cases. The most time consuming task is to solve $Ac_k = b$ and we used the MATLB linear solver "linsolve" to solve the linear equation. We considered the situation that set $C_1 = C_2 = 1$, $\alpha = 0$, and $\beta = \frac{\pi}{2}$ and by a shift, $x_3 \to x_3 - \beta$ so that $(0, 0, 0, 0)$ is an equilibrium of the system

$$\dot{x} = \begin{bmatrix} x_2 \\ -x_2\, x_4\, \cot\left(\frac{x_3 + \frac{\pi}{2}}{2}\right) + \frac{1}{2}\csc^2\left(\frac{x_3 + \frac{\pi}{2}}{2}\right)\sin\left(x_1\right) \\ x_4 \\ \sin\left(x_3 + \frac{\pi}{2}\right)\, x_2^2 + 2\,\sin(x_3) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\csc^2\left(\frac{x_3 + \frac{\pi}{2}}{2}\right) & 0 \\ 0 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (38)$$

and consider the cost index (2) with the matrix R is identity matrix. We approximated the optimal input on the domain $\Omega = [-1.6, 1.6] \times [-1.6, 1.6] \times [-1, 1] \times [-1.6, 1.6]$ and compared system trajectories driven by the approximation input by SAA with two type the penalty of state

$$l_1 = x_2^2 + x_4^2 \quad (39)$$

$$l_2 = x_1^2 + x_3^2 \quad (40)$$

to system trajectories driven by the alternative input by the Ric equitation

$$u_{Ric} = -g(x)^T P_i x \quad (41)$$

where P_i are the solution of Ric equation corresponding l_i,

$$P_1 = \begin{pmatrix} 0.7071 & 0.5000 & 0 & 0 \\ 0.5000 & 0.7071 & 0 & 0 \\ 0 & 0 & 0.6124 & 0.2500 \\ 0 & 0 & 0.2500 & 0.3062 \end{pmatrix} \quad (42)$$

$$P_2 = \begin{pmatrix} 1.4222 & 0.8090 & 0 & 0 \\ 0.8090 & 0.6360 & 0 & 0 \\ 0 & 0 & 1.0056 & 0.4045 \\ 0 & 0 & 0.4045 & 0.2249 \end{pmatrix}. \quad (43)$$

In the both cases, we set the initial input of SAA as

$$u_0 = -10g(x)^T P_1 x. \quad (44)$$

D. Simulation Results

We used Chebyshev polynomial $T_0, \cdots, T_{10}$ and the Chebyshev zeros of T_{11}, so, in the iteration step, we had the matrix $A \in R^{(11^4-1)\times(11^4)}$ and the coefficients $c \in R^{11^4}$. We tested two cases: 1) $l_1 = x_2^2 + x_4^2$ and 2) $l_2 = x_1^2 + x_3^2$. Both the cases, we continued the iteration process to 7 times.

Then our approximation input by SSA showed performance than the Ric input (41).

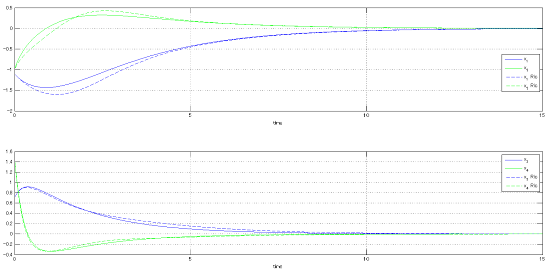
1. in the case $l_1 = x_2^2 + x_4^2$, the table I is the value of the cost index $J(x_0)$ defined by (2) along with system trajectories of two type inputs respectively and each rows are the value of L_2 norm of each variables.

TABLE I

THE VALUE OF $J(x)$ IN THE CASE1

	$J(x_0)$	u_1	u_2	x_2	x_4
u	4.7659	2.7271	1.1660	0.4936	0.3793
u_{ric}	4.9363	2.6207	1.2484	0.7244	0.3427

Fig. 1. The trajectory Approximation vs Riccati in the case 1



In this case, our approximation u didn't show significant difference in the performance.

2. in the case $l_2 = x_1^2 + x_3^2$, the contents of the table II are same as the table I.

TABLE II

THE VALUE OF $J(x)$ IN THE CASE 2

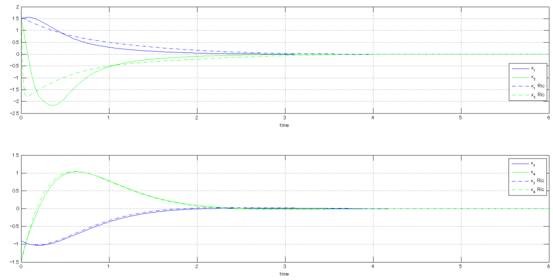
	$J(x_0)$	u_1	u_2	x_1	x_3
u	3.9013	1.0530	1.1445	1.0075	0.6961
u_{ric}	7.2621	4.3390	1.2540	1.0201	0.6489

In this case, u showed better performance than u_{ric}. The L_2 norm of l_2 are almost same, but although x_1 converge to 0 rapidly, the cost of u_1 of our approximation is a four part of the Ric input.

VI. CONCLUSIONS AND FUTURE WORKS

We have constructed using successive approximation method, based on the Chebyshev Collocation with uniformly convergent property, and applied our method to the problem of optimally controlling eye movement as a four dimensional dynamical system. We show that, our feedback controller

Fig. 2. The trajectory Approximation vs Riccati in the case 2



has a relatively better performance than the linear controller based on the well known Riccati approach.

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