SIAM J. CONTROL AND OPTIMIZATION
Vol. 26, No. 5, September 1988

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DIFFERENTIAL GEOMETRIC METHODS IN HYBRID PARAMETRIZATION OF LINEAR DYNAMICAL SYSTEMS*

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Abstract. This paper introduces a non-Euclidean parametrization of linear dynamical systems both in the open loop and in the closed loop. In particular, the various parametrizations introduced have a bundle structure which appears to be of relevance in system identification. Furthermore a quotient topology is obtained on the space of all systems and compared with the well-known graph topology. The article describes many desirable features of the quotient topology. Applications of vector bundle and fiber bundle theory in parametrization of control systems in the closed loop introduced in this paper are new.

Key words. vector bundle, parametrization, characteristic class

AMS(MOS) subject classifications. 14, 30, 93

1. Introduction. In feedback control system design, there has been recent interest in considering collections of linear multi-input multi-output dynamical systems, rather than fixed ones, and recursive schemes to update the parameters of compensators in order to satisfy specific sets of design constraints (e.g., sensitivity minimization, stabilization, etc.) in the closed loop. As has already been noted in the literature [1]-[3], the collection of plants might arise as a result of parameter variation or of our ignorance about the true value of a set of parameters and wherein only a bound on the parameters is known. Likewise the compensator parameters might have to be updated or adjusted in view of more recent information about the plant parameters or as a result of a change in the design requirements.

As a first step in the design of a compensator, it is important to select a suitable parametrized space for systems. Understanding the topology of this space is important since the convergence of algorithms may depend on it. Brockett [27] has studied the space of single-input single-output systems of McMillan degree n (called Rat (n)) and has paved the way for interesting research work by many others. In addition to this, we are clearly interested in obtaining a suitable parametrization of a class of plants and compensators for which recursive adjustment of the parameters is feasible. At the very least, we are interested in both off-line and on-line variations of the parameters of the plant-compensator pair in the closed loop that achieve robust stability. In this paper, we initiate such a program and parametrize plants and compensators that are robustly stable in the closed loop. An advantage of our parametrization is that the McMillan degrees of the family of plants and compensators considered are not assumed to be fixed. This is a very desirable feature, since it is extremely difficult to estimate the McMillan degree in parameter identification. As a special case, we parametrize the space of strictly proper multi-input multi-output systems and show that the associated space is graded (i.e., there is a sequence of subsets $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots$ such that $\bigcup_{n=1}^{\infty} \Omega_i$ is the given space) and each of the graded spaces (i.e., Ω_i's) is diffeomorphic to a Euclidean space. This property is particularly desirable in system identification, as has already been argued in [4], if the identification algorithm is defined by a locally and globally convergent vector field. This fact is justified due to the following theorem of Milnor (see Appendix I and [5]): "If a manifold admits a locally

* Received by the editors November 6, 1986; accepted for publication (in revised form) October 12, 1987. This research was partly supported by National Science Foundation grant ECS-8414220.

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and globally asymptotically stable vector field then it is diffeomorphic to a Euclidean space." Thus a non-Euclidean space does not admit a globally convergent vector field. On the other hand, because of the Euclidean structure of the graded spaces, it may now be possible to globally generalize many known parameter identification algorithms described in [10], where the associated parameter space is assumed to be Euclidean.

In general however, i.e., when the plants and the compensators are not necessarily strictly proper, we show that the proposed graded space is not Euclidean. However, it has the structure of a vector bundle [6], [7]. A vector bundle has the desirable feature that its fibers are isomorphic to a Euclidean space. Thus a point on the fiber may be identified via globally convergent identification schemes defined on a Euclidean space. Identifying points on the non-Euclidean base space may be difficult in general. However, in our parametrization the base space is sufficiently lower-dimensional compared to the dimension of the total space. Existence of separate identification algorithms on the base space and on the fibers is currently under study.

The main results of this paper are now described. In § 3 we introduce the notion of a lag and parametrize the space of $p \times m$ strictly proper systems of lag $\leqq n$. In § 4 we study the vector bundle structure of the space of $p \times m$ proper or improper systems of lag $\leqq n$. In § 5 we parametrize plant-compensator pairs of lag $\leqq n$ and $\leqq q$, respectively, that are stable in the closed loop and show that the parametrized space has the structure of a vector bundle. In § 6, we construct a fiber bundle of the space of proper or improper systems with unbounded lag and describe a quotient topology on the space of systems. This topology is compared with the well-known graph topology [8] and many desirable features of the proposed topology are described. Finally, in § 7 we parametrize plant-compensator pairs of lag $\leqq n$ and $\leqq q$, respectively, that are not necessarily stable in the closed loop and show that such a space has the structure of a fiber bundle.

In this paper we introduce the application of non-Euclidean geometry in parametrization problems and therefore cover some of the work of Byrnes [11], [12], Delchamps [13], and Helmke [14], [15]. The main point of this paper, however, is that, via a suitable over-parametrization, the parameter space we obtain is endowed with the structure of either a vector bundle or a fiber bundle. The base of the bundle so obtained is generally not a Euclidean space. Thus we might consider a set of charts which cover the base. This would agree with some of the works of Hazewinkel [16] on the numerical aspect of the parametrization problem as a consequence of "chart changes."

Here we assume that the reader is familiar with the "characteristic class" theory at the level of [7]. For a good introduction of this subject and its relevance to system theory we refer the reader to [13].

2. Notation.

$LS^n_{m,p}$: Space of $p \times m$ strictly proper systems of lag n.

$LP^n_{m,p}$: Space of $p \times m$ proper systems of lag n.

$L\Omega^n_{m,p}$: A parametrization of the set of $p \times m$ strictly proper systems of lag $\leqq n$.

$L\Pi^n_{m,p}$: A parametrization of the set of $p \times m$ proper systems of lag $\leqq n$.

$L\Omega^{n,q}_{m,p}$: A parametrization of the set of $p \times m$ strictly proper systems of lag $\leqq n$ that can be stabilized by a compensator of lag $\leqq q$.

$L\Pi^{n,q}_{m,p}$: A parametrization of the set of $p \times m$ proper systems of lag $\leqq n$ that can be stabilized by a compensator of lag $\leqq q$.

$LFB^{n,q}_{m,p}$: A parametrization of $p \times m$ systems $G(s)$ of lag $\leqq n$ and $m \times p$ compensators $K(s)$ of lag $\leqq q$ such that the closed-loop system $G(s)[I+K(s)G(s)]^{-1}$ is proper and stable.

$\mathbb{R}$: The real line.
$\mathbf{RP}^n$: The n-dimensional real projective space.
$\mathbb{C}$: The complex plane.
$\bar{\mathbb{C}}$: The complex plane together with the point at infinity.
$\mathbb{D}_s$: The open interior of the unit disc.
$\mathbb{D}_u$: $\bar{\mathbb{C}} - \mathbb{D}_s$.
$L^n_{m,p}$: A parametrization of $p \times m$ proper or improper autoregression moving average (ARMA) systems having a lag of at most n.
$L^{n,q}_{m,p}$: The space of ARMA systems in $L^n_{m,p}$ that can be stabilized by an $m \times p$ ARMA system in $L^q_{p,m}$.
$L^\infty_{m,p}$: A parametrization of $p \times m$ proper or improper ARMA systems having an arbitrary large lag.
$\tilde{L}^\infty_{m,p}$: A quotient space of all $p \times m$ ARMA systems.
H: The ring of rational functions with poles in $\mathbb{D}_s$.
J: Set of invertible elements in H.
$H^{p \times m}$: A $p \times m$ matrix with elements in H.
$LF^{n,q}_{m,p}$: A parametrization of m-input p-output feedback control systems of plants of lag $\leqq n$ and compensators of lag $\leqq q$.

3. A parametrization of the space of $p \times m$ strictly proper systems of lag $\leqq n$. In this section we consider the space $LS^n_{m,p}$ of $p \times m$ strictly proper systems of lag n. The notion of "lag" rather than the notion of "McMillan degree" is frequently considered in economics and statistics in the modeling of autoregressive moving average (ARMA) systems. In particular, we refer to Deistler and Hannan [9] and consider the following difference equation:

$$D_0 y(t) + D_1 y(t-1) + \cdots + D_n y(t-n) = N_0 u(t) + N_1 u(t-1) + \cdots + N_n u(t-n) \tag{3.1}$$

where $y(t)$ is the output p vector, $u(t)$ is the input m vector, and where D_i, N_i for $i = 0, \cdots, n$ are, respectively, $p \times p$ and $p \times m$ real matrices. The above difference equation, (3.1), can be represented in the frequency domain as follows:

$$D(z) y(z) = N(z) u(z) \tag{3.2}$$

where

$$\begin{aligned} D(z) &= D_0 z^n + D_1 z^{n-1} + \cdots + D_n, \\ N(z) &= N_0 z^n + N_1 z^{n-1} + \cdots + N_n. \end{aligned} \tag{3.3}$$

Let us now consider the following definitions.

DEFINITION 3.1. The pair $D(z)$, $N(z)$ is coprime if

$$\operatorname{rank}\,[D(z), N(z)] = p \tag{3.4}$$

for all $z \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

DEFINITION 3.2. The difference equation (3.1) is said to represent an ARMA model if $D(z)$, $N(z)$ are coprime.

DEFINITION 3.3. An ARMA model (3.1) is said to be proper if D_0 is nonsingular. Otherwise it is called improper.

DEFINITION 3.4. An ARMA model (3.1) is said to be strictly proper if D_0 is nonsingular and $N_0 = 0$.

Let us now describe the space $LS^n_{m,p}$ as follows:

$$LS^n_{m,p} \triangleq \{(D_0, \cdots, D_n, N_0, \cdots, N_n): D_0 = I, N_0 = 0 \text{ and } D(z), N(z) \text{ are coprime}\}. \tag{3.5}$$

From (3.5), we clearly have the following proposition which we state without proof.

PROPOSITION 3.5. *$LS^n_{m,p}$ is an open and dense subset of* $\mathbb{R}^{np(m+p)}$.

We now describe the following parametrization problem originally considered in an earlier paper [4].

Problem 3.6. Assume that $D_0 = I$, $N_0 = 0$. Parametrize the set of points $\xi \triangleq (D_1, \cdots, D_n, N_1, \cdots, N_n)$ in $\mathbb{R}^{np(m+p)}$ with the property that there exists a neighborhood $N(\xi)$ of ξ in $\mathbb{R}^{np(m+p)}$ such that $N(\xi) \cap LS^n_{m,p}$ is simultaneously stabilizable by a dynamic compensator.

Throughout this paper, we shall assume that the region of stability is $\mathbb{D}_s$, the open interior of the unit disc. Elementary arguments would show that the set of points ξ, satisfying the property described in Problem 3.6, give rise to the space $L\Omega^n_{m,p}$ defined as follows:

$$L\Omega^n_{m,p} \triangleq \{(D_0, D_1, \cdots, D_n, N_0, N_1, \cdots, N_n): D_0 = I,\ N_0 = 0 \text{ and } \operatorname{rank}[D(z), N(z)] = p \text{ for all } z \text{ in } \mathbb{D}_u\}. \tag{3.6}$$

Clearly the space $L\Omega^n_{m,p}$ parametrizes the set of ARMA systems with lag $\leqq n$. In this parametrization, however, a given system is represented in a nonunique way. To see why (3.6) indeed describes the set of points defined in Problem 3.6 let $N_c(z)D_c(z)^{-1}$ be the transfer function of a dynamic compensator. The return difference polynomial is given by $\det[D(z)D_c(z) + N(z)N_c(z)]$. Thus if there exists a point ξ in $\mathbb{R}^{np(m+p)}$ for which $\operatorname{rank}[D(z)N(z)] < p$ for some z_0 in $\mathbb{D}_u$, then every neighbor $N(\xi)$ of ξ would have a point ξ' in $LS^n_{m,p}$ with the property that if $D(z)^{-1}N'(z)$ is the corresponding transfer function, then $\det[D'(z)D_c(z) + N'(z)N_c(z)]$ vanishes arbitrary close to z_0. Thus $\xi' \notin L\Omega^n_{m,p}$. The sufficiency condition, on the other hand, is clear.

We now generalize the spaces $LS^n_{m,p}$ and $L\Omega^n_{m,p}$ and consider proper systems of lag n and lag $\leqq n$, respectively, as follows:

$$LP^n_{m,p} \triangleq \{(D_0, \cdots, D_n, N_0, \cdots, N_n): D_0 = I \text{ and } D(z), N(z) \text{ are coprime}\}, \tag{3.7}$$

$$L\Pi^n_{m,p} \triangleq \{(D_0, \cdots, D_n, N_0, \cdots, N_n): D_0 = I \text{ and } \operatorname{rank}(D(z), N(z)) = p \text{ for all } z \text{ in } ID_u\}. \tag{3.8}$$

We shall say more about the spaces $LP^n_{m,p}$ and $L\Pi^n_{m,p}$ in subsequent sections. Presently however we need the space $L\Pi^n_{m,p}$ in order to define the stabilizability problem.

In order to study the stabilizability properties of a control system we now propose to restrict the space $L\Omega^n_{m,p}$ and consider the space $L\Omega^{n;q}_{m,p}$ of ARMA systems in $L\Omega^n_{m,p}$ that can be stabilized by an $m \times p$ proper ARMA system of lag $\leqq q$ in $L\Pi^q_{p,m}$. Likewise we consider the space $L\Pi^{n;q}_{m,p}$ of proper ARMA systems in $L\Pi^n_{m,p}$ that can be stabilized by a compensator in $L\Pi^q_{p,m}$. The main result of this section is a generalization of Theorem 2.3 in [4] described as follows.

THEOREM 3.7. *The space $L\Omega^{n;q}_{m,p}$ is diffeomorphic to $\mathbb{R}^{np(m+p)}$ for all $q = 0, \cdots, \infty$.*

Remark 3.8. Using a different argument we shall show subsequently that $L\Pi^{n;q}_{m,p}$ is diffeomorphic to $\mathbb{R}^{np(m+p)+mp}$ for all $q = 0, \cdots, \infty$. It may be noted however that for $q = \infty$, the spaces $L\Omega^{n;\infty}_{m,p}$, $L\Omega^n_{m,p}$ and $L\Pi^{n;\infty}_{m,p}$, $L\Pi^n_{m,p}$ are identical.

Proof of Theorem 3.7. Consider the map

$$\phi : \mathbb{R}^{np(m+p)} \times \mathbb{R} \to \mathbb{R}^{np(m+p)}, \tag{3.9}$$

$$\phi(D_1, \cdots, D_n, N_1, \cdots, N_n, t) = (e^{-t}D_1, \cdots, e^{-nt}D_n, e^{-t}N_1, \cdots, e^{-nt}N_n). \tag{3.10}$$

The map ϕ defines a flow on $\mathbb{R}^{np(m+p)}$. Let X be the corresponding vector field. Clearly X has a unique equilibrium point at the origin of $\mathbb{R}^{np(m+p)}$. Moreover it may be seen that X restricts to a globally asymptotically stable vector field on $L\Omega^{n;q}_{m,p}$. Therefore by Milnor's Theorem (see Appendix I), it follows that $L\Omega^{n;q}_{m,p}$ is diffeomorphic to $\mathbb{R}^{np(m+p)}$. □

In view of Remark 3.8, we have the following corollary.

COROLLARY 3.9. *The space $L\Omega^{n}_{m,p}$ is diffeomorphic to $\mathbb{R}^{np(m+p)}$.*

Remark 3.10. Note that for a parameter identification algorithm defined by a locally and globally convergent vector field, the associated space must be diffeomorphic to a Euclidean space. As a consequence of Theorem 3.7 it may now be possible to generalize many known parameter identification algorithms to $L\Omega^{n;q}_{m,p}$ (see [10]), where the associated space is assumed to be Euclidean. On the basis of a referee's comment, however, we remark that parameter identification algorithms may not be necessarily defined by a locally and globally convergent vector field. Thus the exact correspondence between the existence of convergent on-line identification algorithms and the structure of the associated parameter space is not entirely clear and is a subject of future investigation.

The main result of this section is to obtain a parametrization of the space of strictly proper systems. In particular we show that the parameter space is diffeomorphic to a Euclidean space.

4. A vector bundle of the space of $p\times m$ systems of lag $\leqq n$. In this section, we generalize the notion of a strictly proper and a proper ARMA system and consider the space $L^n_{m,p}$ of ARMA systems with m inputs and p outputs having a lag of at most n described as follows:

$$L^n_{m,p} \triangleq \{[D_0,\cdots,D_n,N_0,\cdots,N_n]\in \text{Grass}\,(p,(n+1)(p+m)):\ \text{rank}\,(D(z),N(z))=p \text{ for all } z\in\mathbb{D}_u\} \tag{4.1}$$

where $[D_0,\cdots,D_n,N_0,\cdots,N_n]$ denotes the subspace spanned by the rows of the matrix $(D_0,\cdots,N_n)$. We will use this notation throughout the paper. Note that left-multiplying (3.1) by a nonsingular $p\times p$ matrix does not change the corresponding input-output relationship. Thus we may view the ARMA system (3.1) as a subset of a Grassmannian.

Of course the space $L^n_{m,p}$ consists of ARMA systems that are not necessarily proper since the matrix D_0 can be singular and in fact can be zero. The improper ARMA systems may be viewed as "infinite objects" that are of interest in the study of system degeneration and high gain compensation (see [12], [17]). Some of the infinite points in $L^n_{m,p}$ are the well-known generalized dynamical systems [18] and may be viewed in $L^n_{m,p}$ as limits of regular systems.

As noted in our earlier paper [4], $L^n_{m,p}$ is an over-parametrization of the space of $p\times m$ ARMA systems of lag $\leqq n$. In fact we have the following definition.

DEFINITION 4.1. Two points $[D^1_0,\cdots,D^1_n,\ N^1_0,\cdots,N^1_n]$ and $[D^2_0,\cdots,D^2_n,\ N^2_0,\cdots,N^2_n]$ in $L^n_{m,p}$ are said to be equivalent if there exist matrices $K_1(z)$ and $K_2(z)$ such that

$$K_2(z)D_2(z)=K_1(z)D_1(z), \tag{4.2}$$

$$K_2(z)N_2(z)=K_1(z)N_1(z) \tag{4.3}$$

and where det $K_1(z)$ and det $K_2(z)$ vanish in $\mathbb{D}$, and where

$$D_i(z)=\sum_{j=0}^{n} D^i_j z^{n-j},\quad N_i(z)=\sum_{j=0}^{n} N^i_j z^{n-j},\quad i=1,2. \tag{4.4}$$

In view of Definition 4.1 we can define the quotient space $L^n_{m,p}/\sim$ and a quotient topology on such a space. We shall describe these in detail in § 6.

In order to study the properties of $L^n_{m,p}$ we begin with the following preliminary result.

LEMMA 4.2. *$L^n_{m,p}$ is an open subset of* Grass $(p, (m+p)(n+1))$.

Proof. Let $\lambda = [D_0, \cdots, D_n, N_0, \cdots, N_n]$ be a point in $L^n_{m,p}$. Since λ is of full row rank, there exists a $p \times p$ submatrix of λ which can be scaled to identity. Assume without loss of generality that λ can be written as

$$\lambda \triangleq [I, D_1, \cdots, D_n, N_0, \cdots, N_n]. \tag{4.5}$$

The argument will be similar for any other structure of λ. Since $\lambda \in L^n_{m,p}$ it follows that

$$\text{rank}\,[D(z), N(z)] = p \tag{4.6}$$

for all $z \in \mathbb{D}_u$. Let $\psi_1(z), \cdots, \psi_t(z)$ be the set of principal minors of the matrix $[D(z), N(z)]$. It follows that $\psi_1(z), \cdots, \psi_t(z)$ do not have a common zero in $\mathbb{D}_u$. Let Ω be a neighborhood of $D_1, \cdots, D_n, N_0, \cdots, N_n$ in $\mathbb{R}^{(m+p)n+m}$. For Ω sufficiently small, corresponding to Ω there exists a neighborhood N of λ in Grass $(p, (m+p) \times (n+1))$ such that $N \in L^n_{m,p}$. □

By analogous arguments we state and prove the subsequent results of this section for the space $L^{n,q}_{m,p}$ defined as follows.

DEFINITION 4.3. Let $L^{n,q}_{m,p}$ be the space of ARMA systems in $L^n_{m,p}$ that can be stabilized by an $m \times p$ ARMA system in $L^q_{p,m}$. The following trivial result is now stated without proof.

LEMMA 4.4. *$L^{n,q}_{m,p}$ is an open subset of* Grass $(p, (m+p)(n+1))$ *for all* $q = 0, \cdots, \infty$.

Note 4.5. $L^{n,\infty}_{m,p}$ is clearly identical to the space $L^n_{m,p}$.

In the subsequent part of this section, we show that $L^{n,q}_{m,p}$ is a vector bundle. As we have argued in § 1, it is this property of $L^{n,q}_{m,p}$ which is of great importance in parameter identification. Thus the parameter identification problem can now be broken up into two distinct parts. The first involves identifying a point on the base; the second involves identifying the corresponding point on the fiber.

We would now detail the structure of $L^{n,q}_{m,p}$ as a vector bundle. Let us remind the reader at this point that if

$$[D_0, \cdots, D_n, N_0, \cdots, N_n] \in L^{n,q}_{m,p} \tag{4.7}$$

then

$$[D_0, N_0] \in \text{Grass}\,(p, m+p). \tag{4.8}$$

This is because

$$\text{rank}\,[D_0, N_0] = \text{rank}\,[D(\infty), N(\infty)] = p. \tag{4.9}$$

Hence we have a well-defined map

$$\psi: L^{n,q}_{m,p} \to \text{Grass}\,(p, m+p), \tag{4.10}$$

$$\psi([D_0, \cdots, D_n, N_0, \cdots, N_n]) = [D_0, N_0]. \tag{4.11}$$

Since for an arbitrary element $[D_0, N_0] \in \text{Grass}\,(p, m+p)$, the corresponding point $[D_0, 0, \cdots, 0, N_0, 0, \cdots, 0]$ is an element in $L^n_{m,p}$ and $\psi([D_0, 0, \cdots, 0, N_0, \cdots, 0]) = [D_0, N_0]$, it follows that ψ is a surjection. Furthermore it is clear that ψ is a smooth mapping. The main result in this section is the following theorem.

THEOREM 4.6.

$$\psi: L^{n,q}_{m,p} \to \text{Grass}\,(p, m+p)$$

can be endowed with the structure of a smooth vector bundle for every $q = 0, 1, \cdots, \infty$.

The proof of Theorem 4.6 is lengthy and will therefore be broken up into several lemmas.

LEMMA 4.7. *For every* $[D_0, N_0] \in \mathrm{Grass}(p, m+p)$, $\psi^{-1}([D_0, N_0])$ *is an embedded submanifold of* $L^{n,q}_{m,p}$ *which is diffeomorphic to* $\mathbb{R}^{(m+p)np}$.

Proof. Since $L^{n,q}_{m,p}$ is an open subset of $\mathrm{Grass}(p, (m+p)(n+1))$ it follows that ψ is a submersion. Hence $\psi^{-1}([D_0, N_0])$ is an embedded submanifold of $L^{n,q}_{m,p}$ of dimension $\mathbb{R}^{(m+p)(np)}$. Let us now consider the local flow

$$\text{(4.12)} \qquad \textcircled{H}: L^{n,q}_{m,p} \times [0, \infty) \to L^{n,q}_{m,p}$$

$$\text{(4.13)} \qquad \textcircled{H}([D(z), N(z)], t) = D(e^t z), N(e^t z)].$$

We leave it to the reader to verify that $\textcircled{H}$ is indeed a local flow on $L^{n,q}_{m,p}$, i.e., to verify the smoothness, the semigroup property, and the fact that $\textcircled{H}([D(z), N(z)], t) \in L^{n,q}_{m,p}$ for all $t \in (-\varepsilon, \infty)$ for some $\varepsilon > 0$ which depends on $[D(z), N(z)]$. Let Y denote the vector field on $L^{n,q}_{m,p}$ generated by this local flow. The key observation is that $\textcircled{H}$ preserves the fiber $\psi^{-1}([D_0, N_0])$ for all $[D_0, N_0] \in \mathrm{Grass}(p, m+p)$. Thus the vector field Y is tangential to the fibers of the submersion ψ. Moreover when restricted to the fiber $\psi^{-1}([D_0, N_0])$, Y has a unique equilibrium point $[z^n D_0, z^n N_0]$ which is a local and global attractor. Thus by Milnor's theorem (Appendix I) it follows that each fiber is diffeomorphic to $\mathbb{R}^{(m+p)np}$. $\square$

Remark 4.8. In order to prove Theorem 4.6 we need to do some more work. In fact we need to show that the diffeomorphisms of Lemma 4.7 can be chosen in such a way that when we endow the fibers $\psi^{-1}([D_0, N_0])$ with vector space structures by using our diffeomorphisms, local triviality of the bundle follows.

Let us now describe the following vector bundle. Denote by Σ the subset of $\mathrm{Grass}(p, (m+p)(n+1))$ defined by

$$\text{(4.14)} \qquad \Sigma \triangleq \{[D_0, \cdots, D_n, N_0, \cdots, N_n] : [D_0, N_0] \in \mathrm{Grass}(p, m+p)\}.$$

Let us define

$$\text{(4.15)} \qquad \Sigma \to \mathrm{Grass}(p, m+p),$$

$$\text{(4.16)} \qquad \pi([D(z), N(z)]) = [D_0, N_0].$$

Clearly π is a smooth submersion. Moreover each fiber $\pi^{-1}([D_0, N_0])$ has a natural vector space structure obtained by defining

$$\text{(4.17)} \qquad \begin{aligned} &[D_0, D_1, \cdots, D_n, N_0, N_1, \cdots, N_n] + [D_0, \bar{D}_1, \cdots, \bar{D}_n, N_0, \bar{N}_1, \cdots, \bar{N}_n] \\ &\quad = [D_0, D_1 + \bar{D}_1, \cdots, D_n + \bar{D}_n, N_0, N_1 + \bar{N}_1, \cdots, N_n + \bar{N}_n], \end{aligned}$$

$$\text{(4.18)} \qquad \begin{aligned} &\lambda [D_0, D_1, \cdots, D_n, N_0, N_1, \cdots, N_n] \\ &\quad = [D_0, \lambda D_1, \cdots, \lambda D_n, N_0, \lambda N_1, \cdots, \lambda N_n]. \end{aligned}$$

Clearly Σ is a vector bundle under the above vector space operations on fibers.

We now proceed to put a smooth Riemannian metric on the vector bundle Σ. In order to do this, when we write an element $[D_0, N_0]$ in $\mathrm{Grass}(p, m+p)$, we shall assume that the representative matrix $[D_0, N_0]$ has orthonormal rows. Hence, if $[D_0, N_0] = [\tilde{D}_0, \tilde{N}_0]$, then there exists an orthogonal matrix g such that

$$\text{(4.19)} \qquad g D_0 = \tilde{D}_0, \qquad g N_0 = \tilde{N}_0.$$

Now if $[D_0, D_1, \cdots, D_n, N_0, N_1, \cdots, N_n] \in \Sigma$ let us denote the columns of D_i as D_{ij}, $j = 1, \cdots, p$ and the columns of N_i as N_{ij}, $j = 1, \cdots, m$. Define a Riemannian metric on the vector bundle Σ by specifying

$$\begin{aligned}&\langle [D_0, D_1, \cdots, D_n, N_0, \cdots, N_n], [D_0, \bar{D}_1, \cdots, \bar{D}_n, N_0, \bar{N}_1, \cdots, \bar{N}_n]\rangle \\ &\quad = \sum_{i=1}^{n} \sum_{j=1}^{p} D_{ij} \cdot \bar{D}_{ij} + \sum_{j=1}^{m} N_{ij} \cdot \bar{N}_{ij}\end{aligned} \tag{4.20}$$

where $D_{ij} \cdot \bar{D}_{ij}$ and $N_{ij} \cdot \bar{N}_{ij}$ denote the usual dot product in $\mathbb{R}^p$. Note that, as a consequence of the restriction on $[D_0, N_0]$ to having orthonormal rows, it follows that the definition of the metric does not depend on the representative element.

Let S_μ denote the sphere bundle [23] over Grass $(p, m+p)$ consisting of vectors in Σ of magnitude equal to $1/\mu$. We now prove the following.

LEMMA 4.9. *For μ sufficiently large, $S_\mu \subseteq L^{n;q}_{m,p}$ and the flow* Ⓗ *is transverse to S_μ.*

Proof. Let W be the space of $p \times (m+p)$ matrices with the property that

$$Q \in W \quad \text{iff } QQ^T = I. \tag{4.21}$$

Elementary arguments show that W is a compact subset in $\mathbb{R}^{p\times(m+p)}$. Hence there exists $\varepsilon > 0$ such that if R is a $p \times m+p$ matrix with columns having magnitude $< \varepsilon$ then $R+Q$ has rank p for all $Q \in W$. Thus for μ large enough we see that if

$$[D_0, D_1, \cdots, D_n, N_0, N_1, \cdots, N_n] \in S_\mu$$

then the matrix

$$\left[\frac{1}{z} D_1 + \frac{1}{z^2} D_2 + \cdots + \frac{1}{z^n} D_n, \frac{1}{z} N_1 + \frac{1}{z^2} N_2 + \cdots + \frac{1}{z^n} N_n\right] \tag{4.22}$$

has columns whose magnitude are less than ε for all $z \in \mathbb{D}_u$ and for all representative elements. Hence it follows that

$$\left[D_0 + \frac{1}{z} D_1 + \cdots + \frac{1}{z^n} D_n, N_0 + \frac{1}{z} N_1 + \cdots + \frac{1}{z^n} N_n\right] \tag{4.23}$$

has rank p for all $z \in \mathbb{D}_u$. This shows that for large μ, $S_\mu \subseteq L^n_{m,p}$.

We now show that for a given q, there exists μ large enough such that $S_\mu \subseteq L^{n;q}_{m,p}$. Notice that since $[D_0, N_0]$ is of rank p there exist matrices D_c, N_c such that

$$D_0 D_c + N_0 N_c = I_p \tag{4.24}$$

where D_c is invertible. In other words there exists a compensator $N_c D_c^{-1}$ which stabilizes the plant $D_0^{-1} N_0$. Moreover for μ large enough the plant

$$\left[D_0 + \frac{1}{z} D_1 + \cdots + \frac{1}{z^n} D_n\right]^{-1} \left[N_0 + \frac{1}{z} N_1 + \cdots + \frac{1}{z^n} N_n\right] \tag{4.25}$$

is stabilizable by the compensator $N_c D_c^{-1}$. It now follows by the compactness of S_μ that for large μ, $S_\mu \subseteq L^{n;q}_{m,p}$. Moreover S_μ is an embedded submanifold of Σ, and hence of $L^{n;q}_{m,p}$ since $L^{n;q}_{m,p} \subseteq_{\text{open}} \Sigma$.

To show that the flow Ⓗ is transverse to S_μ let us define

$$[D(z), N(z)] = [D_0, D_1, \cdots, D_n, N_0, N_1, \cdots, N_n] \in S_\mu \tag{4.26}$$

where we still assume that $[D_0, N_0]$ has orthonormal rows. It follows that

$$\begin{aligned}Ⓗ([D(z), N(z)], t) &= [D(e^t z), N(e^t z)] \\ &= [D_0, e^{-t} D_1, \cdots, e^{-nt} D_n, N_0, e^{-t} N_1, \cdots, e^{-nt} N_n].\end{aligned} \tag{4.27}$$

Hence

$$\|D(e^t z), N(e^t z)\|^2 < \sum_{j=1}^{p} \sum_{i=1}^{n} D_{ij} \cdot D_{ij} + \sum_{j=1}^{m} \sum_{i=1}^{n} N_{ij} \cdot N_{ij} \tag{4.28}$$
$$= \|D(s), N(s)\|^2 = \frac{1}{\mu^2}.$$

Thus if $t > 0$ then

$$\|Ⓗ([D(s), N(s)], t)\| < \frac{1}{\mu}, \tag{4.29}$$

thus proving the assertion. □

Finally, before we prove Theorem 4.6, we need the following technical result from the literature which we state without proof (see [28, p. 85]).

LEMMA 4.10. *Suppose that M is a paracompact C^∞ manifold and X is a smooth vector field. Then there exist a nowhere zero smooth function f such that fX is a complete vector field.*

We now prove Theorem 4.6.

Proof of Theorem 4.6. Let Y be the vector field on $L_{m,p}^{n,q}$ which has been defined earlier and let f be a strictly positive smooth function on $L_{m,p}^{n,q}$ such that fY is complete. Let μ be large enough such that Y is transversal to S_μ. Clearly fY is transversal to S_μ also. Moreover fY restricts to a vector field on $\psi^{-1}([D_0, N_0])$ for each $[D_0, N_0] \in$ Grass $(p, m+p)$. Since fY is transverse to S_μ for large μ and since fY is globally attracting to $[D_0, N_0]$, it follows that $[D_0, N_0]$ is the unique globally asymptotically stable equilibrium point on $\psi^{-1}([D_0, N_0])$ of the vector field fY. Let us now define the map

$$\Phi : \Sigma \to L_{m,p}^{n,q}, \tag{4.30}$$

$$\Phi(\xi) = \xi \quad \text{if } \xi \in \text{Grass}\,(p, m+p) \tag{4.31}$$
$$= \phi^{fY}_{-\log \mu \|\xi\|}\left(\frac{1}{\mu \|\xi\|}\xi\right).$$

It is easy to check that

(4.32) (1) Φ is a diffeomorphism;
(2) the diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{\ \Phi\ } & L_{m,p}^{n,q} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \text{Grass}\,(p, m+p) & \xrightarrow{\ id\ } & \text{Grass}\,(p, m+p) \end{array} \tag{4.33}$$

commutes.

We now use Φ to endow a vector space structure on each fiber of

$$\psi : L_{m,p}^{n,q} \to \text{Grass}\,(p, m+p). \tag{4.34}$$

Thus we have $L_{m,p}^{n,q}$ as a vector bundle isomorphic to Σ. □

Remark 4.11. The vector bundle structure endowed on $L_{m,p}^{n,q}$ is independent of the choice of the function f up to bundle isomorphism.

An important corollary of Theorem 4.6 is described as follows.

COROLLARY 4.12. *$L\Pi_{m,p}^{n,q}$ is isomorphic to $\mathbb{R}^{p[(m+p)n+m]}$.*

Proof. Consider the subspace B of Grass $(p, m+p)$ defined as follows:

$$B \triangleq \{[D_0, N_0] \in \text{Grass}(p, m+p) \colon D_0 = I\}. \tag{4.35}$$

It is clear that B is a chart of Grass $(p, m+p)$ and is diffeomorphic to $\mathbb{R}^{mp}$. We now consider the pullback of the vector bundle (4.34) as follows:

$$\begin{array}{ccc} f^* L_{m,p}^{n,q} & & L_{m,p}^{n,q} \\ {\scriptstyle f^*\psi}\downarrow & & \downarrow{\scriptstyle \psi} \\ B & \xrightarrow[f]{} & \text{Grass}(p, m+p) \end{array}$$

where f is the inclusion of B in Grass $(p, m+p)$. Of course, since B is contractible to a point, the vector bundle $f^*\psi : f^* L_{m,p}^{n,q} \to B$ is trivial [23]. Thus $f^* L_{m,p}^{n,q} \cong B \times \mathbb{R}^{(m+p)np}$. Finally from the definition of $L\Pi_{m,p}^{n,q}$ in § 3, it is clear that $f^* L_{m,p}^{n,q}$ and $L\Pi_{m,p}^{n,q}$ are identical. Thus $L\Pi_{m,p}^{n,q} \cong \mathbb{R}^{mp} \times \mathbb{R}^{(m+p)mp}$. □

We now proceed to describe the structure of the vector bundle $\psi : L_{m,p}^{n,q} \to$ Grass $(p, m+p)$. Define a canonical vector bundle $\gamma^p(\mathbb{R}^{m+p})$ over Grass $(p, m+p)$ as follows. Let

$$E = E(\gamma^p(\mathbb{R}^{m+p})) \tag{4.36}$$

be the set of all pairs

$$(p \text{ plane in } \mathbb{R}^{m+p}, \text{ vector in that } p\text{-plane}).$$

This is to be topologized as a subset of Grass $(p, m+p) \times \mathbb{R}^{m+p}$. The projection map $\pi_1 : E \to$ Grass $(p, m+p)$ is defined by $\pi_1(X, x) = X$ and the vector space structure in the fiber over X is defined by

$$t_1(X, x_1) + t_2(X, x_2) = (X, t_1 x_1 + t_2 x_2).$$

The main result on the structure of $L_{m,p}^{n,q}$ is described as follows.

THEOREM 4.13. *The vector bundle*

$$\psi : L_{m,p}^{n,q} \to \text{Grass}(p, m+p) \tag{4.37}$$

is isomorphic to $(m+p)n$ Whitney sums of canonical vector bundles $\gamma^p(\mathbb{R}^{m+p})$.

$$L_{m,p}^{n,q} \cong \gamma^p(\mathbb{R}^{m+p}) \oplus \cdots \oplus \gamma^p(\mathbb{R}^{m+p}). \tag{4.38}$$

Proof. Let us consider Σ as defined in (4.14), together with the map π given by (4.15), (4.16). In view of the proof of Theorem 4.6, it suffices to show that this map

$$\pi : \Sigma \to \text{Grass}(p, m+p) \tag{4.39}$$

has the structure of a vector bundle which is isomorphic to $n(m+p)$ copies of canonical vector bundle $\gamma^p(\mathbb{R}^{m+p})$.

We now proceed to prove the above claim. Recall from (4.17), (4.18) that π has the structure of a vector bundle.

Let D_{ij} denote the jth column of D_i and let N_{ij} denote the jth column of N_i. Restrict the representative elements $[D_0, D_1, \cdots, N_0, N_1, \cdots, N_n]$ to have the property that the rows of $[D_0, N_0]$ are orthonormal. Let us now define the map σ as follows:

(4.40)
$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \gamma^p \oplus \cdots \oplus \gamma^p (n(m+p) \text{ copies}) \\ \pi \downarrow & & \downarrow \\ \text{Grass}(p, m+p) & \xleftrightarrow{id} & \text{Grass}(p, m+p) \end{array}$$

$$\sigma : \Sigma \to \gamma^p \oplus \cdots \oplus \gamma^p,$$
$$\sigma[D_0 D_1 \cdots D_n N_0 N_1 \cdots N_n]$$
$$= \left(\sum_{j=1}^{p} D_{ij}(D_0, N_0)^j, \sum_{j=1}^{p} D_{2j}(D_0, N_0)^j, \cdots, \sum_{j=1}^{m} D_{nj}(D_0, N_0)^j \right)$$

where $(D_0, N_0)^j$ denotes the jth row of the matrix (D_0, N_0) considered as a vector in $\mathbb{R}^{m+p}$. We need to check, however, that σ is well defined; i.e., if K is an orthogonal $p \times p$ matrix then

(4.41)
$$\sigma([KD_0, KD_1, \cdots, KD_n, KN_0, \cdots, KN_n]) = \sigma([D_0, D_1, \cdots, D_n, N_0, \cdots, N_n]).$$

The above fact, however, follows at once since

(4.42)
$$(KD_0, KN_0)^T KV = (D_0, N_0)^T K^T KV = (D_0, N_0)^T V.$$

The rest of the proof that σ is a diffeomorphism and the diagram commutes is clear and is omitted. □

When $p = 1$, the vector bundle (4.37) reduces to a vector bundle over a projective space. In this situation, it is possible in many cases to decide whether or not the bundle (4.37) is trivial.

Let us assume $p = 1$ and write

(4.43)
$$m + 1 = 2^r t \quad \text{and} \quad n = 2^{r_1} t_1$$

where t, t_1 are odd. We now have the following interesting corollaries.

COROLLARY 4.14. $2^{r_1} < t \Rightarrow L^{n,q}_{m,1}$ *is nontrivial.*

COROLLARY 4.15. $L^{n,q}_{1,1}$, $L^{n,q}_{3,1}$, $L^{n,q}_{7,1}$ *are trivial vector bundles.*

COROLLARY 4.16. $L^{1,q}_{15,1}$, $L^{1,q}_{31,1}$, $L^{1,q}_{63,1}$ $\cdots$ *are nontrivial vector bundles.*

Before we state and prove the corollaries above, we need certain notions from the theory of "characteristic classes," for which we refer to [7]. We would like to summarize some of the main ideas here as follows.

Let $H^i(B, G)$ denote the ith singular cohomology group of B with coefficients in G. We have the following important result.

FACT 4.17. The group $H^i(\mathbb{RP}^n; Z/2)$ is cyclic of order two for $0 \leqq i \leqq n$ and is zero for higher values of i. Furthermore, if a denotes the nonzero element of $H^1(\mathbb{RP}^n; Z/2)$, then each $H^i(\mathbb{RP}^n; Z/2)$ is generated by the i-fold cap product a^i.

Thus $H^*(\mathbb{RP}^n; Z/2)$ can be described as the algebra with unit over $Z/2$ having one generator a and one relation $a^{n+1} = 0$. Moreover the total Stiefel-Whitney class of the canonical line-bundle $\gamma^1(\mathbb{R}^{m+1})$ over $\mathbb{RP}^m$ is given by

(4.44)
$$w(\gamma^1(\mathbb{R}^{m+1})) = 1 + a.$$

FACT 4.18. Let $m + 1 = 2^r t$, $n = 2^{r_1} t_1$, where t, t_1 are odd. Then

(4.45)
$$2^{r_1} < t \Leftrightarrow (1 + a)^{n(m+1)} \neq 1$$

where $a^{m+1} = 0$.

Proof. Assume $2^{r_1} < t$. Clearly we have

$$(4.46)\qquad \begin{aligned}(1+a)^{n(m+1)} &= (1+a^{2^{r+r_1}})^{tt_1} \\ &= 1 + tt_1 a^{2^{r+r_1}} + (\text{higher-order terms}).\end{aligned}$$

Since $2^{r+r_1} < m+1$, it follows that

$$(1+a)^{n(m+1)} \neq 1.$$

Conversely, let $2^{r_1} \geqq t$. It follows that

$$(1+a)^{n(m+1)} = (1+a^{2^{r+r_1}})^{tt_1} = 1. \qquad \square$$

We now proceed to prove the corollaries.

Proof of Corollary 4.14. The Stiefel–Whitney class $w(L^{n,q}_{m,1})$ is given by $(1+a)^{n(m+1)}$, which is not equal to 1 from Fact 4.18. Hence $w(L^{n,q}_{m,1})$ is a nontrivial vector bundle. $\square$

Proof of Corollary 4.15. Let $m \in \{1, 3, 7\}$. We have

$$(4.47)\qquad w(L^{n,q}_{m,1}) = (1+a)^{n(m+1)} = 1.$$

Let $n = 1$; then

$$L^{1,q}_{m,1} \cong \gamma^1(\mathbb{R}^{m+1}) \oplus \cdots \oplus \gamma^1(\mathbb{R}^{m+1}) (m+1 \text{ copies}).$$

By a theorem due to Stiefel [19], $m+1$ Whitney sums of $\gamma^1(\mathbb{R}^{m+1})$ are a trivial vector bundle if $m = 1, 3, 7$. Thus $L^{1,q}_{m,1}$ is a trivial vector bundle. Moreover $L^{n,q}_{m,1}$ is an n-fold Whitney sum of trivial vector bundles and is therefore trivial. $\square$

Proof of Corollary 4.16. From Theorem 4.13 we may conclude that if $m = 15, 31, 63, \cdots$, $L^{1,q}_{m,1}$ is not trivial. This follows from a theorem due to Bott and Milnor [20], Kervaire [21], and Adams [22] that $m+1$ Whitney sums of $\gamma^1(\mathbb{R}^{m+1})$ are nontrivial if $m = 15, 31, 63, \cdots$. $\square$

Example 4.19. Assume $p = 2$, $m = 3$. In this example we show that $L^{n,q}_{3,2}$ is nontrivial if n is not a multiple of 8.

First of all, note that $L^{n,q}_{3,2}$ is isomorphic to $5n$ Whitney sums of canonical vector bundles $\gamma^2(\mathbb{R}^5)$. In fact, it follows from [7, Problem 7-B, p. 87] that $H^*(G_2(\mathbb{R}^5))$ over $Z/2$ is generated by the Stiefel–Whitney classes w_1, w_2 of $\gamma^2(\mathbb{R}^5)$ and the dual classes $\bar{w}_1$, $\bar{w}_2$, $\bar{w}_3$ subject only to the $n+k$ defining relations

$$(4.48)\qquad (1+w_1+w_2)(1+\bar{w}_1+\bar{w}_2+\bar{w}_3) = 1.$$

The restriction (4.48) defines the relation

$$(4.49)\qquad w_2 w_1^3 = 0$$

and

$$(4.50)\qquad w_2^2 + w_1^4 + w_2 w_1^2 = 0.$$

The total Stiefel–Whitney class of $\gamma^2(\mathbb{R}^5)$ is given by $1 + w_1 + w_2$. Hence

$$(4.51)\qquad w(L^{n,q}_{3,2}) = (1+w_1+w_2)^{5n}.$$

It can be checked easily that the minimum positive integer such that $(1+w_1+w_2)^j = 1$ is given by $j = 8$, i.e.,

$$(4.52)\qquad (1+w_1+w_2)^8 = 1.$$

Thus if n is not a multiple of 8,

$$(4.53)\qquad (1+w_1+w_2)^{5n} \neq 1$$

so that $L^{n,q}_{3,2}$ is nontrivial if n is not a multiple of 8.

To summarize, in this section we show that the space $L^{n,q}_{m,p}$, of $p\times m$ proper or improper ARMA systems of lag $\leqq n$ that can be stabilized by an $m\times p$ ARMA system of lag $\leqq q$, has the structure of a vector bundle over Grass $(p, m+p)$ for all $n=0,1,\cdots$ $q=0,1,\cdots$. The vector bundle above is isomorphic to an $n(m+p)$-fold Whitney sum of canonical vector bundle γ^p over Grass $(p, m+p)$. Finally the vector bundles $L^{n,q}_{m,1}$ are trivial for $m=1,3,7$.

5. A parametrization of stable feedback control systems. In this section we parametrize the space of $p\times m$ ARMA systems $G(z)$ of lag $\leqq n$, and $m\times p$ ARMA systems $K(z)$ of lag $\leqq q$, such that the closed-loop system $G(z)[I+K(z)G(z)]^{-1}$ is stable. Of course in the above definition $G(z)$ is assumed to represent the plant, $K(z)$ is assumed to represent the compensator, and the region of stability is assumed to be $\mathbb{D}_s$, the open interior of the unit disc.

Thus we define the space

(5.1)
$$LFB^{n,q}_{m,p}\triangleq\{([D_0,\cdots,D_n,N_0,\cdots,N_n],[L_0,\cdots,L_q,M_0,\cdots,M_q])\in L^n_{m,p}\times L^q_{p,m}|\text{ the pair of plants } D(z)^{-1}N(z) \text{ and } L(z)^{-1}M(z) \text{ is a stable pair}\}.$$

In (5.1) we define $D(z)$, $N(z)$ as in (3.3) and define

$$L(z)=L_0z^q+L_1z^{q-1}+\cdots+L_q,$$
$$M(z)=M_0z^q+M_1z^{q-1}+\cdots+M_q.$$

In order to justify the need to consider the space $LFB^{n,q}_{m,p}$ let us consider the following. In system design problems wherein we are designing a compensator for an unknown plant, frequently we assume an initial value of the plant and its stabilizing compensator. Subsequently both the plant and the compensator parameters are updated and the design objective is to ensure that the plant parameters converge to its true value. In this technique of identifying the parameters of a closed loop system, we define a recursive algorithm on the space $LFB^{n,q}_{m,p}$. Likewise, in adaptive control problems we are frequently interested in updating the parameters of an unknown plant and compensator in real time, with the added hypothesis that if the adaptive algorithm is switched off, the corresponding plant-compensator pair is stable. Thus we are interested in adaptively updating points on $LFB^{n,q}_{m,p}$ in real time.

The main result we now show in this section is described as follows.

MAIN THEOREM 5.1. *$LFB^{n,q}_{m,p}$ has the structure of a vector bundle over the base space* Grass $(p, m+p)$.

The ideas behind the proof of the main theorem can be most easily explained by considering the following preliminary lemmas. First, however, we must set up the following notation.

Define W to be

$$W\triangleq\{(A,B)\in \text{Grass}\,(p,m+p)\times\text{Grass}\,(m,m+p)|A\oplus B=\mathbb{R}^{m+p}\}. \tag{5.2}$$

Let ψ denote the projection

$$\psi^n_{m,p}: L^n_{m,p}\to\text{Grass}\,(p,m+p) \tag{5.3}$$

as in (4.34). We now state and prove the following lemmas.

LEMMA 5.2. *The map*

$$\phi: LFB^{n,q}_{m,p}\to\text{Grass}\,(p,m+p)\times\text{Grass}\,(m,m+p) \tag{5.4}$$

defined by

$$\phi([D(z),N(z)],[L(z),M(z)])=(\psi^n_{m,p}[D(z),N(z)],\psi^q_{p,m}[L(z),M(z)]) \tag{5.5}$$

has W as its image and ϕ is a submersion onto W.

Proof. Suppose that $([D(z), N(z)], [L(z), M(z)])$ is in $LFB^{n,q}_{m,p}$. It now follows that

$$\begin{bmatrix} z^{-n}D(z) & z^{-n}N(z) \\ z^{-q}M(z) & z^{-q}L(z) \end{bmatrix} \tag{5.6}$$

has full rank for all $z \in \mathbb{D}_u$. In particular, since $\infty \in \mathbb{D}_u$ it follows that the matrix

$$\begin{bmatrix} D_0 & N_0 \\ M_0 & L_0 \end{bmatrix} \tag{5.7}$$

has full rank. This shows that im $(\phi) \subseteq W$. Since for each pair $([D_0 \quad N_0], [L_0 \quad M_0]) \in W$, the corresponding closed loop system is stable, it now follows that im $(\phi) = W$. Finally since $LFB^{n,q}_{m,p}$ is an open subset of $L^n_{m,p} \times L^q_{p,m}$ and since

$$\psi^n_{m,p} \times \psi^q_{p,m} : L^n_{m,p} \times L^q_{p,m} \to \text{Grass}\,(p, m+p) \times \text{Grass}\,(m, m+p) \tag{5.8}$$

is a submersion, it follows that (5.4) is also a submersion. □

LEMMA 5.3. *The map* (5.4) *has the structure of a fiber bundle with fibers diffeomorphic to* $\mathbb{R}^{(m+p)(np+mq)}$.

Proof. Consider the local flow

$$Ⓗ^{n,q}_{m,p}: LFB^{n,q}_{m,p} \times [0, \infty) \to LFB^{n,q}_{m,p} \tag{5.9}$$

defined by

$$Ⓗ^{n,q}_{m,p}(([D(z), N(z)], [L(z), M(z)]); t)([D(e^tz), N(e^tz)], [L(e^tz), M(e^tz)]). \tag{5.10}$$

It is easy to check that the above local flow is well defined and smooth. Clearly, for each $(A, B) \in W$, the fiber $\phi^{-1}(A, B)$ of the submersion (5.4) at (A, B) is invariant under the flow $Ⓗ^{n,q}_{m,p}$. Moreover, restricted to $\phi^{-1}(A, B)$, this flow has exactly one equilibrium point which is globally and locally attracting. Since ϕ is a submersion, it follows that $\phi^{-1}(A, B)$ is a smooth Hausdorff manifold of dimension $(m+p)(mq+np)$. Thus using Milnor's theorem [5] we conclude that $\phi^{-1}(A, B)$ is diffeomorphic to $\mathbb{R}^{(m+p)(np+mq)}$.

The proof of local triviality of $LFB^{n,q}_{m,p}$ is similar to the proof sketched in the Theorem 4.6. Let $X^{n,q}_{m,p}$ be the vector field generated by the local flow $Ⓗ^{n,q}_{m,p}$ and let g be a nowhere zero C^∞ function on $LFB^{n,q}_{m,p}$ such that the vector field $Y \triangleq gX^{n,q}_{m,p}$ is complete. Let us define $\Sigma^n_{m,p}$ as the space defined by (4.14) and consider the vector bundle

$$\psi^n_{m,p}: \Sigma^n_{m,p} \to \text{Grass}\,(p, m+p) \tag{5.11}$$

defined via equations (4.15), (4.16). Endow a Riemannian metric on this vector space via (4.20). Similarly consider the vector bundle

$$\psi^q_{p,m}: \Sigma^q_{p,m} \to \text{Grass}\,(m, m+p) \tag{5.12}$$

and let

$$\psi: \Sigma \to \text{Grass}\,(p, m+p) \times \text{Grass}\,(m, m+p) \tag{5.13}$$

denote the Cartesian product of the two vector bundles (5.11) and (5.12). Endow the vector bundle Σ in (5.13) with the Cartesian product of the metrics on $\Sigma^n_{m,p}$ and $\Sigma^q_{p,m}$. Let S_μ denote the sphere bundle over Grass $(p, m+p) \times$ Grass $(m, m+p)$ consisting of vectors of Σ of magnitude μ. If U is an open subset of Grass $(p, m+p) \times$ Grass $(m, m+p)$, we will denote the restrictions of Σ and S_μ to U by $\Sigma | U$ and $S_\mu | U$, respectively.

Now let (A, B) be in W, and let U be a relatively compact open neighborhood of (A, B) in W. It follows easily that for μ sufficiently small, $S_\mu | U \subseteq LFB^{n,q}_{m,p}$. Moreover the vector field $X^{n,q}_{m,p}$ (and hence the vector field Y also) is transverse to $S_\mu | U$. Now by an argument analogous to that used in the proof of Theorem 4.6, we conclude that $LFB^{n,q}_{m,p}|U$ is a trivial fiber bundle with fibers diffeomorphic to $\mathbb{R}^{(m+p)(np+mq)}$. This proves that local triviality of

$$\phi : LFB^{n,q}_{m,p} \to W. \tag{5.14}$$

Hence $LFB^{n,q}_{m,p}$ is a fiber bundle over W with fibers diffeomorphic to $\mathbb{R}^{(m+p)(np+mq)}$. □

The above argument is not powerful enough to show that $LFB^{n,q}_{m,p}$ is a vector bundle. Before proceeding to prove this stronger result, we need to set up some notation.

If $A \in \text{Grass}\,(p, m+p)$, we will denote the orthogonal projection of A in $\mathbb{R}^{m+p}$ by $A^\perp$. Let

$$BW \triangleq \{(A, B) \in W | A \in \text{Grass}\,(p, m+p),\ B = A^\perp\}. \tag{5.15}$$

Let

$$i : BW \to W \tag{5.16}$$

be the inclusion map. We now consider the pullback of the fiber bundle (5.14) via the diagram

$$\begin{array}{ccc} i^*LFB^{n,q}_{m,p} & & LFB^{n,q}_{m,p} \\ \big\downarrow{\scriptstyle i^*\phi} & & \big\downarrow{\scriptstyle \phi} \\ BW & \xrightarrow{\ \ i\ \ } & W \end{array} \tag{5.17}$$

to obtain the locally trivial fiber bundle

$$i^*\phi : i^*LFB^{n,q}_{m,p} \to BW. \tag{5.18}$$

We now have the following two lemmas.

LEMMA 5.4. *$i^*LFB^{n,q}_{m,p}$ can be endowed with the structure of a vector bundle.*

Proof. Let Σ and S_μ be as in the proof of Lemma 5.3. Since BW is a compact subset of W, it follows that there exists a μ small enough such that $S_\mu | BW$ is contained in $i^*LFB^{n,q}_{m,p}$. Now by constructing a transverse and complete flow and using the argument in the proof of Theorem 4.6, we conclude that $i^*LFB^{n,q}_{m,p}$ can be endowed with the structure of a vector bundle which is isomorphic to $i^*\Sigma$. □

LEMMA 5.5. *Let us consider the map*

$$\rho : W \to BW \tag{5.19}$$

defined by

$$\rho(A, B) = (A, A^\perp); \tag{5.20}$$

then W can be endowed with the structure of a vector bundle over BW.

Proof. It is clear that $\rho : W \to BW$ is a submersion. Identify BW with $\text{Grass}\,(p, m+p)$ via the mapping

$$(A, A^\perp) \to A. \tag{5.21}$$

Let us put coordinates on BW by using standard coordinate charts on $\text{Grass}\,(p, m+p)$, i.e., let $i_1, \cdots, i_p$ be integers such that $1 \leq i_1 < i_2 < \cdots < i_p \leq m+p$. If V is in $\text{Grass}\,(p, m+p)$, denote by $V^{(i_1, \cdots, i_p)}$ the subspace in $\mathbb{R}^p$ obtained by projecting V along the $i_1, \cdots, i_p$ coordinate directions. Let

$$U^{(i_1, \cdots, i_p)} \triangleq \{V \in \text{Grass}\,(p, m+p) | V^{(i_1, \cdots, i_p)} = \mathbb{R}^p\}. \tag{5.22}$$

For each V in $U^{(i_1,\cdots,i_p)}$ find the unique basis of V consisting of vectors $\{v_1, \cdots, v_p\}$ with the property that

$$V_j = e_{i_j} + \sum_{k \notin \{i_1,\cdots,i_p\}} x_{j_k} e_k \tag{5.23}$$

where $\{e_1, \cdots, e_{m+p}\}$ is the standard basis of $\mathbb{R}^{m+p}$. Now let us define a map

$$\phi^{(i_1,\cdots,i_p)}: U^{(i_1,\cdots,i_p)} \to \mathbb{R}^{mp} \tag{5.24}$$

by

$$V \mapsto (x_{jk};\ 1 \leqq j \leqq p;\ k \notin \{i_1, \cdots, i_p\}. \tag{5.25}$$

We can easily see that $(u^{(i_1,\cdots,i_p)}, \phi^{(i_1,\cdots,i_p)})$ is a coordinate chart of Grass $(p, m+p)$. Such charts will be referred to as standard charts.

Let us consider a standard chart $(U^{(i_1,\cdots,i_p)}, \phi^{(i_1,\cdots,i_p)})$. Let $(A, B) \in W$ be such that $A \in U^{(i_1,\cdots,i_p)}$. With respect to this chart we can identify B with a unique element of $\mathbb{R}^{mp}$ in the following way.

Let $(v_1, \cdots, v_p)$ be the unique basis associated with A as described above. Let $j_1, \cdots, j_m$ be integers such that $1 \leqq j_1 < j_2 < \cdots < j_m \leqq m+p$ and $\{i_1, \cdots, i_p, j_1, \cdots, j_m\} = \{1, 2, \cdots, m+p\}$. Now begin with the basis $\{v_1, v_2, \cdots, v_p, e_{j_1}, \cdots, e_{j_m}\}$ and use the Gram-Schmidt procedure to define an orthonormal basis $\{u_1, \cdots, u_{m+p}\}$. Now B can be written uniquely as span $\{z_1, \cdots, z_m\}$, where

$$z_i = u_{p+i} + \sum_{k=1}^{p} y_{i_k} u_k. \tag{5.26}$$

Now identify B with the matrix $[y_{ik}]$. Let

$$\phi : \rho^{-1}(u^{(i_1,\cdots,i_p)}) \to u^{(i_1,\cdots,i_p)} \times \mathbb{R}^{mp} \tag{5.27}$$

be the map

$$(A, B) \mapsto (A, [y_{ik}]) \tag{5.28}$$

where $[y_{ik}]$ is constructed as above. It is easily seen that (5.27) is a local trivialization of the map $\rho: W \to BW$ defined in (5.19), (5.20).

Finally, if A belongs to two standard coordinate neighborhoods, $U^{(i_1,\cdots,i_p)}$ and $U^{(\hat{i}_1,\cdots,\hat{i}_p)}$, and if ϕ and $\hat{\phi}$ are the local trivializations constructed as in (5.27), then the map

$$\phi \cdot \hat{\phi}^{-1}|_{A \times \mathbb{R}^{m+p}}: \mathbb{R}^{mp} \to \mathbb{R}^{mp} \tag{5.29}$$

is easily seen to be a linear map.

Thus $\rho: W \to BW$ has the structure of a vector bundle. □

We now prove a theorem which may be considered as a generalization of Lemma 5.2.

THEOREM 5.6. *The map ϕ defined in* (5.4) *has the structure of a vector bundle.*

Proof. Refer to the following diagram:

$$\begin{array}{ccc} LFB^{n,q}_{m,p} & & i^* LFB^{n,q}_{m,p} \\ \phi \big\downarrow & & \big\downarrow i^*\phi \\ W & \underset{\rho}{\overset{i}{\rightleftarrows}} & BW \end{array} \tag{5.30}$$

By Lemma 5.3 we know that $\phi : LFB^{n,q}_{m,p} \to W$ is a fiber bundle with fibers diffeomorphic to $\mathbb{R}^{(m+p)(np+qm)}$. By Lemma 5.5 we know that $\rho: W \to BW$ is a homotopy equivalence.

Finally by Lemma 5.4 we know that $i^*\phi : i^*LFB^{n,q}_{m,p} \to BW$ is a vector bundle. It therefore follows from the homotopy property of vector bundles, that $\phi : LFB^{n,q}_{m,p} \to W$ is a vector bundle (see [23, p. 57]). □

Finally we sketch the proof of Main Theorem 5.1.

Proof of Main Theorem 5.1. From the Lemma 5.5 and Theorem 5.6 it follows that $LFB^{n,q}_{m,p}$ has the structure of a vector bundle over W and W has the structure of a vector bundle over BW. It therefore follows from [24] that $LFB^{n,q}_{m,p}$ is a vector bundle over BW. Finally, since BW can be identified with Grass $(p, m+p)$, it follows that

$$\rho \cdot \phi : LFB^{n,q}_{m,p} \to \text{Grass}\,(p, m+p) \tag{5.31}$$

has the structure of a vector bundle. □

The following corollary is immediate from the proof of the Main Theorem 5.1.

COROLLARY 5.7. *The vector bundle* $LFB^{n,q}_{m,p}$ *over* BW *is a Whitney sum of the two vector bundles* (5.18) *and* (5.19).

The proof of Corollary 5.7 follows immediately from [24] and is omitted.

In general, the structure of the vector bundle (5.31) is not known. The following theorem summarizes a partial result in this direction.

THEOREM 5.8. *Assume that* $\min(m, p) = 1$. *The vector bundle* (5.31) *is trivial if and only if* $\max(m, p) = 1, 3,$ *or* 7.

Before we sketch the proof of Theorem 5.8 we state and prove the following lemma.

LEMMA 5.9. *Assume that* $\min(m, p) = 1$. *The vector bundle* $\rho : W \to BW$ *is isomorphic to the tangent bundle of the projective space* $\mathbb{RP}^{\max(m,p)}$.

Proof. Assume that $p = 1$ without loss of generality. The vector bundle $\rho : W \to BW$ is therefore defined on the base space $\mathbb{RP}^m$. Let τ be the tangent bundle over $\mathbb{RP}^m$. We now consider the following diagram:

$$\begin{array}{ccc} D\mathbb{RP}^m & & W \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \rho} \\ \mathbb{RP}^m & \xleftarrow[id]{\quad} & \mathbb{RP}^m \end{array}.$$

It is well known (see [7, p. 44]) that the tangent manifold $D\mathbb{RP}^m$ can be identified with the set of all pairs

$$\{(x, v), (-x, -v) : x, v \in \mathbb{R}^{m+1}, x.x = 1, x.v = 0\}. \tag{5.32}$$

Similarly the space W can be identified with the set of all pairs

$$\{(x, v), (-x, -v) : x, v \in \mathbb{R}^{m+1}, x.x = 1, x.v > 0\}. \tag{5.33}$$

The spaces W or $D\mathbb{RP}^m$ are endowed with the vector space structure in the following way. Let $[x, v]$ denote the pair $(x, v), (-x, -v)$; then

$$\alpha[x, v_1] + \beta[x, v_2] = [x, \alpha v_1 + \beta v_2]. \tag{5.34}$$

We therefore consider an isomorphism between the spaces $D\mathbb{RP}^m$ and W via

$$(x, v) \mapsto (x, x+v). \tag{5.35}$$

This completes the proof. □

We shall now prove Theorem 5.8.

Proof of Theorem 5.8. Consider the vector bundle (5.18). In Lemma 5.4 we have shown that this vector bundle is isomorphic to $i^*\Sigma$, where Σ is described in (5.13) and the inclusion map i is given by (5.16). Let us now include W in Grass $(p, m+p)\times$ Grass $(m, m+p)$ via the inclusion map α. We now have the following diagram:

$$\begin{array}{ccccc} i^*LFB^{n,q}_{m,p} & & LFB^{n,q}_{m,p} & & L^n_{m,p}\times L^q_{p,m} \\ \downarrow{\scriptstyle i^*\phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi^n_{m,p}\times\psi^q_{p,m}} \\ BW & \xrightarrow[i]{} & W & \xrightarrow[\alpha]{} & \text{Grass}\,(p, m+p)\times\text{Grass}\,(m, m+p) \end{array}$$

where the vector bundle

(5.36) $$i^*LFB^{n,q}_{m,p}\to BW$$

is a pullback of the vector bundle

(5.37) $$L^n_{m,p}\times L^q_{p,m}\to \text{Grass}\,(p, m+p)\times\text{Grass}\,(m, m+p).$$

From Corollaries 4.14–4.16 we know that the vector bundle (5.36) is trivial if $\min(m, p)=1$, 3, or 7. It therefore follows that the vector bundle (5.18) is trivial if $\min(m, p)=1$, 3, or 7. Finally from Theorem 5.8 it follows that the vector bundle (5.19) is trivial if and only if $\min(m, p)=1$, 3, or 7. Thus in view of Corollary 5.7 we have the proof. □

6. A fiber bundle of the space of proper systems with unbounded lag: quotient topology. The various spaces that we have considered so far in this paper assume that the ARMA systems are of bounded lag. This assumption enabled us to parametrize spaces that are finite-dimensional vector bundles. Frequently, however, it is unreasonable to assume under the presence of high frequency parasitics, that the family of systems under consideration is of bounded lag. In this section we now parametrize the space $L^\infty_{m,p}$ of all $p\times m$ ARMA systems of arbitrary large lag.

Let $\mathbb{R}^\infty$ denote the vector space consisting of those infinite sequences

(6.1) $$x=(x_1, x_2, x_3, \cdots)$$

of real numbers for which all but a finite number of x_i are zero. For fixed k, the subspace consisting of all

(6.2) $$x=(x_1, x_2, \cdots, x_k, 0, 0, 0, \cdots)$$

will be identified with the coordinate space $\mathbb{R}^k$. Thus $\mathbb{R}^1\subset\mathbb{R}^2\subset\cdots$ with union $\mathbb{R}^\infty$.

DEFINITION 6.1. The infinite Grassmann manifold

(6.3) $$G_n=\text{Grass}\,(n, \infty)$$

is the set of all n-dimensional linear subspaces of $\mathbb{R}^\infty$, topologized as the direct limit of the sequence

(6.4) $$\text{Grass}\,(n, n)\subset\text{Grass}\,(n, n+1)\subset\cdots.$$

In other words, a subset of G_n is open if and only if its intersection with Grass $(n, n+k)$ is open as a subset of Grass $(n, n+k)$ for each $k\geqq n$. The following assertion is rather trivial to check and its proof is omitted.

PROPOSITION 6.2. *If* Grass $(p, (m+p)(n+1))$ *is included in* Grass $(p, (m+p)\times(n+2))$ *as follows*:

(6.5)
$$[D_0, D_1, \cdots, D_n, N_0, N_1, \cdots, N_n] \mapsto [D_0, D_1, \cdots, D_n, 0, N_0, N_1, \cdots, N_n, 0],$$

we have

$$\begin{array}{ccccc} \text{Grass}\,(p, (m+p)(n+1)) & \subset & \text{Grass}\,(p, (m+p)(n+2)) & \subset \cdots \subset & \text{Grass}\,(p, \infty) \\ \cup & & \cup & & \cup \\ L^n_{m,p} & \subset & L^{n+1}_{m,p} & \subset \cdots \subset & L^\infty_{m,p} \end{array} \tag{6.6}$$

where $L^\infty_{m,p}$ *is defined to be the direct limit of the sequence*

$$L^n_{m,p} \subset L^{n+1}_{m,p} \subset \cdots. \tag{6.7}$$

One of the results that we would now like to state and prove in this section is about the structure of $L^\infty_{m,p}$.

THEOREM 6.3. $L^\infty_{m,p}$ *can be endowed with the structure of a fiber bundle over the base* Grass $(p, m+p)$ *and with fibers homeomorphic to* $\mathbb{R}^\infty$.

Proof. We know from the Theorem 4.6 that $L^n_{m,p}$ can be endowed with the structure of a vector bundle over Grass $(p, m+p)$ with fibers isomorphic to $\mathbb{R}^{np[m+p]}$. Consider a trivializing chart U of Grass $(p, m+p)$. Let ψ_n be the bundle map

$$\psi_n : L^n_{m,p} \to \text{Grass}\,(p, m+p). \tag{6.8}$$

It follows that $\psi_n^{-1}(U)$ is isomorphic to $U \times \mathbb{R}^{np[m+p]}$. It is easy to check the validity of the following diagram:

$$\begin{array}{ccccc} L^n_{m,p} & \subset & L^{n+1}_{m,p} & \subset \cdots \subset & L^\infty_{m,p} \\ \cup & & \cup & & \cup \\ \psi_n^{-1}(U) & \subset & \psi_{n+1}^{-1}(U) & \subset \cdots \subset & \psi_\infty^{-1}(U) \end{array} \tag{6.9}$$

where $\psi_n^{-1}(U)$ has been included in $\psi_{n+1}^{-1}(U)$ by including $\mathbb{R}^{np(m+p)}$ within $\mathbb{R}^{(n+1)=(m+p)}$ via the construction of $\mathbb{R}^\infty$. It follows that the space $\psi_\infty^{-1}(U)$ obtained by considering the direct limit of $\psi_\infty^{-1}(U) \subset \psi_{n+1}^{-1}(U) \subset \cdots$ is homeomorphic to $U \times \mathbb{R}^\infty$.

By considering various trivializing charts U_α which cover the base Grass $(p, m+p)$, it follows that $L^\infty_{m,p}$ can be endowed with the structure of a fiber bundle over Grass $(p, m+p)$. $\square$

We now define an equivalence relation $\sim$ on $L^\infty_{m,p}$ as follows: Let $\omega_1, \omega_2 \in L^\infty_{m,p}$. Then $\omega_1 \sim \omega_2$ if and only if both ω_1 and ω_2 correspond to equivalent ARMA systems in the sense of Definition 4.1. Consider the quotient space $L^\infty_{m,p}/\sim \triangleq \tilde{L}^\infty_{m,p}$ together with the natural map

$$\psi : L^\infty_{m,p} \to \tilde{L}^\infty_{m,p} \tag{6.10}$$

which sends a point in $L^\infty_{m,p}$ to its equivalence class. We now endow $\tilde{L}^\infty_{m,p}$ with the quotient topology.

The space $\tilde{L}^\infty_{m,p}$ so constructed is a hybrid topology on the space of all ARMA systems. We now compare the space $\tilde{L}^\infty_{m,p}$ with the well-known graph topology [8] on the space of all systems. Let $P_{m,p}$ be the space of all multi-input multi-output ARMA systems endowed with the graph topology. We now consider the map

$$\phi_\infty : L^\infty_{m,p} \to P_{m,p} \tag{6.11}$$

which sends a point in $L^{\infty}_{m,p}$ to the corresponding ARMA system in $P_{m,p}$. Finally we consider the map ψ_{∞} which makes the diagram

$$\begin{array}{ccc} L^{\infty}_{m,p} & \xrightarrow{\phi_{\infty}} & P_{m,p} \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\psi_{\infty}} & \\ & \tilde{L}^{\infty}_{m,p} & \end{array} \tag{6.12}$$

commute. We now state and prove the following interesting results.

THEOREM 6.4. *ϕ_{∞} is a continuous function.*

Proof. Clearly it is enough to show that

$$\phi_{\infty}: L^{n}_{m,p} \to P_{m,p} \tag{6.13}$$

is continuous for all n. Let $\tilde{p}$ be a point in $P_{m,p}$ with the coprime factorization given by $\tilde{p}_1^{-1}\tilde{p}_2$, where $\tilde{p}_1, \tilde{p}_2$ are matrices with elements in H, the ring of stable and proper rational functions. Define an open neighborhood $N_{\varepsilon_1,\varepsilon_2}(\tilde{p}_1, \tilde{p}_2)$ of $\tilde{p}$ in $P_{m,p}$ as follows:

$$N_{\varepsilon_1,\varepsilon_2}(\tilde{p}_1, \tilde{p}_2) = \{p \in P_{m,p}: p = p_1^{-1}p_2,\ p_1 \text{ and } p_2 \text{ are coprime}, \quad \|p_1 - \tilde{p}_1\|_{\infty} < \varepsilon_1,\ \|p_2 - \tilde{p}_2\|_{\infty} < \varepsilon_2\}. \tag{6.14}$$

The norm in (6.14) denotes the H^{∞} norm.

Let p_0 be a point in $L^{n}_{m,p}$ such that

$$\phi_{\infty}(p_0) = \tilde{p}. \tag{6.15}$$

We would now show that there exists an open neighborhood $N(p_0)$ of p_0 in $L^{n}_{m,p}$ such that

$$\phi_{\infty}(N(p_0)) \subset N_{\varepsilon_1,\varepsilon_2}(\tilde{p}_1, \tilde{p}_2). \tag{6.16}$$

Let

$$p_0 = [\bar{D}_0\bar{D}_1 \cdots \bar{D}_n\bar{N}_0 \cdots \bar{N}_n] \tag{6.17}$$

so that $\tilde{p}$ may be written as

$$[\bar{D}_0 z^n + \bar{D}_1 z^{n-1} + \cdots + \bar{D}_n]^{-1}[\bar{N}_0 z^n + \bar{N}_1 z^{n-1} + \cdots + \bar{N}_n]. \tag{6.18}$$

The coprime representation $\tilde{p}_2^{-1}\tilde{p}_1$ of $\tilde{p}$ may be written as

$$\tilde{p}_1 = \Delta(z)[(\delta_0 z^n + \cdots + \delta_n)I]^{-1}[\bar{D}_0 z^n + \cdots + \bar{D}_n], \tag{6.19}$$

$$\tilde{p}_2 = \Delta(z)[(\delta_0 z^n + \cdots + \delta_n)I]^{-1}[\bar{N}_0 z^n + \cdots + \bar{N}_n] \tag{6.20}$$

where $\Delta(z) \in H^{p\times p}$ $\det \Delta(z) \in J$ and $\delta_0 z^n + \cdots + \delta_n$ is a Hurwitz polynomial of degree n.

Since $p_0 \in \text{Grass}(p, (n+1)(m+p))$, without loss of generality assume that $\bar{D}_0$ is nonsingular. In particular assume that $\bar{D}_0 = I$. For μ sufficiently small, we define $N_{\mu}(p_0)$ of p_0 in $L^{n}_{m,p}$ as follows:

$$N_{\mu}(p_0) = \{I, D_1, \cdots, D_n, N_0, \cdots, N_n]: \sum (\bar{d}_{ijk} - d_{ijk})^2 + \sum (\bar{n}_{ijk} - n_{ijk})^2 < \mu\} \tag{6.21}$$

where $\bar{d}_{ijk}$ is the ijth entry of $\bar{D}_k$ and d_{ijk} is the ijth entry of D_k. We define $\bar{n}_{ijk}$ and n_{ijk} similarly.

For any $p \in N_{\mu}(p_0)$, a coprime factorization of $\phi_{\infty}(p_0)$ is given by $p_2^{-1}p_1$, where

$$p_1 = \Delta(z)[(\delta_0 z^n + \cdots + \delta_n)I]^{-1}[Iz^n + D_1 z^{n-1} + \cdots + D_n], \tag{6.22}$$

$$p_2 = \Delta(z)[(\delta_0 z^n + \cdots + \delta_n)I]^{-1}[N_0 z^n + N_1 z^{n-1} + \cdots + N_n]. \tag{6.23}$$

We now compute

$$\|p_1-\tilde{p}_1\|_\infty \leqq \|\Delta(z)\|_\infty \|[(\delta_0 z^n+\cdots+\delta_n)I]^{-1}[(D_1-\bar{D}_1)z^{n-1}+\cdots+(D_n-\bar{D}_n)]\|_\infty \tag{6.24}$$

and

$$\|p_2-\tilde{p}_2\|_\infty \leqq \|\Delta(z)\|_\infty \|[(\delta_0 z^n+\cdots+\delta_n)I]^{-1}[(N_0-\bar{N}_0)z^n+\cdots+(N_n-\bar{N}_n)]\|_\infty. \tag{6.25}$$

For μ sufficiently small, it is clear that

$$\|p_1-\tilde{p}_1\|_\infty<\varepsilon_1 \quad \text{and} \quad \|p_2-\tilde{p}_2\|_\infty<\varepsilon_2 \tag{6.26}$$

implying that

$$\phi_\infty(N_\mu(p_0))\subset N_{\varepsilon_1,\varepsilon_2}(\tilde{p}_1,\tilde{p}_2). \tag{6.27}$$

□

Our next result is the following.

THEOREM 6.5. *Let $\tilde{\omega}_i$, $i=1,2,\cdots$ be a sequence of points in $\tilde{L}^\infty_{m,p}$. Then the following two conditions are equivalent*:

(1) *The sequence $\{\tilde{\omega}_i, i=1,2,\cdots$ be a sequence of points in $\tilde{L}^\infty_{m,p}$. Then*

(2) (a) *The sequence $\{\psi_\infty(\tilde{\omega}_i), i=1,2,\cdots\}$ converges in $P_{m,p}$.*

(b) *There exists n sufficiently large such that the lag of $\psi_\infty(\tilde{\omega}_i)\leqq n$ for all $i=1,2,\cdots$.*

Before we prove Theorem 6.5, we state the following interesting corollary.

COROLLARY 6.6. *ψ_∞^{-1} is not a continuous function.*

The proof of Corollary 6.6 is trivial and relies on the existence of a sequence δ_i, $i=1,2,\cdots$ of plants which converges in $P_{m,p}$ and is such that lag $(\delta_i)<$ lag (δ_{i+1}) for $i=1,2,\cdots$.

Remark 6.7. The graph topology and the hybrid topology introduced in this paper are not identical. However, if we define $P^n_{m,p}$ to be the set of all $p\times m$ systems in $P_{m,p}$ of lag$\leqq n$, then it follows from Theorem 6.5 that $\tilde{L}^n_{m,p}$ is homeomorphic to $P^n_{m,p}$, where $\tilde{L}^n_{m,p}=\psi(L^n_{m,p})$. In general a set S is open in $\tilde{L}^\infty_{m,p}$ if $S\cap\tilde{L}^n_{m,p}$ is open in $\tilde{L}^n_{m,p}$ for all n. This property is unfortunately not true for the graph topology.

Proof of Theorem 6.5. $(2\Rightarrow 1)$ Let $\tilde{p}_1,\tilde{p}_2,\cdots$ be a sequence of plants in $P_{m,p}$ which converges to $\tilde{p}_0$ in $P_{m,p}$. Let n be such that deg $\tilde{p}_i\leqq n$ for $i=1,2,\cdots$. It follows that deg $(\tilde{p}_0)\leqq n$; otherwise there exists an open neighborhood U of $\tilde{p}_0$ in $P_{m,p}$ which does not contain any $\tilde{p}_i$. Let $\tilde{\omega}_i$, $i=1,2,\cdots$ be the sequence in $\tilde{L}^\infty_{m,p}$ such that $\psi_\infty(\tilde{\omega}_i)=\tilde{p}_i$, $i=1,2,\cdots$ and let $\tilde{\omega}_0$ be the point in $\tilde{L}^\infty_{m,p}$ such that $\psi_\infty(\tilde{\omega}_0)=\tilde{p}_0$. Let us write $\tilde{\omega}_i$ to be the equivalence class of

$$[D_{i0},D_{i1},\cdots,D_{in},N_{i0},N_{i1},\cdots,N_{in}] \tag{6.28}$$

so that $\tilde{p}_i$ may be written as

$$[D_{i0}z^n+\cdots+D_{in}]^{-1}[N_{i0}Z^n+\cdots+N_{in}]. \tag{6.29}$$

A coprime factorization of $\tilde{p}_i$ may be written as $\tilde{p}_{i1}/\tilde{p}_{i2}$, where

$$\tilde{p}_{i1}=[D_{i0}z^n+\cdots+D_{in}]/z^n, \tag{6.30}$$

$$\tilde{p}_{i2}=[N_{i0}z^n+\cdots+N_{in}]/z^n. \tag{6.31}$$

By assumption, there exists a coprime factorization $\tilde{p}_{01}/\tilde{p}_{02}$ of $\tilde{p}_0$ in $P_{m,p}$, where

$$\tilde{p}_{01}=\Delta(z)[D_{00}z^n+\cdots+D_{0n}]/z^n, \tag{6.32}$$

$$\tilde{p}_{02}=\Delta(z)[N_{00}z^n+\cdots+N_{0n}]/z^n \tag{6.33}$$

such that

$$\mathrm{Lt}_{i\to\infty} \|\tilde{p}_{i1} - \tilde{p}_{01}\| = 0, \tag{6.34}$$

$$\mathrm{Lt}_{i\to\infty} \|\tilde{p}_{i2} - \tilde{p}_{02}\| = 0 \tag{6.35}$$

where we assume that $\Delta(z) \in H^{p\times p}$, $\det \Delta(z) \in J$. Writing $\Delta(z) = \Delta_2(z)^{-1}\Delta_1(z)$, where Δ_1, Δ_2 are matrices with polynomial entries such that $\det \Delta_1(z)$ and $\det \Delta_2(z)$ are strictly Hurwitz polynomials. We may therefore conclude that in the space of polynomials of degree $n+d$ (for some d) with matrix coefficients, the polynomials

$$\Delta_2(z)[N_{i0}z^n + \cdots + N_{in}] \tag{6.36}$$

converge to the polynomial

$$\Delta_1(z)[N_{00}z^n + \cdots + N_{0n}] \tag{6.37}$$

and the polynomials

$$\Delta_2(z)[D_{i0}z^n + \cdots + D_{in}] \tag{6.38}$$

converge to the polynomial

$$\Delta_1(z)[N_{00}z^n + \cdots + N_{0n}] \tag{6.39}$$

as $i \to \infty$. It therefore follows that in $\tilde{L}^{\infty}_{m,p}$, the equivalence class of points which correspond to the function

$$[D_{i0}z^n + \cdots + D_{in}]^{-1}[N_{i0}z^n + \cdots + N_{in}] \tag{6.40}$$

converges to the function

$$[D_{00}z^n + \cdots + D_{0n}]^{-1}[N_{00}z^n + \cdots + N_{0n}]. \tag{6.41}$$

Thus $\tilde{\omega}_i$ converges to $\tilde{\omega}_0$ in $\tilde{L}^{\infty}_{m,p}$ as $i \to \infty$.

$(1 \Rightarrow 2)$ Since ϕ_∞ is continuous and $\tilde{L}^{\infty}_{m,p}$ has quotient topology, it follows that ψ_∞ is continuous. Thus $1 \Rightarrow 2$(a). Finally, in order to show that $1 \Rightarrow 2$(b), let p_i, $i = 1, 2, \cdots$ be a sequence of plants in $P_{m,p}$ with the property that $\operatorname{lag} p_{i+1} > \operatorname{lag} p_i$ and $\operatorname{lag} p_1 = 1$, $i = 1, 2, 3, \cdots$. It now follows from the proof of Theorem 3.4 in [4] that $\psi_\infty^{-1}(p_i)$, $i = 1, 2, \cdots$ does not converge in $\tilde{L}^{\infty}_{m,p}$. This is because if there exists $\tilde{\omega}_0$ in $\tilde{L}^{\infty}_{m,p}$ such that $\{\psi_\infty^{-1}(p_i)\}$ converges to $\tilde{\omega}_0$, define $\psi_\infty(\tilde{\omega}_0) = p_0$. Let U_1 be an open neighborhood of p_0 in $P_{m,p}$ which does not contain p_1. Define

$$S_1 \triangleq \psi_\infty^{-1}(U_1) \cap \tilde{L}^{1}_{m,p} \tag{6.42}$$

and

$$V_1 \triangleq \psi_\infty(S_1). \tag{6.43}$$

Note that V_1 does not contain $\{p_i\}$. Assume that U_i, S_i, V_i are defined where V_i does not contain the above sequence $\{p_i, i = 1, 2, \cdots\}$ in $P_{m,p}$ and where

$$S_i = \psi_\infty^{-1}(U_i) \cap \tilde{L}^{i}_{m,p}. \tag{6.44}$$

Define U_{i+1} to be an open neighborhood of V_i which does not contain $p_1, \cdots, p_{i+1}$. Define

$$S_{i+1} \triangleq \psi_\infty^{-1}(U_{i+1}) \cap \tilde{L}^{i+1}_{m,p} \tag{6.45}$$

and

$$V_{i+1} \triangleq \psi_\infty(S_{i+1}). \tag{6.46}$$

It is clear that

$$S_i \subset S_{i+1} \quad \forall i = 1, 2, \cdots. \tag{6.47}$$

Define

$$S_\infty = \bigcup_{i=1}^{\infty} S_i. \tag{6.48}$$

It may be concluded that S_∞ is an open neighborhood of $\tilde{\omega}_0$ which does not contain the sequence $\{\psi_\infty^{-1}(p_i)\}$. □

7. Some further remarks on the parametrization of feedback control systems. Continuing our earlier discussion in § 5, in this section we once again address the problem of parametrizing feedback control systems. However we do not assume that the closed loop system is stable. In particular we consider the space of $p \times m$ ARMA systems $G(z)$ of lag $\leqq n$ and $m \times p$ ARMA systems $K(z)$ of lag $\leqq q$ and parametrize in the product space $L^n_{m,p} \times L^q_{p,m}$ those pairs of plant/compensator that corresponds to a closed loop system in $L^{n+q}_{m,p}$.

Stated more precisely, let us consider the product space

$$L^n_{m,p} \times L^q_{p,m} \tag{7.1}$$

of plant/compensator pairs $G(z)$, $K(z)$, where $G(z) \in L^n_{m,p}$, $K(z) \in [^q_{p,m}$. Suppose that $G(z)$ defines the input-output system

$$D_p(z)y = N_p(z)u \tag{7.2}$$

and $K(z)$ defines the input-output system

$$\tilde{D}_c(z) = \tilde{N}_c(z)u \tag{7.3}$$

where

$$\tilde{D}_c^{-1}\tilde{N}_c = N_c D_c^{-1}. \tag{7.4}$$

Here we assume that the pairs (D_p, N_p), (D_c, N_c) and $(\tilde{D}_c, \tilde{N}_c)$ are coprime. We may define the space $LF^{n,q}_{m,p}$ as follows:

$$LF^{n,q}_{m,p} \triangleq \{[D_p, N_p], [\tilde{D}_c, \tilde{N}_c] \in L^n_{m,p} \times L^q_{p,m}: [D_pD_c + N_pN_c, N_pN_c] \in L^{n+q}_{m,p}\}. \tag{7.5}$$

Note that multiplying $[D_p, N_p]$ by a nonsingular matrix to the left and $[\tilde{D}_c, \tilde{N}_c]$ by a nonsingular matrix to the right does not change the condition in (7.5). Thus (7.5) is well defined.

The space $LF^{n,q}_{m,p}$ parametrizes plant/compensator pairs that define an ARMA system in the closed loop, and is clearly of interest in control system design. Furthermore, in off-line identification of parameters in $L^n_{m,p} \times L^q_{p,m}$, the closed-loop stability of the intermediate parameter values of the plant/compensator pair is not required (as opposed to an onpline identification problem and adaptive control). Thus whereas the space $LFB^{n,q}_{m,p}$ considered in § 5 is of importance in on-line recursive algorithms, the space $LF^{n,q}_{m,p}$ appears to be of importance in off-line recursive identification problems.

Remark 7.1. As an example of a plant-compensator pair that does not belong to $LF^{n,q}_{m,p}$ consider the pair

$$\frac{1}{z+1}, \frac{2z+3}{5}. \tag{7.6}$$

Define

(7.7) $$D_p = \frac{z+1}{z}, \quad N_p = \frac{1}{z}, \quad \tilde{D}_c = \frac{5}{z}, \quad \tilde{N}_c = \frac{2z+3}{z}$$

and note that

(7.8) $$D_pD_c + N_pN_c = (7z+8)/(z^2),$$

(7.9) $$N_pN_c = (2z+3)/(z^2)$$

which vanishes at $z = \infty$.

The main result that we show in this section is described below.

Main Theorem 7.2. *$LF^{n,q}_{m,p}$ is a fiber bundle over $L^{0,0}_{m,p}$.*

Before we proceed to prove the Main Theorem it may be worthwhile to study the structure of $L^{0,0}_{1,1}$.

Example 7.3. In this example we assume that $m = p = 1$, $n = q = 0$, and describe $L^{0,0}_{1,1}$. Clearly, $L^{0,0}_{1,1}$ is a subset of $L^0_{1,1} \times L^0_{1,1}$ described by

(7.10) $$\{[a, b], [c, d]: [ac + bd, bd] \in \mathbb{RP}^1\}.$$

Equivalently, $L^{0,0}_{1,1}$ is described by

(7.11) $$\mathbb{RP}^1 \times \mathbb{RP}^1 - \{([0, 1], [1, 0]), ([1, 0], [0, 1])\}.$$

Thus, $L^{0,0}_{1,1}$ is homeomorphic to a torus with two distinct points removed.

Remark 7.4. In view of Example 7.3, it appears that the structure of $LF^{0,0}_{m,p}$ is more complicated in comparison with the structure of $LFB^{0,0}_{m,p}$. Recall from § 5 that $LFN^{0,0}_{m,p}$ is unknown.

Proof of Main Theorem 7.3. The proof of this theorem is analogous to the technique of transverse and complete flow argument used in the proof of Theorem 4.6. Therefore we sketch only the main points here. First note that $L^{0,0}_{m,p}$ is a manifold since it is an open subset of Grass $(m, m+p) \times$ Grass $(p, m+p)$. Consider the map

(7.12) $$\psi: LF^{n,q}_{m,p} \to LF^{0,0}_{m,p}$$

given by

(7.13) $$\psi([D_p(z), N_p(z)], [\tilde{D}_c(z), \tilde{N}_c(z)]) = ([D_p(\infty), N_p(\infty), [\tilde{D}_c(\infty), \tilde{N}_c(\infty)]).$$

Consider the local flow

(7.14) $$Ⓗ^{n,q}_{m,p}: LF^{n,q}_{m,p} \times [0, \infty) \to LF^{n,q}_{m,p}$$

defined by

(7.15) $$\begin{aligned} Ⓗ^{n,q}_{m,p}(([D_p(z), N_p(z)], [\tilde{D}_c(z), \tilde{N}_c(z)]), t) \\ = ([D_p(e^tz), N_p(e^tz)], [\tilde{D}_c(e^tz), \tilde{N}_c(e^tz)]). \end{aligned}$$

It is easily seen that the above flow is well defined and smooth. Moreover if $Z \in LF^{0,0}_{m,p}$, the fiber $\psi^{-1}(Z)$ of the submersion (7.12) is invariant under the above flow. By argument similar to the proof of Lemma 5.3, $\psi^{-1}(Z)$ is diffeomorphic to $\mathbb{R}^{(np+qm)(m+p)}$. The proof of the local triviality of $LF^{n,q}_{m,p}$ is similar to the proof sketched in Lemma 5.3 and Theorem 4.6. □

8. Conclusion. To conclude, in this paper we have studied the problem of parametrizing linear dynamical systems, both in the open loop and in the closed loop. Although we have restricted our attention to discrete-time ARMA systems, consideration of state-space, continuous-time systems would be analogous. The main result of this paper is that the various parametrizations obtained have a bundle structure, which we believe would be particularly useful in system identification defined by a locally and globally convergent vector field wherein the structure of the fibers, diffeomorphic to a Euclidean space, would be exploited.

Appendix I. Milnor's theorem.

THEOREM. *Let M be an n-dimensional manifold. Suppose that M admits a vector field X with a unique locally and globally attracting singularity. Then M is diffeomorphic to $\mathbb{R}^n$.*

Remark. Among the geometric control theory community this is known as Milnor's theorem. (Perhaps this name is due to Chris Byrnes.) This is essentially contained in [5] and can be proved using several methods. We sketch a proof based on [5].

The following lemma is easy (see [5], [25], or [26] for a stronger version). In what follows, D_k is the closed disk of radius k in $\mathbb{R}^n$.

LEMMA 1. *Let N be an n-dimensional oriented manifold, and let $f_1, f_2: D_1 \to N$ be orientation-preserving embeddings into the interior of N. Then there exists a diffeomorphism $h: N \to N$ such that $h \cdot f_1 = f_2$.*

LEMMA 2 [5]. *Let N be an n-dimensional manifold such that each compact subset is contained in an open set diffeomorphic to $\mathbb{R}^n$. Then N is diffeomorphic to $\mathbb{R}^n$.*

Proof. We can easily construct a sequence $W_1 \subset W_2 \subset \cdots \subset N$ such that $\bigcup_{k=1}^{\infty} W_k = N$ and W_k is diffeomorphic to D_k. Say $g_k: D_k \to W_k$ is a diffeomorphism. We are now going to define a new sequence of diffeomorphisms $f_k: D_k \to W_k$ by induction. Set $f_1 = g_1$.

Now suppose that $f_1, \cdots, f_k$ have already been defined. Orient W_{k+1} and modify g_{k+1} if necessary such that

$$D_k \xrightarrow{f_k} W_k \xhookrightarrow{i_k} W_{k+1} \quad \text{and} \quad D_k \xhookrightarrow{j_k} D_{k+1} \xrightarrow{g_{k+1}} W_{k+1}$$

are orientation-preserving, where i_k, j_k are inclusions. By Lemma 1, find a diffeomorphism $h_{k+1}: W_{k+1} \to W_{k+1}$ such that $h_k \cdot g_k \cdot j_k = i_k \cdot f_k$ and define $f_{k+1} = h_{k+1} \cdot g_{k+1}$. Then $f_{k+1}|D_k = f_k$, and hence pass to the direct limit to obtain the diffeomorphism $\lim_{\to} f_k: \mathbb{R}^n \to N$. □

Proof of the theorem. By Lemma 4.10, without loss of generality we assume that X is complete. Let U be a neighborhood of p which is diffeomorphic to $\mathbb{R}^n$. Define $U_k = \phi^X_{-k}(U)$, $k = 1, 2, \cdots$. Then each U_k is diffeomorphic to $\mathbb{R}^n$, and $M = \bigcup_{k=1}^{\infty} U_k$. Hence by Lemma 2, M is also diffeomorphic to $\mathbb{R}^n$.

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