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A hybrid parameterization of linear single input single output systems *

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Abstract: We obtain a parameterization of the set of finite dimensional linear dynamical systems of unbounded McMillan degree. In this parameterization a given system is represented in a nonunique way. However, a certain notion of order is available, thereby providing a graded parameterization. An important feature is that each graded space is diffeomorphic to a Euclidean space. In view of this fact we hope that many of the local results in identification (valid only if the parameter space is Euclidean) an actually be generalized globally. Moreover we show that the topology induced on the space of all plants by this space is finer than the graph topology.

Keywords: Graded parameterization, Graph topology.

1. Introduction

In 1976, Brockett [1] introduced a geometric theory of linear systems and showed that the space Rat n of strictly proper transfer functions of a given McMillan degree n is a disjoint union of $n+1$ connected components. Frequently, in system identification and control, it is of interest to study a family of plants for which the McMillan degree is not fixed. In particular in system identification, as the parameters of the plant vary or in adaptive control, as the parameters of the controller vary, it is possible that the degrees of the associated transfer functions may degenerate to a lower value. Therefore, in these situations, the Rat n geometry is not directly applicable and, roughly speaking, one has to glue the various components of Rat n by including plants of order less than n.

To illustrate the idea we assume $n = 2$ and consider the transfer function

$$[p_1 s + p_2]/[s^2 + q_1 s + q_2]. \tag{1.1}$$

Of course (1.1) may be parameterized in $\mathbb{R}^4$ via the co-ordinates (p_1, q_1, p_2, q_2) and following [1], Rat 2 is defined as a subset of $\mathbb{R}^4$ given by

$$\text{Rat } 2 \triangleq \{(p_1, q_1, p_2, q_2): \det S_2 \neq 0\} \tag{1.2}$$

where

$$S_2 = \begin{bmatrix} 0 & p_1 & p_2 & 0 \\ 1 & q_1 & q_2 & 0 \\ 0 & 0 & p_1 & p_2 \\ 0 & 1 & q_1 & q_2 \end{bmatrix}. \tag{1.3}$$

Note that $\det S_2$ is the resultant of the numerator and the denominator polynomials of (1.1) (see Kailath [8]). It follows that $\det S_2 \neq 0$ is a necessary and sufficient condition that the rational function (1.1) is of McMillan degree 2.

We now pose the following problem:

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Problem 1.1. Parameterize the set Ω_2 of points p in $\mathbb{R}^4$ with the property that there exists a neighborhood $N(p)$ of p in $\mathbb{R}^4$ such that $N(p) \cap \text{Rat } 2$ is simultaneously stabilizable by a dynamic compensator.

Clearly, the space Ω_2 is of interest in the problem of parameterization of robustly stabilizable plants. Elementary arguments show that the space Ω_2 corresponds to those points in $\mathbb{R}^4$ for which in (1.1) there is no unstable pole–zero cancellation. Throughout in this paper we assume that the region of stability is the interior of the unit disc. The space Ω_2 is described as follows:

$$\Omega_2 = \{(p_1, q_1, p_2, q_2): (\det S_2 \neq 0) \text{ or } (|p_1| > |p_2|) \text{ or } (|q_2| < 1 \text{ and } |q_1| < q_2 + 1)\}. \tag{1.4}$$

It is easily seen that Ω_2 is a connected, open, dense and semialgebraic subset of $\mathbb{R}^4$.

2. Robust parameterization of the space of systems of bounded degree

Following the reasoning developed in the introduction, we define the polynomials

$$p(s) = p_1 s^{n-1} + \cdots + p_n, \tag{2.1}$$

$$q(s) = s^n + q_1 s^{n-1} + \cdots + q_n, \tag{2.2}$$

and consider rational functions of the type $p(s)/q(s)$. These rational functions can be parameterized as points in $\mathbb{R}^{2n}$ via the co-efficients $(p_1, q_1, \ldots, p_n, q_n)$.

Definition 2.1. Let Ω_n be the subset of $\mathbb{R}^{2n}$ described as follows:

$$\Omega_n \triangleq \{(p_1, q_1, \ldots, p_n, q_n): p(s) \text{ and } q(s) \text{ do not have any common roots in the closed exterior of the unit disc}\}. \tag{2.3}$$

Clearly Ω_n contains Rat n and elementary arguments would suggest:

Proposition 2.2. *Ω_n is a semialgebraic, open, connected and dense subset of $\mathbb{R}^{2n}$.*

The proof of Proposition 2.2 is trivial and is omitted. We remark however that the semialgebraic description of Ω_n in terms of the standard coördinates of $\mathbb{R}^{2n}$ is explicit and can be obtained via the Euclidean long division. Let us now prove the following theorem which is quite surprising.

Theorem 2.3. *Ω_n is diffeomorphic to $\mathbb{R}^{2n}$.*

Proof. Consider the map

$$\phi: \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}^{2n}, \tag{2.4}$$

$$\phi(p_1, q_1, \ldots, p_n, q_n, t) = (e^{-t}p_1, e^{-t}q_1, \ldots, e^{-nt}p_n, e^{-nt}q_n). \tag{2.5}$$

The map ϕ defines a flow on $\mathbb{R}^{2n}$. Let X be the corresponding vector field. Clearly X has a unique equilibrium point at the origin of $\mathbb{R}^{2n}$.

Note that the polynomial

$$s^n + e^{-t}q_1 s^{n-1} + e^{-2t}q_2 s^{n-2} + \cdots + e^{-nt}q_n \tag{2.6}$$

can be factored as

$$(s + e^{-t}a_1)(s + e^{-t}a_2) \cdots (s + e^{-t}a_n) \tag{2.7}$$

where $\{a_1, \ldots, a_n\}$ is a self conjugate set of complex numbers. On the other hand the polynomial

$$e^{-t}p_1 s^{n-1} + e^{-2t}p_2 s^{n-2} + \cdots + e^{-nt}p_n \tag{2.8}$$

can be factored as

$$e^{-mt}b_0(s+e^{-t}b_1)(s+e^{-t}b_2)\cdots(s+e^{-t}b_{n-m}), \tag{2.9}$$

where b_0 is a nonzero real number; m is a positive integer and $\{b_1, b_2, \ldots, b_{n-m}\}$ is a self conjugate set of complex numbers. Clearly for all $t \in [0, \infty]$, the polynomials (2.7) and (2.9) either do not have a common root or if they do, it remains in the closed unit disk. Hence X restricts to a globally asymptotically stable vector field on Ω_n and therefore, by Milnor's theorem [Theorem 1, p. 166, 6] it follows that Ω_n is diffeomorphic to $\mathbb{R}^{2n}$. □

Remark 2.4. It may be pointed out that various other spaces that are of interest in system identification have been shown by Byrnes [9] to be diffeomorphic to a Euclidean space.

Remark 2.5. As a consequence of Theorem 2.3, it may now be possible to generalize to Ω_n, many known parameter identification algorithms (see for example [2]) where the associated parameter space is assumed to be Euclidean.

3. A parameterization of the space of all strictly proper rational functions

The space Ω_n introduced in the last section is useful in the study of a structured family of linear dynamical systems. Frequently however, one is also interested in the study of a family of systems possibly with some unmodelled dynamics. For example in the presence of high frequency parasitics, it is unreasonable to assume that the McMillan degree of a family of systems is bounded by n. In this section, we now describe a space Ω_∞ of all strictly proper transfer functions of arbitrary large degree.

Let $\mathbb{R}^\infty$ denote the vector space consisting of those infinite sequences

$$x = (x_1, x_2, x_3, \ldots) \tag{3.1}$$

of real numbers for which all but a finite number of x_i are zero. For fixed k, the subspace consisting of all

$$x = (x_1, x_2, x_3, \ldots, x_k, 0, 0, 0, \ldots) \tag{3.2}$$

will be identified with the co-ordinate space $\mathbb{R}^k$. Thus one obtains a sequence

$$\mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \tag{3.3}$$

with union

$$\mathbb{R}^\infty \triangleq \bigcup_{i=1}^{\infty} \mathbb{R}^i. \tag{3.4}$$

The space $\mathbb{R}^\infty$ is topologized as the direct limit of the sequence (3.3) (see [3], pp. 62–63). Thus in particular, a subset S of $\mathbb{R}^\infty$ is open iff $S \cap \mathbb{R}^n$ is open in $\mathbb{R}^n$ for all $n = 1, 2, \ldots$. The following assertion is rather trivial to check and its proof is omitted.

Proposition 3.1. *If $\mathbb{R}^{2k}$ is included in $\mathbb{R}^{2k+2}$ for $k = 1, 2, 3, \ldots$ via*

$$(x_1, \ldots, x_{2k}) \mapsto (x_1, \ldots, x_{2k}, 0, 0), \tag{3.5}$$

we have

$$\begin{array}{ccccccc} \mathbb{R}^2 & \subset & \mathbb{R}^4 & \subset & \cdots & \subset & \mathbb{R}^\infty \\ \cup & & \cup & & & & \cup \\ \Omega_1 & \subset & \Omega_2 & \subset & \cdots & \subset & \Omega_\infty \end{array} \tag{3.6}$$

where Ω_∞ is defined to be the direct limit of the sequence $\Omega_1 \subset \Omega_2 \subset \cdots$.

In order to study various properties of Ω_∞, we consider the space P of all single input single output strictly proper plants endowed with the graph topology (see [4]). Let $\tilde{p}$ be a point in P with the coprime factorization

$$\tilde{p} = \tilde{p}_1/\tilde{p}_2 \tag{3.7}$$

where $\tilde{p}_1, \tilde{p}_2 \in H$, the ring of stable and proper rational functions. Define an open neighborhood $N_{\varepsilon_1,\varepsilon_2}(\tilde{p}_1, \tilde{p}_2)$ of $\tilde{p}$ in P as follows:

$$N_{\varepsilon_1,\varepsilon_2}(\tilde{p}_1, \tilde{p}_2) = \{ p \in p\colon\ p = p_1/p_2,\ p_1\&p_2 \text{ are coprime},\ \| p_1 - \tilde{p}_1 \|_\infty < \varepsilon_1,\ \| p_2 - \tilde{p}_2 \|_\infty < \varepsilon_2 \}. \tag{3.8}$$

The norm in (3.8) denotes the H^∞ norm. The set of all open neighborhoods of the type (3.8) defines a basis for the graph topology of P. Consider the map

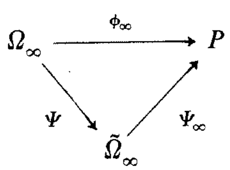
$$\phi_\infty : \Omega_\infty \to P \tag{3.9}$$

which sends a point in Ω_∞ to the corresponding plant in P.

Furthermore in Ω_∞ let us define an equivalence ration $\sim$ as follows: $\omega_1 \sim \omega_2$ iff both ω_1 and ω_2 correspond to the same plant, i.e. $\phi_\infty(\omega_1) = \phi_\infty(\omega_2)$. Consider the quotient space $\Omega_\infty/\sim \triangleq \tilde{\Omega}_\infty$ together with the natural map

$$\Psi : \Omega_\infty \to \tilde{\Omega}_\infty \tag{3.10}$$

which sends a point in Ω_∞ to its equivalence class. Of course $\tilde{\Omega}_\infty$ can be endowed with the quotient topology. Consider the map Ψ_∞ which makes the diagram



commute.

The main results of this paper are now summarized below.

Theorem 3.2. *ϕ_∞ is a continuous function.*

Corollary 3.3. *Let $\tilde{\omega}$ be a point in $\tilde{\Omega}_\infty$ and let p be the corresponding plant. Suppose that c is a dynamic compensator which stabilizes p. Then there exists an open neighborhood of $\tilde{\omega}$ in $\tilde{\Omega}_\infty$ such that c is a stabilizing compensator for every point in this neighborhood.*

Proof. Consider an open neighborhood W of $\Psi_\infty(\tilde{\omega})$ in P which can be stabilized by the dynamic compensator c. Existence of W follows from [4, Theorem 7.2.29]. Since ϕ_∞ is continuous, it follows that Ψ_∞ is continuous. Thus we conclude that $\Psi_\infty^{-1}(W)$ is an open neighborhood of $\tilde{\omega}$ in $\tilde{\Omega}_\infty$. Moreover every plant corresponding to points in $\Psi_\infty^{-1}(W)$ can be stabilized by c. □

Theorem 3.4. *Let $\tilde{\omega}_i$, $i = 1, 2, \ldots,$ be a sequence of points in $\tilde{\Omega}_\infty$. The following two conditions are equivalent:*

1. *The sequence $\{\tilde{\omega}_i,\ i = 1, 2, \ldots\}$ in $\tilde{\Omega}_\infty$ converges.*
2. (a) *The sequence $\{\Psi_\infty(\tilde{\omega}_i),\ i = 1, 2, \ldots\}$ in P converges.*
   (b) *There exists N sufficiently large such that the degree of $\Psi_\infty(\tilde{\omega}_i) \le N$ for all $i = 1, 2, \ldots$*

Theorem 3.5. *Ψ_∞^{-1} is not a continuous function.*

The consequences of the above two theorems are now described.

Remark 3.6. Let ω_i, $i=1, 2, \ldots$, be a convergent sequence of points in Ω_∞; then there exists an integer n sufficiently large such that $\omega_i \in \Omega_n$ for all $i=1, 2, \ldots$ This follows at once from the topology of Ω_∞.

Remark 3.7. It may be pointed out that $\tilde{\Omega}_k$ is homeomorphic to the space P_k of plants in P of McMillan degree $\leq k$. In fact, whereas a subset U in $\tilde{\Omega}_\infty$ is open if $U \cap \tilde{\Omega}_k$ is open in $\tilde{\Omega}_k$ for all $k=0, 1, \ldots$, the corresponding statement is not true in the Graph Topology.

This property of Ω_∞ suggests that in a parameter identification problem, if it is known apriori that the parameters converge, then the identification algorithm can be defined on Ω_N for N chosen sufficiently large.

In the following example we illustrate that non-convergence of a sequence in Ω_∞ does not necessarily imply non-convergence of the associated sequence in $\tilde{\Omega}_\infty$.

Example 3.8. Consider the sequence $(1/n, \frac{1}{2}(-1)^n, 0, 0, \ldots)$, $n=1, 2, \ldots$, of points in Ω_∞ which clearly does not converge in Ω_∞. On the other hand the above sequence converges to the equivalence class of $(0, 0, \ldots)$ in $\tilde{\Omega}_\infty$. This is because every neighborhood of $(0, 0, \ldots)$ in Ω_∞ which is also the inverse image of an open set in $\tilde{\Omega}_\infty$ contains the points $(1/n, \frac{1}{2}(-1)^n, 0, \ldots)$ for all $n > N$, N chosen sufficiently large.

The proof of Theorem 3.2 is sketched in Section 4. We now consider the following.

Proof of Theorem 3.4. $(2 \Rightarrow 1)$ Let $\tilde{p}_1, \tilde{p}_2, \ldots$, be a sequence of plants in P which converges to $\tilde{p}_0$ in P. Let N be such that $\deg \tilde{p}_i \leq N$ for $i=1, 2, \ldots$. It follows that $\deg \tilde{p}_0 \leq N$ for otherwise there exists an open neighborhood U of $\tilde{p}_0$ in P which does not contain any $\tilde{p}_i$. Let $\tilde{\omega}_i$, $i=1, 2, \ldots$, be the sequence in $\tilde{\Omega}_\infty$ such that $\Psi_\infty(\tilde{\omega}_i) = \tilde{p}_i$, $i=1, 2, \ldots$, and let $\tilde{\omega}_0$ be the point in $\tilde{\Omega}_\infty$ such that $\Psi_\infty(\tilde{\omega}_0) = \tilde{p}_0$. Let us write $\tilde{\omega}_i$ to be the equivalence class of

$$(\alpha_{i0}, \beta_{i0}, \alpha_{i1}, \beta_{i1}, \ldots, \alpha_{iN-1}, \beta_{iN-1}, 0, 0, \ldots)$$

so that $\tilde{p}_i$ may be written as

$$\frac{\alpha_{i0}s^{N-1} + \alpha_{i1}s^{N-2} + \cdots + \alpha_{iN-1}}{s^N + \beta_{i0}s^{N-1} + \cdots + \beta_{iN-1}}. \tag{3.11}$$

A coprime factorization of $\tilde{p}_i$ may be written as $\tilde{p}_{i1}/\tilde{p}_{i2}$ where

$$\tilde{p}_{i1} = \left[\alpha_{i0}s^{N-1} + \alpha_{i1}s^{N-2} + \cdots + \alpha_{iN-1}\right]/s^N, \tag{3.12}$$

$$\tilde{p}_{i2} = \left[s^N + \beta_{i0}s^{N-1} + \cdots + \beta_{iN-1}\right]/s^N. \tag{3.13}$$

By assumption, there exists a coprime factorization $\tilde{p}_{01}/\tilde{p}_{02}$ of $\tilde{p}_0$ in P where

$$\tilde{p}_{01} = \frac{\Delta_1(s)}{\Delta_2(s)} \frac{\alpha_{00}s^{N-1} + \alpha_{01}s^{N-2} + \cdots + \alpha_{0n-1}}{s^N}, \tag{3.14}$$

$$\tilde{p}_{02} = \frac{\Delta_1(s)}{\Delta_2(s)} \frac{s^N + \beta_{00}s^{N-1} + \cdots + \beta_{0N-1}}{s^N} \tag{3.15}$$

such that

$$\lim_{i \to \infty} \|[\tilde{p}_{i1} - \tilde{p}_{01}]\|_\infty = 0 \quad \text{and} \quad \lim_{i \to \infty} \|\tilde{p}_{i2} - \tilde{p}_{02}\|_\infty = 0 \tag{3.16}$$

where we assume that $\Delta_1(s)$, $\Delta_2(s)$ are stable polynomials of degree d. It may therefore be concluded that in the space of polynomials of degree $N+d-1$, the polynomials

$$\Delta_2\left[\alpha_{i0}s^{N-1} + \alpha_{i1}s^{N-2} + \cdots + \alpha_{iN-1}\right] \tag{3.17}$$

converge to the polynomial

$$\Delta_1\left[\alpha_{00}s^{N-1}+\alpha_{01}s^{N-2}+\cdots+\alpha_{0N-1}\right] \tag{3.18}$$

and in the space of polynomials of degree $N+d$, the polynomials

$$\Delta_2\left[s^N+\beta_{i0}s^{N-1}+\cdots+\beta_{iN-1}\right] \tag{3.19}$$

converge to the polynomial

$$\Delta_1\left[s^N+\beta_{00}s^{N-1}+\cdots+\beta_{0N-1}\right] \tag{3.20}$$

as $i\to\infty$. It therefore follows that in $\tilde{\Omega}_\infty$, the equivalence class of points which corresponds to the rational function

$$\frac{\Delta_2\left[\alpha_{i0}s^{N-1}+\alpha_{i1}s^{N-2}+\cdots+\alpha_{iN-1}\right]}{\Delta_2\left[s^N+\beta_{i0}s^{N-1}+\cdots+\beta_{iN-1}\right]} \tag{3.21}$$

converges to the equivalence class of points which corresponds to the rational function

$$\frac{\Delta_1}{\Delta_1}\,\frac{\left[\alpha_{00}s^{N-1}+\alpha_{01}s^{N-2}+\cdots+\alpha_{0N-1}\right]}{\left[s^N+\beta_{00}s^{N-1}+\cdots+\beta_{0N-1}\right]}. \tag{3.22}$$

Thus $\tilde{\omega}_i$ converges to $\tilde{\omega}_0$ in $\tilde{\Omega}_\infty$ as $i\to\infty$.

($1\Rightarrow 2$) Since ϕ_∞ is continuous and $\tilde{\Omega}_\infty$ has quotient topology, it follows that Ψ_∞ is continuous. Therefore if the sequence $\tilde{\omega}_i$, $i=1,2,\ldots,$ converges in $\tilde{\Omega}_\infty$, it follows that $\psi_\infty(\tilde{\omega}_i)$, $i=1,2,\ldots,$ converges in P also.

On the other hand let p_i, $i=1,2,\ldots,$ be a sequence of plants in P with the property that $\deg p_{i+1}>\deg p_i$ and $\deg p_1=1$. We would now show that $\Psi_\infty^{-1}(p_i)$, $i=1,2,\ldots,$ does not converge in $\tilde{\Omega}_\infty$. For otherwise there exists $\tilde{\omega}_0$ in $\tilde{\Omega}_\infty$ such that $\{\Psi_\infty^{-1}(p_i)\}$ converges to $\tilde{\omega}_0$. Define $\Psi_\infty(\tilde{\omega}_0)=p_0$. Let U_1 be an open neighborhood of p_0 in P which does not contain p_1. Define

$$S_1\triangleq\Psi_\infty^{-1}(U_1)\cap\tilde{\Omega}_1,\quad\text{and}\quad V_1\triangleq\Psi_\infty(S_1). \tag{3.23}$$

Note that V_1 does not contain the sequence $\{p_i\}$. Assume that U_i, S_i and V_i are defined where V_i does not contain the above sequence $\{p_i,\ i=1,2,\ \ldots\}$ in P and where

$$S_i=\Psi_\infty^{-1}(U_i)\cap\tilde{\Omega}_i. \tag{3.24}$$

Define U_{i+1} to be an open neighborhood of V_i which does not contain $p_1,\ldots,p_{i+1}$. Define

$$S_{i+1}\triangleq\Psi_\infty^{-1}(U_{i+1})\cap\tilde{\Omega}_{i+1}\quad\text{and}\quad V_{i+1}\triangleq\Psi_\infty(S_{i+1}). \tag{3.25}$$

It is clear that

$$S_i\subset S_{i+1}\quad\forall i=1,2,\ \ldots. \tag{3.26}$$

Define

$$S_\infty=\bigcup_{i=1}^{\infty}S_i. \tag{3.27}$$

We conclude that S_∞ is open since S_i is open in $\tilde{\Omega}_i$ for $i=1,2,\ \ldots.$ Moreover $\psi_\infty^{-1}p_0\in S_i\ \forall i$ implying that $\psi_\infty^{-1}p_0\triangleq\tilde{\omega}_0\in S_\infty$. Finally since

$$\psi_\infty^{-1}p_n(s)\notin S_i\quad\forall n=1,2,\ldots,\ \text{and}\ i=1,2,\ldots, \tag{3.28}$$

it follows that the sequence $\{\psi_\infty^{-1}p_i\}$ is not contained in S_j for all $j=1,2,\ \ldots$ and is therefore not contained in S_∞. This, however, contradicts the hypothesis that $\psi_\infty^{-1}p_i$ converges to $\tilde{\omega}_0$. □

Proof of Theorem 3.5. Consider the sequence of plants

$$\delta_n(s) = \frac{s^n}{s^{n+1} + 1/(n+2)} \tag{3.29}$$

which converges in P to $1/s$ as $n \to \infty$. From Theorem 3.4 it follows that $\delta_n(s)$ does not converge in $\tilde{\Omega}_\infty$. □

The main results of this section are now summarized. Whereas every point in Ω_∞ or $\tilde{\Omega}_\infty$ is robustly stabilizable by a dynamic compensator, in parameter identification and adaptive control problems, if it is apriori known that the parameters converge, it is enough to work in Ω_N rather than Ω_∞ for N chosen sufficiently large. The advantage of updating parameters in Ω_N is that Ω_N is diffeomorphic to $\mathbb{R}^{2N}$.

4. Proof of Theorem 3.2

Proof. Clearly it is enough to show that

$$\phi : \Omega_N \to P \tag{4.1}$$

is continuous for all N. Let $\tilde{p}$ be a point in p with the coprime factorization given by (3.7). For a given pair ε_1, $\varepsilon_2 > 0$ let $N_{\varepsilon_1,\varepsilon_2}(\tilde{p}_1, \tilde{p}_2)$ be a neighborhood of $\tilde{p}$ as described in (3.8). Let p_0 be a point in Ω_N such that

$$\phi(p_0) = \tilde{p}. \tag{4.2}$$

We would now show that there exists an open neighborhood $N(p_0)$ of p_0 in Ω_N such that

$$\phi(N(p_0)) \subset N_{\varepsilon_1,\varepsilon_2}(\tilde{p}_1, \tilde{p}_2). \tag{4.3}$$

Let

$$p_0 = (\bar{b}_0, \bar{a}_0, \ldots, \bar{b}_{N-1}, \bar{a}_{N-1}), \tag{4.4}$$

$$\alpha_0(s) = \bar{b}_0 s^{N-1} + \cdots + \bar{b}_{N-1}, \tag{4.5}$$

$$\beta_0(s) = s^N + \bar{a}_0 s^{N-1} + \cdots + \bar{a}_{N-1}, \tag{4.6}$$

so that $\tilde{p}$ may be written as

$$\frac{\bar{b}_0 s^{N-1} + \cdots + \bar{b}_{N-1}}{s^N + \bar{a}_0 s^{N-1} + \cdots + \bar{a}_{N-1}}. \tag{4.7}$$

The coprime representation $\tilde{p}_1/\tilde{p}_2$ of $\tilde{p}$ may be written as

$$\tilde{p}_1 = \Delta \frac{\bar{b}_0 s^{N-1} + \cdots + \bar{b}_{N-1}}{\delta(s)}, \qquad \tilde{p}_2 = \Delta \frac{s^N + \bar{a}_0 s^{N-1} + \cdots + \bar{a}_{N-1}}{\delta(s)}, \tag{4.8}$$

where $\Delta \in J$, the set of units of H and $\delta(s)$ is a Hurwitz polynomial of degree N, i.e. a polynomial with roots in the open interior of the unit disc.

For sufficiently small μ, define open neighborhoods $N_\mu(p_0)$ of p_0 in Ω_N as follows:

$$N_\mu(p_0) = \left\{ (b_0, a_0, \ldots, b_{N-1}, a_{N-1}) \,\middle|\, \sum_{i=0}^{n-1} (b_i - \bar{b}_i)^2 + \sum_{i=0}^{N-1} (a_i - \bar{a}_i)^2 < \mu \right\}. \tag{4.9}$$

For any $p \in N_\mu(p_0)$, $\phi(p)$ may be written as

$$\frac{b_0 s^{N-1} + \cdots + b_{N-1}}{s^N + a_0 s^{N-1} + \cdots + a_{N-1}} \triangleq \frac{\alpha(s)}{\beta(s)}. \tag{4.10}$$

A coprime factorization of $\phi(p)$ is therefore given by p_1/p_2 where

$$p_1 = \Delta \frac{b_0 s^{N-1} + \cdots + b_{N-1}}{\delta(s)}, \qquad p_2 = \Delta \frac{s^N + a_0 s^{N-1} + \cdots + a_{N-1}}{\delta(s)}. \tag{4.11}$$

We now compute

$$\| p_1 - \tilde{p}_1 \|_\infty \le \| \Delta \|_\infty \left\| \left[\sum_{j=0}^{N-1} (b_j - \bar{b}_j) s^{N-j-1} \right] \Big/ \delta(s) \right\|_\infty \tag{4.12}$$

and

$$\| p_2 - \tilde{p}_2 \|_\infty \le \| \Delta \|_\infty \left\| \left[\sum_{j=0}^{N-1} (a_j - \bar{a}_j) s^{N-j-1} \right] \Big/ \delta(s) \right\|_\infty . \tag{4.13}$$

For μ sufficiently small it is clear that

$$\| p_1 - \tilde{p}_1 \|_\infty < \varepsilon_1 \quad \text{and} \quad \| p_2 - \tilde{p}_2 \|_\infty < \varepsilon_2, \tag{4.14}$$

implying that

$$\phi(N_\mu(p_0)) \subset N_{\varepsilon_1,\varepsilon_2}(\tilde{p}_1, \tilde{p}_2). \quad \square \tag{4.15}$$

5. An interesting example

In this section we consider an example which further illustrates the properties of the parameterization Ω_∞. Consider the point

$$(2, 4, 6, 1, 4, -6, 0, 0, \ldots) \tag{5.1}$$

in $\mathbb{R}^\infty$. Since the two polynomials $2s^2 + 6s + 4$ and $s^3 + 4s^2 + s - 6$ vanish at $s = -2$, it follows that the point (5.1) is not in Ω_∞. We now define two points $p \triangleq (b_0, a_0, \ldots, b_{n-1}, a_{n-1})$ and $q \triangleq (c_0, d_0, \ldots, c_{m-1}, d_{m-1})$ in $\mathbb{R}^\infty$ to be equivalent and denote it as $p \sim q$ if there exists a rational function $\alpha(s)$ with finite poles and zeros in the open interior of the unit disc such that

$$(c_0 s^{m-1} + \cdots + c_{m-1}) \alpha(s) = b_0 s^{n-1} + \cdots + b_{n-1} \tag{5.2}$$

and

$$(s^m + d_0 s^{m-1} + \cdots + d_{m-1}) \alpha(s) = s^n + a_0 s^{n-1} + \cdots + a_{n-1}. \tag{5.3}$$

Furthermore, we shall call an open subset U of $\mathbb{R}^\infty$ to be saturated if for every $p, q \in \mathbb{R}^\infty$ we have $p \in U$, $p \sim q \Rightarrow q \in U$.

We now show, rather surprisingly, that for every integer n and for every saturated open neighborhood U of the point (5.1) in $\mathbb{R}^\infty$, there exists a point in $U' \triangleq U \cap \Omega_\infty$ that corresponds to a plant which cannot be stabilized by a compensator of degree $< n$. Since U' is saturated in Ω_∞ it follows that $\phi_\infty(U')$ contains a sequence of plants τ_i, $i = 1, 2, \ldots$, such that if d_i is the minimum degree of stabilizing compensator for τ_i then $d_i < d_{i+1}$, $i = 1, 2, \ldots$. Since U' is open and saturated in Ω_∞, it follows that $\Psi(U')$ is open in $\tilde{\Omega}_\infty$.

Therefore it follows from [5] that the open set of points $\Psi(U')$ in $\tilde{\Omega}_\infty$, which corresponds via ψ_∞ to the open set of plants $\phi_\infty(U')$ in P, is not simultaneously stabilizable even by a universal adaptive controller.

In order to prove the above claim, consider the plant

$$\frac{(z+\varepsilon)^n(z-1)(z-3)}{z(z+\varepsilon)^n(z-2)(z-3)+\mu(z-2)} \tag{5.4}$$

where $n = 1, 2, \ldots$ and $0 < \varepsilon < 3/n$ and $\mu < 0$ chosen in such a way that the degree of the plant in (5.4) is $n+3$. It follows from the argument presented by Smith [7] that there does not exist a proper compensator of degree $< n$ which places the poles of the plant (5.4) in the open left half of the complex plane. We now consider the conformal transformation

$$z = (s-1)/(s+1) \tag{5.5}$$

which maps the open left half of the complex plane to the open interior of the unit disc. The plant (5.4) reduces to the plant

$$\frac{4(s+2)(s+1)[(\varepsilon+1)s+(\varepsilon-1)]^n}{2(s+2)(s+3)(s-1)[(\varepsilon+1)s+(\varepsilon-1)]^n-\mu(s+3)(s+1)^{n+2}} \tag{5.6}$$

for which there does not exist a proper compensator of degree $< n$ which places the poles of (5.6) in the open interior of the unit disc. It is trivial to check that for ε, μ arbitrarily close to 0, the transfer function (5.6) corresponds to points in $\mathbb{R}^{2n+6}$ arbitrarily close to $(2, 4, 6, 1, 4, -6, 0, 0, \ldots, 0)$, i.e. if U_s is a saturated neighborhood of (5.1) in $\mathbb{R}^\infty$, the open set $U_s \cap \Omega_{n+3}$ contains a point that corresponds to a plant of the type (5.6) for ε, μ arbitrary small.

6. Summary and conclusion

In this paper we obtain a parameterization of the set of linear dynamical systems of arbitrary large McMillan degree. Such a parameterization we hope would be useful in system identification and adaptive control. The advantage of the proposed parameterization Ω_∞ over the presently existing 'graph-topology' is that Ω_∞ is obtained by piecing together systems of bounded McMillan degree. In fact in parameter identification it is enough to work with Ω_N for N sufficiently large. Since Ω_N is diffeomorphic to $\mathbb{R}^{2N}$, we hope that many of the local results in identification (valid only if the parameter space is Euclidean) can be generalized globally. Lastly we show that the quotient topology on $\tilde{\Omega}_\infty$ is finer than the graph topology.

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