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SIMULTANEOUS COEFFICIENT ASSIGNMENT OF DISCRETE-TIME MULTI-INPUT MULTI-OUTPUT LINEAR TIME-VARYING SYSTEM: A NEW APPROACH FOR COMPENSATOR DESIGN*

BIJOY K. GHOSH† AND PAUL R. BOUTHELLIER†‡

Abstract. In this paper, a linear time-varying input-output system is considered and its realization as a linear time-varying autoregressive moving average system (ARMA) is studied. A time-varying z-transform is also introduced and its properties are studied. Furthermore a time-varying version of the coefficient assignment problem well known in time invariant system theory as the pole placement problem is posed and analyzed. A r-tuple of discrete time, linear time-varying plants with m inputs and p outputs are considered together with a single p input m output linear time-varying compensator. The design objective is to construct a single compensator that "coefficient assigns," and hence "bounded input bounded output stabilizes" under suitable additional technical assumptions, the set of r plants simultaneously in the closed loop. Such a problem is useful in robust design of linear time-varying control systems in the closed loop. Among the results, it is shown that a generic r-tuple of $p \times m$ plants (in a suitable topology) is simultaneously coefficient assignable, provided that $r < m/p$. The design procedure involves splitting the closed-loop system into an ARMA system in cascade with a moving average system. The coefficient assignment problem consists of assigning the coefficients of the autoregressive part of the ARMA subsystem. Thereby an algorithm is obtained that is nonrecursive and involves solving for each time instant a system of linear equations with time-varying coefficients. The associated time-varying matrix has the "Sylvester matrix structure." Such a structure is well-known in pole placement of time-invariant systems by dynamic compensation. Additionally the problem of coefficient assignment of the autoregressive part of the ARMA system is considered in the closed loop, without splitting up into a cascade of two subsystems as before. A new recursive algorithm to analyze this problem has been introduced. The proposed algorithm has no counterpart in the time-invariant system design and thus represents a new design procedure. A special case of this algorithm for the single-input single-output system has been described in detail. An interesting feature of the proposed recursive algorithm is that one obtains a nonlinear recursion on the compensator parameters that would assign a prespecified sequence of coefficients for the closed-loop system. For a specific design problem it is shown that the dynamics of this nonlinear recursion is chaotic.

Key words. coefficient assignment, recursive algorithm, simultaneous design, time-varying system, chaotic dynamics

AMS(MOS) subject classifications. 14, 93

1. Introduction. Motivated by earlier successes in the field of simultaneous system design (see [1]–[7]) for linear time-invariant systems, in this paper the same general idea is applied to linear time-varying (LTV) systems as well. The motivation for considering time-varying systems is as follows. Many systems are time varying because they switch modes frequently (namely, high-performance aircrafts, power systems undergoing several modes of failure, etc.). Time-varying systems also arise from nonlinear systems linearized along a nominal trajectory. Furthermore, time-varying systems also arise from linear systems, where the parameters are perturbed by a time-varying function. A feedback design strategy that leads to a time-varying system is *adaptive control*, wherein the time variation is a result of real time adaptation. An important pair of problems in the design of time-varying systems is described as follows.

PROBLEM 1.1 (stability criterion). Given a class of linear time-varying systems. What condition on the parameters of the systems would guarantee bounded-input bounded-output (BIBO) stability?

PROBLEM 1.2 (stability criterion). If a linear time-varying system is not already BIBO stable, find a linear time-varying output feedback compensator such that the closed-loop system is BIBO stable.

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† Department of Systems Science and Mathematics, Campus Box 1040, Washington University, One Brookings Drive, Saint Louis, Missouri 63130-4899.
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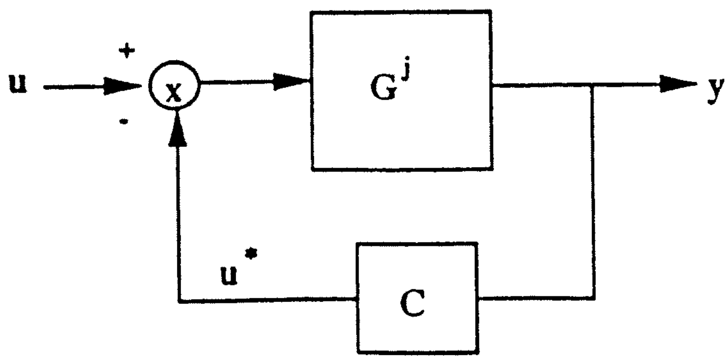

FIG. 1.1. *Closed-loop system corresponding to* (G^j, C).

The main problem of ascertaining stability of a linear time-varying system is that the problem is not equivalent to localizing the eigenvalues of a possibly time-varying matrix to a certain region of the complex plane. In fact, a time-varying system stable in frozen time (i.e., stable for each time instance) is not necessarily a stable time-varying system (see [8] and [9]). Stability can be ascertained, however, if the parameters of the system are varying sufficiently slowly. It has been shown by Desoer [10] that there exists open regions of the parameter space with the property that if the parameter vector of the system resides in such a region for all times, then the associated time-varying system is indeed stable (see also [11]–[13]). A sufficiency criterion for stabilizing a time-varying system is, therefore, one of choosing a compensator that localizes the coefficients of the closed-loop system to within such an open region. For reasons of robustness and fault tolerance, however, one is interested in stabilizing, not just a single plant, but an entire r-tuple of plants. This problem is now described as follows.

PROBLEM 1.3 (simultaneous stabilizability problem). Given an r-tuple of linear time-varying plants $G^1, G^2, \ldots, G^r$, find, if possible, a linear time-varying output feedback compensator C that simultaneously stabilizes each one of the closed-loop systems $(G^j, C), j = 1, \ldots, r$.

In Problem 1.3 (G^j, C) denotes the closed-loop system described in Fig. 1.1. For an introduction to the simultaneous stabilization problem for time-invariant systems, we refer to [1]–[3]. The main idea is that G^1 is a nominal plant, which, as a result of sudden component failures, may take up $r-1$ additional different modes $G^2, \ldots, G^r$. The design goal is to construct a compensator that stabilizes the nominal plant together with all its failed modes simultaneously. To obtain a tighter control on the response of the closed-loop system, we consider the following problem.

PROBLEM 1.4 (simultaneous coefficient assignment problem). Given an r-tuple of m input p output *autoregressive moving average* (ARMA) models of lag ℓ, denoted by $\{G^j\}$ and defined by

$$y_k + \sum_{i=1}^{\ell} D_k^j(i) y_{k-i} = \sum_{i=1}^{\ell} N_k^j(i) v_{k-i}. \tag{1.1}$$

When does there exist a compensator C of lag q, defined by

$$\begin{aligned} u_k^* + \sum_{i=1}^{q} \bar{D}_k(i) u_{k-i}^* &= \sum_{i=0}^{q} \bar{N}_k(i) y_{k-i} \\ v_k &= u_k - u_k^*, \end{aligned} \tag{1.2}$$

that will assign the coefficients $\Delta_k^j(i)$ of the equations of the closed-loop systems that (G^j, C) described as

$$\Delta_k^j(0)y_{1,k} + \sum_{i=1}^{\ell+q} \Delta_k^j(i)y_{1,k-i} = \sum_{i=0}^{\ell+q} r_k^j(i)u_{k-i}, j = 1, \ldots, r. \tag{1.3}$$

The main contributions of this paper are now described. In §2 we consider a general linear input–output map and obtain a necessary and sufficient condition as to when such a map is realizable as a linear time-varying ARMA model of finite lag. We introduce left and right fraction representation of an LTV system and show via examples that, unlike the time-invariant case, the existence of one does not imply the existence of the other. However, in a suitable topology on the space of LTV ARMA systems, a generic system is shown to admit either of the two representations. In §3, we pose and analyze a simplified coefficient assignment problem. The proposed problem consists of splitting the closed-loop system into a cascade of ARMA and a moving average subsystem. The problem considered is to assign the coefficients of the autoregressive part of the ARMA subsystem. We show that the solution to the problem consists of analyzing linear equations with time-varying matrices. Under a suitable topology we consider an r-tuple of m-input p-output plants and show that a sufficient condition for the proposed coefficient assignment problem is given by $rp < m+p$. Under an additional technical condition (3.29) the above inequality is shown to be sufficient for simultaneous stabilization of the r-tuple of plants as well. In §4 we consider the closed-loop system as a single ARMA system and propose assigning the coefficients of the autoregressive part. The coefficient assignment problem is considered when $r = 1$ and m and p are arbitrary. A nonlinear time-varying iteration scheme that recursively assigns the coefficients of such time-varying systems in the closed-loop is described. Such an iterative scheme appears to be new in the literature and has no counterpart in the time-invariant system theory.

The main technique that we use in this paper is that of a time-varying version of the z-transform as part of an operational algebra for discrete-time, linear time-varying systems. Such an operational algebra has also been used previously by Kamen, Khargonekar, Poolla, and Hwang [14]–[17]. More recently a continuous-time version of the above operational algebra is being used by Tsakalis and Ioannou [18] and [19].

The main idea of this paper is to describe a time-varying version of the return difference matrix and to ensure that the coefficients of this matrix can be assigned arbitrarily under a sufficiency condition. Such a sufficiency condition in principal is a generalization of the results on "pole placement by dynamic compensation" for linear time-invariant systems (see [20]–[22]). The time-varying nature of the problem considered here imposes restrictions that did not exist in the literature concerning time-invariant systems. For example, we see in this paper that a coefficient-assigning compensator for a single-input single-output plant can be obtained by solving a nonlinear difference equation recursively. When restricted to time-invariant parameters, the difference equation reduces to the well-known linear algebraic equation of the type $Sx = b$, where S is a Sylvester matrix. Stability analysis of the proposed nonlinear difference equation has not been carried out in general and is a subject of future research.

There are other approaches [23] and [24] to stabilization, simultaneous stabilization of linear time-varying systems in the literature. For example, [23] deals with continuous-time systems wherein the input–output time-varying plant is modeled as an operator between two suitable function spaces. Among the results, it is shown that if an r-tuple of linear time-varying plants is internally stabilizable individually, then the r-tuple is simultaneously stabilizable by a stable linear time-varying compensator. The main difference between this paper and the approach presented in [23] is now described. In this paper, we deal with linear time-varying systems in discrete time. Moreover, the time-varying system is described as a parametric variation on the space of time-invariant systems. Thus, this paper addresses problems in system design that pertain to real time adaptation of the compensator parameters as a result of real time changes in the plant parameters. The main system design problem that we consider is coefficient assignment, wherein no assumption is made about the parameters of the plants and compensators for all future times. In particular, the parameters of the plants and compensators are not assumed to be known completely. In fact, in this paper we assume that the future values of the plant parameters are unknown. Of course, to implement the coefficient assignment algorithms presented, we need to know the values of the plant parameters for an a priori fixed span of time (depending upon the lags of the systems) in the future. This adds a new twist to the problem of compensator design for a time-varying plant. Estimating the parameters of a time-varying plant in the immediate future appears to be an integral part of compensating a time-varying system, in discrete time, and to the best of our knowledge has never been considered before in the literature.

2. Representations of time-varying input–output maps. In this section, we consider a linear time-varying input–output map and study the problem of realizing the map as an impulse response of a linear time-varying *autoregressive moving average* (LTV ARMA) model of finite lag q. LTV ARMA models are of interest because the plants and compensators considered in subsequent sections of this paper are modeled as LTV ARMA systems, i.e., as ARMA systems with time-varying parameters.

Linear time-varying input–output maps are described by their impulse response sequence. We derive condition on the impulse response parameters so that it is realizable as an impulse response of an LTV ARMA system of a given lag q. To derive the realizability condition and also in later sections to describe the compensator that assigns coefficients of the closed-loop system, we find it convenient to introduce the notion of transfer function for an LTV input–output map. Such a transfer function is an obvious generalization of the z-transform methods well known in linear time-invariant discrete-time system design. To describe the transfer function, we need to introduce an operational algebra on the space of infinite power series with time-varying coefficients. Establishing connection between LTV input–output system, LTV ARMA system, and LTV transfer functions form the core of the main results described in this section.

Consider a m-input p-output LTV input–output map described by its impulse response sequence $H_j(i)$, where $H_j(i)$ is a $p \times m$ matrix defined to be the output at time j corresponding to a unit impulse at time $j - i$. To impose causality, we set $H_j(i) = 0$, the zero matrix, for all $i > j$.

Using linearity, it is clear that the impulse response sequence $H_j(i)$ completely specifies the input–output map. In fact, if $u_j, u_{j-1}, \dots$ is a sequence of m vector inputs at time $j, j-1, \dots$, respectively, we have

$$y_j = \sum_{\ell=0}^{\infty} H_j(\ell) u_{j-\ell}, \tag{2.1}$$

where y_j is the p-vector output at the time instant j. Equation (2.1) will be referred to as the LTV input-output map. The realization problem that we now consider is described as follows.

PROBLEM 2.1. Given a time-varying ARMA model of finite lag ℓ described by

$$y_k + \sum_{i=1}^{\ell} D_k(i) y_{k-i} = \sum_{i=0}^{\ell} N_k(i) u_{k-i}, \tag{2.2}$$

where $D_k(i), i = 1, 2, \ldots, \ell$ are $p \times p$ matrices and $N_k(i), i = 0, 1, \ldots, \ell$ are $p \times m$ matrices. When is it true that the impulse response of a LTV input–output map described by (2.1) coincides with the impulse response of an ARMA model of type (2.2)?

We will see subsequently in this section that a necessary and sufficient condition for the above realization is given by a sequence of recursive conditions on the impulse response sequence H_j^i. The procedure is in principal similar to checking ranks of Hankel matrices in the theory of linear time-invariant systems.

Before we proceed to study Problem 2.1, we introduce an operational algebra and consider the notion of a transfer function for LTV input–output systems and LTV ARMA systems. This is done as follows.

Let y_k be a discrete-time vector sequence. Define a shift operator z^{-i} as follows:

$$z^{-i} y_k = y_{k-i}. \tag{2.3}$$

In the notation of (2.3), extending the operator z^{-i} linearly, we can write (2.1) as

$$y_k = \left[\sum_{\ell=0}^{\infty} H_k(\ell) z^{-\ell} \right] u_k. \tag{2.4}$$

The infinite power series

$$\mathcal{H}(z^{-1}) = \sum_{\ell=0}^{\infty} H_k(\ell) z^{-\ell} \tag{2.5}$$

is defined to be the transfer function of the LTV input–output map described by (2.1). We now define an operation of multiplication of two infinite power series of the type (2.5). Denote the multiplication operation by $\circ$.

Let

$$\mathcal{J}(z^{-1}) = \sum_{\ell=0}^{\infty} J_k(\ell) z^{-\ell} \tag{2.6}$$

be another infinite power series. We define

$$\mathcal{H}(z^{-1}) \circ \mathcal{J}(z^{-1}) = \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} [(H_k(\ell_1) z^{-\ell_1}) \circ (J_k(\ell_2) z^{-\ell_2})], \tag{2.7}$$

where

$$(H_k(\ell_1) z^{-\ell_1}) \circ (J_k(\ell_2) z^{-\ell_2}) = H_k(\ell_1) J_{k-\ell_1}(\ell_2) z^{-(\ell_1+\ell_2)}. \tag{2.8}$$

Of course, we assume that $H_k(\ell), J_k(\ell)$ are matrices of compatible dimension so that the product $H_k(\ell_1) J_{k-\ell_1}(\ell_2)$ is defined. The following straightforward properties of the multiplication operation are now stated without proof.

PROPOSITION 2.2. *Let $\mathcal{H}(z^{-1}), \mathcal{J}(z^{-1}), \mathcal{L}(z^{-1})$ be a set of three infinite power series. Assuming that $\mathcal{H}(z^{-1}) \circ \mathcal{J}(z^{-1})$ and $\mathcal{J}(z^{-1}) \circ \mathcal{L}(z^{-1})$ are defined, we have the following:*

1. *The multiplication operation* $\circ$ *is associative, i.e.,*

$$[\mathcal{H}(z^{-1}) \circ \mathcal{J}(z^{-1})] \circ \mathcal{L}(z^{-1}) = \mathcal{H}(z^{-1}) \circ [\mathcal{J}(z^{-1}) \circ \mathcal{L}(z^{-1})].$$

2. *The multiplication operation is not commutative, i.e.,*

$$\mathcal{H}(z^{-1}) \circ \mathcal{J}(z^{-1}) \neq \mathcal{J}(z^{-1}) \circ \mathcal{H}(z^{-1})$$

in general, even when the right-hand side is defined.

3. $[\mathcal{H}(z^{-1}) \circ \mathcal{J}(z^{-1})]y_k = \mathcal{H}(z^{-1})[\mathcal{J}(z^{-1})y_k]$.

We now consider the following definition.

DEFINITION 2.3 (existence of inverse). Let $\mathcal{Q}(z^{-1})$ be an infinite power series with square coefficient matrices of size $\alpha \times \alpha$. If there exists $\mathcal{W}(z^{-1})$, an infinite power series with square coefficient matrices of size $\alpha \times \alpha$ such that

$$\mathcal{Q}(z^{-1}) \circ \mathcal{W}(z^{-1}) = \mathcal{W}(z^{-1}) \circ \mathcal{Q}(z^{-1}) = I_\alpha,$$

where I_α is an identity matrix of size $\alpha \times \alpha$, then $\mathcal{W}(z^{-1})$ is called an *inverse* of $\mathcal{Q}(z^{-1})$ and we write

$$\mathcal{W}(z^{-1}) = \mathcal{Q}^{-1}(z^{-1}).$$

Not all infinite power series would have an inverse. The following proposition is important, but involves straightforward checking. Hence, the proof is omitted.

PROPOSITION 2.4. *Let*

$$\mathcal{D}(z^{-1}) = I_p + \sum_{i=1}^{\ell} D_k(i)z^{-1},$$

where $D_k(i)$*-s are* $p \times p$ *matrices. Then* $\mathcal{D}(z^{-1})$ *has an unique inverse given by*

$$D^{-1}(z^{-1}) = A_k(0) + A_k(1)z^{-1} + A_k(2)z^{-2} + \cdots,$$

where

$$\begin{aligned} A_k(0) &= I_p \\ A_k(1) &= -A_k(0)D_k(1) \\ A_k(2) &= -A_k(1)D_{k-1}(1) - A_k(0)D_k(2), \text{ etc.} \end{aligned}$$

The LTV ARMA model (2.2) can be written as

$$\left[I + \sum_{i=1}^{\ell} D_k(i)z^{-i}\right] y_k = \left[\sum_{i=0}^{\ell} N_k(i)z^{-i}\right] u_k. \tag{2.9}$$

Using Proposition 2.4, we can now write

$$y_k = \left[I + \sum_{i=1}^{\ell} D_k(i)z^{-i}\right]^{-1} \left[\sum_{i=0}^{\ell} N_k(i)z^{-i}\right] u_k \tag{2.10}$$

$$= \Psi(z^{-1})u_k. \tag{2.11}$$

The power series $\Psi(z^{-1})$ is defined to be the transfer function of an LTV ARMA model of lag ℓ. We would now define the left and right representations of LTV systems as follows.

DEFINITION 2.5. The transfer function (2.5) of the input–output map (2.1) is said to have a left factorization

$$\Psi_L(z^{-1}) = \left(I + \sum_{i=1}^{\ell} D_k(i)z^{-i}\right)^{-1} \circ \left(\sum_{j=0}^{\ell} N_k(j)z^{-j}\right) \tag{2.12}$$

of lag ℓ if

$$\left(\sum_{i=0}^{\infty} H_k(i)z^{-i}\right) = \Psi_L(z^{-1}) \tag{2.13}$$

and a right factorization

$$\Psi_R(z^{-1}) = \left(\sum_{j=0}^{\ell} \bar{N}_k(j)z^{-j}\right) \circ \left(I + \sum_{i=1}^{\ell} \bar{D}_k(i)z^{-1}\right)^{-1} \tag{2.14}$$

of lag ℓ, if

$$\left(\sum_{i=0}^{\infty} H_k(i)z^{-i}\right) = \Psi_R(z^{-1}). \tag{2.15}$$

It is clear from (2.9) and (2.10) that Problem 2.1 involves finding conditions under which the transfer function (2.5) admits a left factorization of lag ℓ. The following theorem completely solves the problem.

THEOREM 2.6. *The infinite power series* (2.5) *admits a left factorization of lag ℓ if and only if there exists $p \times p$ matrices $D_k(1), \ldots, D_k(\ell)$ such that*

$$\begin{bmatrix} H_k^T(\ell+1) \\ H_k^T(\ell+2) \\ \vdots \end{bmatrix} + \begin{bmatrix} H_{k-1}^T(\ell) & H_{k-1}^T(\ell-1) & \cdots & H_{k-\ell}^T(1) \\ H_{k-1}^T(\ell+1) & H_{k-2}^T(\ell) & \cdots & H_{k-\ell}^T(2) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} D_k(1)^T \\ D_k(2)^T \\ \vdots \\ D_k(\ell)^T \end{bmatrix}. \tag{2.16}$$

The infinite power series (2.5) *admits a right factorization of lag ℓ if and only if there exist $p \times p$ matrices $D_k(1), \ldots, D_k(\ell)$ such that*

$$\begin{aligned} &[H_k^T(\ell+1)H_{k+1}^T(\ell+2)\cdots] \\ &= [D_{k-\ell}^T(1)D_{k-\ell+1}^T(2)\cdots D_{k-1}^T(\ell)] \begin{bmatrix} H_k^T(\ell) & H_{k+1}^T(\ell+1) & \cdots \\ H_k^T(\ell-1) & H_{k+1}^T(\ell) & \cdots \\ \vdots & \vdots & \vdots \\ H_k^T(1) & H_{k+1}^T(2) & \cdots \end{bmatrix}. \end{aligned} \tag{2.17}$$

Proof. We prove Theorem 2.6 for the case of left factorizations. The case of right factorizations is similar and is omitted.

The impulse response matrix (2.1) admits a left factorization of lag ℓ if and only if there exists $D_k(i). N_k(i)$ such that (2.13) is satisfied. It follows that

$$\left[\left(I + \sum_{i=1}^{\ell} D_k(i)z^{-i}\right) \circ \left(\sum_{i=0}^{\infty} H_k(i)z^{-i}\right)\right] \left(\sum_{i=0}^{\ell} N_k(i)z^{-i}\right). \tag{2.18}$$

Expanding both sides of (2.18) and equating like powers of z^{-1} yields the desired result. □

Remark 2.7. For linear time-invariant systems, the notion of left and right factorization is well known [25]. For time-varying systems, these were introduced in [14] and [17]. The realizability condition introduced in Theorem 2.6 is new. Note that if the impulse response sequence is arranged in the form of a matrix

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & H_0(0) & H_1(1) & H_2(2) & H_3(3) & H_4(4) & \cdot & \cdot \\ \cdot & \cdot & O & H_1(0) & H_2(1) & H_3(2) & H_4(3) & \cdot & \cdot \\ \cdot & \cdot & O & O & H_2(0) & H_3(1) & H_4(2) & \cdot & \cdot \\ \cdot & \cdot & O & O & O & H_3(0) & H_4(1) & \cdots & \\ \cdot & \cdot & O & O & O & O & H_4(0) & \cdots & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \end{bmatrix}, \tag{2.19}$$

the conditions for left (right) factorization can be viewed as a rank condition on the columns (rows) of the above matrix. In time-invariant system theory the above rank condition would reduce to checking ranks of a suitable Hankel matrix. The results are quite standard [26], and, therefore, the details are omitted.

Remark 2.8. It may be noted that, unlike the results in [14] and [17], wherein left and right representations for input–output maps of the form (2.1) have also been given, we do not have to construct state-space realizations of minimal order for (2.1) as an intermediary step.

The following two examples serve to illustrate the important fact that, unlike the time-invariant case, the existence of a left factorization of finite lag for a time-varying system does not imply the existence of a right factorization of finite lag and vice-versa. Both examples are for single-input single-output systems.

Example 2.9. Consider an input–output system defined by the following impulse response sequence

$$H_k(0) = 1 \quad k = 0, 1, 2, \ldots. \tag{2.20}$$

For $i = 1, 2, 3, \ldots$

$$\begin{aligned} H_{k+i}(i) &= 1 \text{ for } k \geq 0, k \text{ even} \\ &= 0 \text{ for } k \geq 0, k \text{ odd.} \end{aligned}$$

Let $H_j(i) = 0$ for all other values of i, j. It is easy to see that (2.20) admits a left factorization of the form (2.12) of lag 1 given by $N_k(0) = 1, N_k(1) = 0, D_k(1) = -1$. On the other hand, (2.20) does not admit a right factorization of finite lag.

Example 2.10. Consider an input-output system defined by the following impulse response sequence:

$$H_k(0) = 1 \quad k = 0, 1, 2, \ldots. \tag{2.21}$$

For $j = 1, 2, \ldots$

$$H_k(j) = 1 k \geq 0, k \text{ even} \tag{2.22}$$

$$= 0 k \geq 0, k \text{ odd.} \tag{2.23}$$

Let $H_j(i) = 0$ for all other values of i, j. It is straightforward to check that (2.21) admits no left factorization, but admits a right factorization of lag 1 given by

$$(1 + N_k(1)z^{-1})(1 - z^{-1})^{-1},$$

where

$$N_k(1) = -1 + H_k(1).$$

Remark 2.11. Left factorizations of finite lag always admit ARMA representations of finite lag as well as state-space realizations of finite order [14], [27].

Example 2.10 raises the concern that right factorizations, unlike left factorizations, may be difficult to implement. For this reason, we will conclude this section by showing that a right factorization

$$\mathcal{N} \circ \mathcal{D}^{-1} \equiv \left(\sum_{i=0}^{\bar{\ell}} \bar{N}_k(i) z^{-i} \right) \circ \left(I + \sum_{i=1}^{\bar{\ell}} \bar{\mathcal{D}}_k(i) z^{-i} \right)^{-1} \tag{2.24}$$

of finite lag (i) generically admits a left factorization of finite lag and (ii) can always be realized in state space form.

We now consider the following topology for the space of right factorization of lag $\leq \bar{\ell}$. Note that the vector of matrices

$$(\bar{N}_k(0), \ldots, \bar{N}_k(\ell), \bar{D}_k(1), \ldots, \bar{D}_k(\ell)) \in \mathbb{F}^N,$$

where

$$N = \bar{\ell} p^2 + (\bar{\ell} + 1) pm$$

for each $k = 0, 1, 2, \ldots$. Thus, every right factorization of the form (2.24) is a point in the product space

$$\prod_{j=0}^{\infty} \mathbb{F}^N = \mathcal{P}. \tag{2.25}$$

We now equip $\mathcal{P}$ with the product topology (see [28]).

DEFINITION 2.12. A set $\mathcal{G}$ of right factorizations is said to be generic if $\mathcal{G}$ can be written as an intersection of a countable number of open and dense sets in $\mathcal{P}$.

The ARMA realization of a system of the form (2.24) is given by the following theorem.

THEOREM 2.13. *Consider a generic element in the space of right factorizations of lag* $\leq \bar{\ell}$ *with* m *inputs and* p *outputs. There always exists a left factorization of lag* ℓ, *where* ℓ *is the smallest integer satisfying* $\ell p \geq \bar{\ell} m$ *such that the two factorizations correspond to the same infinite power series for all* $k \geq \ell$.

Proof. See Appendix I.

Note 2.14. The basic interpretation of Theorem 2.13 is that almost all LTV transfer functions with a right factorization also have a left factorization and, therefore, can be realized as an LTV ARMA system. Of course, we could also define a generic set of left factorizations to show that generically a transfer function has left factorization (right factorization) if it has a right factorization (respectively, left factorization).

We will now state that right factorizations of finite lag (2.24) can always be realized in state-space form. This realization has not been used subsequently in this paper. It is stated only to satisfy our curiosity that even though a right factorization may not have a left factorization, it can still be realized as a state-space system, but possibly not as an ARMA system. The result follows with modifications from [14] and [16].

THEOREM 2.15 (state-space realization). *The right factorization given by* (2.24) *can always be realized as an* ℓm*th-order state-space system*

$$x(k+1) = F(k)x(k) + G(k)u(k); x(0) = 0$$
$$y(k) = H(k)x(k) + J(k)u(k),$$

where

$$F(k) \equiv \begin{bmatrix} O & O & & O & O & -\bar{D}_k(\ell) \\ I & O & & O & O & -\bar{D}_{k-1}(\ell-1) \\ O & I & & O & O & -\bar{D}_{k-2}(\ell-2) \\ & & \cdots & \vdots & \vdots & \vdots \\ O & O & & I & O & -\bar{D}_{k-\ell+2}(2) \\ O & O & & O & I & -\bar{D}_{k-\ell+1}(1) \end{bmatrix}, \tag{2.26}$$

$$G \equiv \text{col}\,[I, O, O, \ldots, O], \tag{2.27}$$

$$H(k) \equiv [W_k(1), W_k(2), \ldots, W_k(\ell-1), W_k(\ell)], \tag{2.28}$$

and

$$J(k) \equiv W_k(0), \tag{2.29}$$

where $W_k(i)$ *is the coefficient of* z^{-i} *in the formal power series expansion of* (2.24).

Proof. We refer the interested reader to [14] and [16].

The main contributions of this section are now summarized. Starting from the impulse response sequence of an input–output map, we introduce a formal infinite power series. We then completely answer the question as to when such a power series can be represented as a left (right) factorization. Existence of a left factorization enables us to construct an LTV ARMA system that realizes the impulse response sequence. Right factorization, on the other hand, can be realized as a state-space system. This fact is of independent interest, but is not used subsequently in this paper. Finally, we show that the existence of left (right) factorization in general does not imply the existence of respectively right (left) factorization, although, for a generic transfer function, that is indeed the case. This fact should be contrasted with linear time-invariant transfer functions, wherein existence of one implies the existence of the other.

3. A nonrecursive compensator design technique for simultaneous coefficient assignment. In this section, we shall consider Problem 1.4 regarding simultaneous coefficient assignment of an r-tuple of m input p output LTV ARMA systems by a single LTV ARMA compensator. We also show that the coefficient assignment problem can be used to analyze Problem 1.3 as well.

To introduce the problem, let us consider the r-tuple of plants defined in (1.1). Assume that the transfer function of the jth plant is given by

$$G^j(z^{-1}) = \mathcal{D}^{j^{-1}}(z^{-1})\mathcal{N}^j(z^{-1}), \tag{3.1}$$

where

$$\begin{aligned} \mathcal{D}^j(z^{-1}) &= I + \sum_{i=1}^{\ell} D_k^j(i) z^{-i}, \\ \mathcal{N}^j(z^{-1}) &= \sum_{i=1}^{\ell} N_k^j(i) z^{-i}, \end{aligned} \tag{3.2}$$

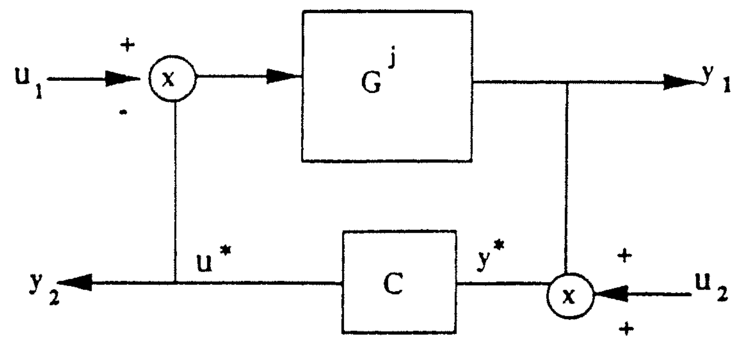

FIG. 3.1. *A two-input two-output configuration of the closed-loop system.*

$j = 1, 2, \ldots, r$. Furthermore, consider a compensator with transfer function

$$\mathcal{C}(z^{-1}) = \bar{\mathcal{N}}(z^{-1})\bar{\mathcal{D}}^{-1}(z^{-1}), \tag{3.3}$$

where

$$\begin{aligned} \bar{\mathcal{D}}(z^{-1}) &= I + \sum_{i=1}^{q} \bar{D}_k(i)z^{-i}, \\ \bar{\mathcal{N}}(z^{-1}) &= \sum_{i=0}^{q} \bar{N}_k(i)z^{-i}. \end{aligned} \tag{3.4}$$

It may be noted that the compensator (3.3) may not have a left representation, and therefore, may not have an LTV ARMA realization unless it satisfies the generic conditions of Theorem 2.13. Assume that the plants and the compensator are put in the configuration given by Fig. 3.1.

The following is the transfer function of the closed-loop system with respect to the jth plant.

$$\begin{bmatrix} y_{1,k} \\ y_{2,k} \end{bmatrix} = \begin{bmatrix} (\bar{\mathcal{D}} \circ (\mathcal{D}^j \circ \bar{\mathcal{D}} + \mathcal{N}^j \circ \bar{\mathcal{N}})^{-1} \circ \mathcal{N}^j) & (-I + \bar{\mathcal{D}} \circ (\mathcal{D}^j \circ \bar{\mathcal{D}} + \mathcal{N}^j \circ \bar{\mathcal{N}})^{-1} \circ \mathcal{D}^j) \\ (\bar{\mathcal{N}} \circ (\mathcal{D}^j \circ \bar{\mathcal{D}} + \mathcal{N}^j \circ \bar{\mathcal{N}})^{-1} \circ \mathcal{N}^j) & (\bar{\mathcal{N}} \circ (\mathcal{D}^j \circ \bar{\mathcal{D}} + \mathcal{N}^j \circ \bar{\mathcal{N}})^{-1} \circ \mathcal{D}^j) \end{bmatrix} \cdot \begin{bmatrix} u_{1,k} \\ u_{2,k} \end{bmatrix} j = 1, 2, \ldots, r. \tag{3.5}$$

Note that in (3.5), $\bar{\mathcal{D}}, \bar{\mathcal{N}}, \mathcal{D}^j, \mathcal{N}^j$ stands for $\bar{\mathcal{D}}(z^{-1}), \bar{\mathcal{N}}(z^{-1}), \mathcal{D}^j(z^{-1}), \mathcal{N}^j(z^{-1})$ defined in (3.2) and (3.4). Whenever convenient, in the future we will suppress z^{-1}. We now consider the following simultaneous coefficient assignment problem.

PROBLEM 3.1 (coefficient assignment problem). Given an r-tuple of plants described by (3.1), find a compensator of the type (3.3) such that

$$\mathcal{D}^j \circ \bar{\mathcal{D}} + \mathcal{N}^j \circ \bar{\mathcal{N}} = I + \sum_{i=1}^{\ell+q} \Delta_k^j(i) z^{-i} \tag{3.6}$$

for a prespecified set of coefficients $\Delta_k^j(i), j = 1, \ldots, r; i = 1, \ldots, l + q$.

For time-invariant systems, the quantity in the left-hand side of (3.6) is the *return difference*, determinant of which is the closed-loop characteristic polynomial. Thus, Problem 3.1 is the analogue of the pole placement problem for time-invariant systems.

It is not a priori clear how the coefficients $\Delta_k^j(i)$ are related to the input–output response of the closed-loop system (3.5). In particular, we might ask the following question: *If a plant is coefficient assignable by some choice of a compensator, is the closed loop-system stable in the bounded input bounded output sense?* For a time-invariant system, the answer to this question is always affirmative. For a time-varying system, the BIBO stability is not necessarily guaranteed. To ascertain the BIBO stability of the closed-loop system, the compensator coefficients have to be uniformly bounded. To examine this question, let us consider the transfer function between $y_{1,k}$ and $u_{1,k}$ for the jth plant given by

$$y_{1,k} = [\bar{\mathcal{D}} \circ (\mathcal{D}^j \circ \bar{\mathcal{D}} + \mathcal{N}^j \circ \bar{\mathcal{N}})^{-1} \circ \mathcal{N}^j] u_{1,k}, \tag{3.7}$$

which may be written as a cascade of two interconnected subsystems given by

$$y_{1,k} = \bar{\mathcal{D}} u_{1,k}^*, \tag{3.8}$$

$$[\mathcal{D}^j \circ \bar{\mathcal{D}} + \mathcal{N}^j \circ \bar{\mathcal{N}}] u_{1,k}^* = \mathcal{N}^j u_{1,k}. \tag{3.9}$$

Clearly, (3.7) is BIBO stable if each of the two subsystems (3.8) and (3.9) are BIBO stable. If we assume that the coefficients of the plants $D_k^j(i)$ and $N_k^j(i)$ are bounded uniformly in k, then an important question to ask is whether or not Problem 3.1 can be solved by a compensator with coefficients $\bar{D}_k(i), \bar{N}_k(i)$, bounded uniformly in k. We therefore consider the following problem.

PROBLEM 3.2 (bounded coefficient assignment problem). Given an r-tuple of plants described by (3.1), find a compensator with coefficients $\bar{D}_k(i), \bar{N}_k(i)$ uniformly bounded in time k such that (3.6) is satisfied for a prespecified set of coefficients $\Delta_k^j(i)$.

Note that if N^j is uniformly bounded, then for an appropriate choice of $\Delta_k^j(i)$, the input–output system (3.9) can be made BIBO stable provided that the coefficients $\Delta_k^j(i)$ are assignable. This fact follows easily from Desoer [10] and has been subsequently studied in detail by Bouthellier [27]. The basic idea is to choose $\Delta_k^j(i)$ such that they are slowly varying in between any two consecutive times. Following [10] and [27], we could construct a chain of open neighborhood Ω_k in the space of coefficients such that for all k, we have

$$(\Delta_k^j(1), \Delta_k^j(2), \ldots, \Delta_k^j(\ell + q)) \in \Omega_k.$$

For the above choice of coefficients, Problem 3.2 would guarantee simultaneous BIBO stabilizability of the r-tuple of plants. The main result of this section is described below.

THEOREM 3.3. *A generic r-tuple of $p \times m$ plant is coefficient assignable if and only if*

$$p + m > rp. \tag{3.10}$$

Furthermore, if (3.10) *is satisfied, then the r-tuple is coefficient assignable by a compensator of lag q where q is the smallest integer satisfying*

$$q[m + p - rp] \geq rp\ell - m, \tag{3.11}$$

where ℓ is the lag of each of the r plants.

To get an idea as to how tight the bounds (3.10) and (3.11) are, we consider the following theorem.

THEOREM 3.4. *A generic r-tuple of $p \times m$ plants can be assigned with bounded coefficients uniformly for all k and for all plants in the generic set by some choice of feedback* *compensator (where the compensator can depend on the choice of the r-tuple of plants) if and only if* (3.10) *is satisfied.*

Thus, for a generic set of r-tuple of plants, if (3.10) is not satisfied, then not only is it not possible to assign coefficients simultaneously, but it is also not possible to restrict the coefficients to a bounded set uniformly in k for all plants in a generic set. We now consider the proofs of Theorems 3.3 and 3.4.

Proof of Theorem 3.3. Consider the r-tuple of plants (3.1) together with the compensator (3.3). In the notation described by (3.2), (3.4) we can equate the like powers of z^{-1} in (3.6) to obtain the following linear equations:

$$M_k^j \nu_k = \Delta_k^j, \tag{3.12}$$

where

$$\nu_k = \text{col}\,[I, \bar{D}_{k+1}(1), \bar{D}_{k+2}(2), \ldots, \bar{D}_{k+q}(q), \bar{N}_k(0), \bar{N}_{k+1}(1), \ldots, \bar{N}_{k+q}(q)], \tag{3.13}$$

$$\Delta_k^j = \text{col}\,[\Delta_{k+1}^j(1), \Delta_{k+2}^j(2), \ldots, \Delta_{k+\ell+q}^j(\ell+q)], \tag{3.14}$$

$M_k^j =$

$$\begin{bmatrix} D_{k+1}^j(1) & I & & O & N_{k+1}^j(1) & O & & O \\ D_{k+2}^j(2) & D_{k+2}^j(1) & & \vdots & N_{k+2}^j(2) & N_{k+2}^j(1) & & \vdots \\ \vdots & D_{k+3}^j(2) & & O & \vdots & N_{k+3}^j(2) & & \vdots \\ \cdot & & \cdots & I & & \cdot & \cdots & O \\ D_{k+\ell}^j(\ell) & \vdots & & D_{k+q+1}^j(1) & N_{k+\ell}^j(\ell) & \vdots & & N_{k+q+1}^j(1) \\ O & D_{k+\ell+1}^j(\ell) & & D_{k+q+2}^j(2) & O & N_{k+\ell+1}^j(\ell) & & N_{k+q+2}^j(2) \\ O & O & & \vdots & \vdots & O & & \cdot \\ O & \vdots & & \cdot & \cdot & \vdots & & \vdots \\ O & O & & D_{k+q+\ell}^j(\ell) & O & O & & N_{k+q+\ell}^j(\ell) \end{bmatrix} \tag{3.15}$$

for $j = 1, 2, \ldots, r$. If we now define the matrix

$$M_k = \text{col}\,(M_k^1, M_k^2, \ldots, M_k^r) \tag{3.16}$$

and the matrix

$$\Delta_k = \text{col}\,(\Delta_k^1, \Delta_k^2, \ldots, \Delta_k^r) \tag{3.17}$$

we can combine the r linear equations (3.12) as

$$M_k \nu_k = \Delta_k. \tag{3.18}$$

It is easy to check that M_k is a $rp(\ell+q) \times (q+1)(m+p)$ matrix, ν_k is a $(q+1) \times (m+p) \times p$ matrix, and Δ_k is a $rp(\ell+q) \times p$ matrix. It follows that given M_k and Δ_k we can solve (3.18) for a suitable ν_k if and only if

$$rp(\ell+q) \le (q+1)(m+p) - p, \tag{3.19}$$

and M_k is of full column rank for each k. The inequality (3.19) follows from the requirement that (3.18) is solvable if and only if the matrix M_k after deleting the first p columns has more columns than rows. Note that the inequality (3.19) is the same as the inequality (3.11). The proof of this theorem is now complete by noting that generically the rows of M_k^* obtained from M_k by deleting the first p columns are all independent. The proof of this last statement is technical, and we refer to [27] for details. The basic idea is to choose a minor for M_k with nonidentically vanishing determinant. □

Proof of Theorem 3.4. The sufficiency part of Theorem 3.2 follows from Theorem 3.1. We now have to prove the necessity part.

Let λ be a variable that takes on values 1, 2, 3, ... Assume that the r-tuple of plants (3.1) is generically coefficient assignable by the compensator (3.3) and assume that the coefficients $\Delta_k^j(i)$ are all bounded with respect to some matrix norm; i.e., there exists $M > 0$ such that

$$\|\Delta_k^j(i)\| \leq M \tag{3.20}$$

for all $j = 1, \ldots, r; i = 1, 2, \ldots, l + q$ and $k = 0, 1, \ldots$. Let $\mathcal{P}^r$ be the space of r-tuples of plants equipped with the product topology similar to that described in (2.25). We now describe a map Ψ_λ for each λ given by

$$\Psi_\lambda : \mathcal{P}^r \to \mathcal{P}^r \tag{3.21}$$

described as

$$D_k^j(i) \mapsto \lambda^{-i} D_k^j(i),$$
$$N_k^j(i) \mapsto \lambda^{-i} N_k^j(i).$$

It follows that Ψ_λ maps a generic set of r-tuples of plants to a generic set of r-tuples of plants. For each λ we now define the compensator

$$\mathcal{C}_\lambda(z^{-1}) = \bar{\mathcal{N}}_\lambda(z^{-1})\bar{\mathcal{D}}_\lambda^{-1}(z^{-1}),$$

where

$$\begin{aligned} \bar{\mathcal{D}}_\lambda(z^{-1}) &= I + \sum_{i=1}^{q} \lambda^{-i}\bar{\mathcal{D}}_k(i) z^{-i}, \\ \bar{\mathcal{N}}_\lambda(z^{-1}) &= \sum_{i=0}^{q} \lambda^{-i}\bar{\mathcal{N}}_k(i) z^{-i}. \end{aligned} \tag{3.22}$$

Thus, we conclude that for each λ, there exists an open and dense set S_λ such that every r-tuple of plants in S_λ can be assigned with coefficients $\Delta_k^j(i)$, by a compensator of type (3.22), such that

$$\|\lambda^i \Delta_k^j(i)\| \leq M \tag{3.23}$$

for $i = 1, 2, \ldots, \ell + q$. Define

$$U = \cap_{\lambda=1}^{\infty} S_\lambda. \tag{3.24}$$

Thus, for every r-tuple of plants in U, there is a sequence of compensators such that the corresponding closed-loop system has coefficients in an arbitrary small neighborhood of 0. However, the map from the space of compensators to the space of coefficients is a linear map described by (3.18). It follows that the image of this linear map is closed. Hence, for every r-tuple of plants in U, there exists a compensator that places the coefficients $\Delta_k^j(i)$ at 0. In other words, we can solve the system of equation

$$M_k \nu_k = 0. \tag{3.25}$$

The proof of Theorem 3.4 is now completed by showing that there exist open sets of r-tuples of plants for which (3.25) is not satisfied if $rp \geq p + m$. This is done as follows.

Define

$$\begin{aligned} D_j^*(i) &= \text{col } [D_j^1(i), D_j^2(i), \ldots, D_j^r(i)], \\ N_j^*(i) &= \text{col } [N_j^1(i), N_j^2(i), \ldots, N_j^r(i)]. \end{aligned} \tag{3.26}$$

We now make specific choices of $D_j^*(i), N_j^*(i)$ as follows. As $m + p \leq rp$, which implies $rp\ell > m$, we set

(i) the $rp\ell \times m$ matrix

$$\text{col } [N_{k+1}^*(1), N_{k+2}^*(2), \ldots, N_{k+\ell}^*(\ell)] \equiv [e_1, e_2, \ldots, e_m], \tag{3.27}$$

where e_i is the ith standard basis vector in $\mathbb{F}^{rp\ell}, i = 1, \ldots, m$;

(ii) the first column of the matrix

$$\text{col } [-D_{k+1}^*(1), -D_{k+2}^*(2), \ldots, -D_{k+\ell}^*(\ell)]$$

to be e_{m+1}, where e_{m+1} is the $m+$ 1st standard basis vector in $\mathbb{F}^{rp\ell}$; and

(iii) the $rp \times (p + m)$

$$[D_{k+\ell+j}^*(\ell), N_{k+\ell+j}^*(\ell)] \equiv [e_1, e_2, \ldots, e_{m+p}], \tag{3.28}$$

where e_i is the ith standard basis vector in $\mathbb{F}^{rp}, i = 1, \ldots, m + p$ for $j = 1, \ldots, q$.

For the above choice of r-tuple of plants, it can be shown that (3.25) cannot be solved (see [27] for details). Furthermore, in any neighborhood of the coefficient space of the above r-tuple of plants, (3.25) has no solution. This concludes the proof. □

Remark. The proof of Theorem 3.4 is an adaptation of a technique due originally to Anderson and Byrnes [29].

Remark 3.5. We now state and prove a result that addresses Problem 3.2.

THEOREM 3.6. *A bounded set S of r-tuple of $p \times m$ plants is coefficient assignable simultaneously by a compensator with bounded coefficients if* (3.10) *is satisfied and if*

$$\det [M_k M_k^T] > \epsilon \tag{3.29}$$

for some $\epsilon > 0$, which is independent of k, and for all r-tuples of plants in S.

Note that as a result of condition (3.29), S fails to be a generic set. The proof of Theorem 3.6 follows from the following simple and well-known proposition.

PROPOSITION 3.7. *Let z be an m vector and A be an m by n matrix of rank m. The n vector x such that $Ax = z$ and $x^T x$ is minimum is given by*

$$x = A^T (AA^T)^{-1} z \tag{3.30}$$

For a proof of the above proposition see Brockett [30, p. 127].

Proof of Theorem 3.6. Our basic problem is to solve (3.18) for a uniformly bounded ν_k. Clearly, in view of Proposition 3.7, the solution

$$\nu_k^* = M_k^T(M_k M_k^T)^{-1}\Delta_k \tag{3.31}$$

has the property that the columns of ν_k^* have minimum norm. Since the r-tuple of plants and the coefficients Δ_k are all bounded uniformly in k, it follows that under the assumption (3.29), ν_k^* is uniformly bounded as well. □

To conclude this section, we reiterate the three important questions that we address in this section.

(a) For a generic set of r-tuple of $p \times m$ plants, when is it possible to coefficient assign simultaneously?

(b) For a generic set of r-tuple of $p \times m$ plants, when is it possible to assign a bounded set of coefficients simultaneously?

(c) For a set of r-tuple of $p \times m$ plants, when is it possible to assign coefficients simultaneously by a compensator with coefficients bounded uniformly in k?

4. A recursive formulation of the coefficient assignment problem. We begin this section with the remark that, so far in this paper, Problem 1.4 has not been considered. Instead, in § 3, the closed-loop system was decomposed into ARMA and moving average subsystems. The design problem considered has been to assign the coefficients of the ARMA subsystem while maintaining an uniform bound on the coefficients of the compensators. Such a design problem leads to a simplified algorithm. To implement the algorithm, we need to solve linear equations.

We now consider Problem 1.4 for a single plant. The case for an r-tuple of plants is analogous and is not described in detail. Assume that the plants and the compensator are put in the configuration given by Fig. 3.1. For simplicity, we only consider the transfer function between y_1 and u_1. However, unlike that in § 3, we will not decompose the transfer function (3.7) into a cascade of two transfer functions (3.8), (3.9). We will see shortly in this section that this introduces new problems, namely, the compensator parameters are not obtained by solving static linear equations one for each time. In general, Problem 1.4 reduces to a nonlinear discrete iteration on the parameter space of compensators. The algorithm, although more complicated, iteratively solves this coefficient assignment problem.

Consider the transfer function (3.7) for a single plant (i.e., assume $j = 1$). Define

$$\mathcal{X}(z^{-1}) = \sum_{i=0}^{q} X_k(i)z^{-i}. \tag{4.1}$$

$$\Delta(z^{-1}) = \sum_{i=0}^{\ell+q} \Delta_k(i)z^{-i} \tag{4.2}$$

such that

$$\bar{\mathcal{D}} \circ (\mathcal{D} \circ \bar{\mathcal{D}} + \mathcal{N} \circ \bar{\mathcal{N}})^{-1} = \Delta^{-1} \circ \mathcal{X}. \tag{4.3}$$

The transfer function (3.7) can be written as

$$\Delta(z^{-1})y_k = \mathcal{X}(z^{-1}) \circ \mathcal{N}(z^{-1})u_k \tag{4.4}$$

PROBLEM 4.1 (the coefficient assignment problem). Given $\mathcal{N}(z^{-1})$ and $\mathcal{D}(z^{-1})$, find, if possible, an $\bar{\mathcal{N}}(z^{-1}), \bar{\mathcal{D}}(z^{-1})$ such that $\Delta_k(i), i = 1, \ldots, \ell+q$ can be assigned a prespecified set of coefficients.

Problem 4.1 can be stated equivalently by rewriting (4.3) as

$$\mathcal{X} \circ (\mathcal{D} \circ \bar{\mathcal{D}} + \mathcal{N} \circ \bar{\mathcal{N}}) = \Delta \circ \bar{\mathcal{D}} \tag{4.5}$$

To solve (4.5) we equate like powers of $z^{-1}, i = 0, 1, \ldots, \ell + 2q$ for all $k \geq i$ and solve for $\bar{\mathcal{D}}$ and $\bar{\mathcal{N}}$. This will be accomplished by considering two sets of equations.

(A) The set of equations derived by equating like powers of $z^{-1}, i = \ell + q, \ldots, \ell + 2q$ in (4.5), and

(B) The set of equations derived by equating like powers of $z^{-1}, i = 0, \ldots, \ell + q - 1$ in (4.5).

Using the above two sets of equations, we derive an iterative scheme that will allow us to solve (4.5). From the set of equations (A) as described above, we get the matrix equation

$$S_k \phi_k = \psi_k, \tag{4.6}$$

where S_k is a $(q+1)p \times (q+1)p$ matrix defined by

$$\begin{bmatrix} \zeta^T_{k+\ell+q-1}(\ell+q) & \zeta^T_{k+\ell+q-2}(\ell+q-1) & \zeta^T_{k+\ell+q-3}(\ell+q-2) & \cdots & \zeta^T_{k+\ell}(\ell+1) & \zeta^T_{k+\ell-1}(\ell) \\ 0 & \zeta^T_{k+\ell+q-2}(\ell+q) & \zeta^T_{k+\ell+q-3}(\ell+q-1) & \cdots & \zeta^T_{k+\ell}(\ell+2) & \zeta^T_{k+\ell-1}(\ell+1) \\ 0 & 0 & \zeta^T_{k+\ell+q-3}(\ell+q) & \cdots & \zeta^T_{k+\ell}(\ell+3) & \zeta^T_{k+\ell-1}(\ell+2) \\ & & \vdots & & & \\ 0 & 0 & 0 & 0 & \zeta^T_{k+\ell}(\ell+q) & \zeta^T_{k+\ell-1}(\ell+q-1) \\ 0 & 0 & 0 & 0 & 0 & \zeta^T_{k+\ell-1}(\ell+q) \end{bmatrix} \tag{4.7}$$

and where we define

$$\mathcal{D}_k \circ \bar{\mathcal{D}}_k + \mathcal{N}_k \circ \bar{\mathcal{N}}_k = \sum_{i=0}^{\ell+q} \zeta_k(i) z^{-i}. \tag{4.8}$$

Moreover, ϕ_k and ψ_k are defined as follows. Note that ϕ_k is a $(q+1)p \times p$ matrix given by

$$\phi_k = \text{col}\,[X^T_{k+\ell+q-1}(0), X^T_{k+\ell+q-1}(1), \ldots, X^T_{k+\ell+q-1}(q-1), X^T_{k+\ell+q-1}(q)] \tag{4.9}$$

and ψ_k is a $(q+1)p \times p$ matrix given by

$$\psi_k = \text{col}\left[\sum_{j=0}^{q} \bar{D}^T_{k+q-j-1}(q-j)\Delta^T_{k+\ell+q-1}(\ell+j), \right.$$
$$\cdot \sum_{j=0}^{q-1} \bar{D}^T_{k+q-j-2}(q-j)\Delta^T_{k+\ell+q-1}(\ell+j+1), \ldots,$$
$$\left. \cdot \sum_{j=0}^{q-i} \bar{D}^T_{k+q-j-i-1}(q-j)\Delta^T_{k+\ell+q-1}(\ell+j+i), \ldots \bar{D}^T_{k-1}(q)\Delta^T_{k+\ell+q-1}(\ell+q)\right]. \tag{4.10}$$

Similarly, from the set of equations (B) we obtain

$$M_k \nu_k = O, \tag{4.11}$$

where ν_k is defined as follows:

$$\nu_k = \text{col}\,[\bar{D}_k(0), \bar{D}_{k+1}(1), \dots, \bar{D}_{k+q}(q); \bar{N}_k(0), \bar{N}_{k+1}(1), \dots, \bar{N}_{k+q}(q)].$$

The matrix M_k is a $(\ell + q)p \times (q+1)(p+m)$ matrix that can be shown to be a function of x_k, Δ_k and the plant parameters. If we assume that

$$(q+1)m > p(\ell - 1),$$

it follows that (4.11) can be solved for a nonzero solution.

Using (4.6) and (4.11), we are now in a position to solve (4.5), and hence, the coefficient assignment problem, in an iterative fashion. However, before we consider the following coefficient assignment algorithm, for the sake of clarity, we will provide an overview of the basic idea. First, we will initialize the algorithm by choosing suitable (timewise) values of the plant, compensator, and parameters to be assigned (i.e., the $\Delta_j(i)$). Having obtained these values, we are then able to solve for the next set (timewise) of compensator parameters ν_k via (4.11). These values of ν_k are then used in (4.6) to solve for the next set of $X_k(i) i = 0, \dots, q$. These values of X_k are then substituted into (4.11) to solve for ν_{k+1}, which is used in turn to solve for X_{k+1} etc. ... It will be assumed that in the following coefficient assignment algorithm (4.6) and (4.11) admit solutions for all times k and that $\det \bar{D}_k(0) \neq 0$ for all k so that $\bar{D}^{-1}$ always exists.

The coefficient assignment algorithm

Step I (initialize the algorithm). Choose values for

(i) $\bar{D}_j(i), \bar{N}_j(i) i = 0, \dots, q, j \leq i - 1$

(ii) $D_j(i), N_j(i) i = 0, \dots, \ell - 1, j \leq q + i - 1; i = \ell, j \leq \ell + q - 1$

(iii) $\Delta_j(i) i = 0, 1, \dots, \ell - 1, j \leq q + i; i = \ell, \dots, \ell + q, j \leq \ell + q - 1$

Step II. Solve (4.6) for $X_j(i) i = 0, \dots, q, j = 0, 1, \dots, \ell + q - 1$ and

Step III. Using the values of $X_j(i)$ computed in Step II compute ν_0 using (4.11).

Step IV. Set $k = 0$

Step V. Obtain an estimate of the future values of the plant parameters

(1) $D_{k+q+i+1}(i), N_{k+q+i+1}(i), i = 0, 1, \dots, \ell - 1$ and $D_{k+q+\ell}(\ell), N_{k+q+\ell}(\ell)$ and choose values for

(2) $\Delta_{k+q+i+1}(i) i = 0, 1, \dots, \ell - 1$ and $\Delta_{k+q+\ell}(i) i = \ell, \dots, \ell + q$

Step VI. Solve (4.6) for $X_{k+\ell+q}(i) i = 0, \dots, q$

Step VII. Solve (4.11) for ν_{k+1}

Step VIII. Set $k = k + 1$ and return to Step V.

Remark 4.2. The values required in Step I could be based on the available knowledge of the plant parameters $D_k(i), N_k(i) i = 0, \dots, \ell$ at time $k = 0$ and the values of $\Delta_k(i) i = 0, 1, \dots, \ell + q$, which have been specified. It should also be noted that the lag q of the compensator computed via the above algorithm is the smallest nonnegative integer, which satisfies $q > [p(\ell - 1) - m]/m$.

Remark 4.3. It can be shown that, using techniques similar to those of §3, that the above algorithm can be extended to simultaneously coefficient assign an r-tuple of $p \times m$ systems, where $r < m/p + 1$. We will not elaborate on this further and refer the interested reader to [31].

We now consider two illustrative examples of the above coefficient assignment algorithm.

Example 4.4. Consider the closed-loop system given by Fig. 1.1 (assume $j = 1$) where the plant G is given by

$$y_k + a_k y_{k-1} = b_k u_k + c_k u_{k-1}$$

and the compensator C is given by the gain feedback

$$u_k^* = \bar{d}_k^{-1} y_k.$$

Writing the plant as

$$y_k = (d_k^{-1} \circ n_k) u_k,$$

where

$$d_k(z^{-1}) = 1 + a_k z^{-1}; n_k(z^{-1}) = b_k + c_k z^{-1},$$

we obtain by Theorem 3.3 the equation of the closed-loop system as follows:

$$([d_k \circ \bar{d}_k + n_k] \circ \bar{d}_k^{-1}) y_k = n_k u_k. \tag{4.12}$$

Writing

$$(d_k \circ \bar{d}_k + n_k) \circ \bar{d}_k^{-1} = f_k^{-1} \circ (1 + e_k z^{-1}), \tag{4.13}$$

(4.12) reduces to

$$y_k + e_k y_{k-1} = f_k b_k u_k + f_k c_k u_{k-1}.$$

From (4.13) it follows that

$$f_k \circ (d_k \circ \bar{d}_k + n_k) = (1 + e_k z^{-1}) \circ \bar{d}_k,$$

i.e., equating like powers of z^{-1}

$$f_k(\bar{d}_k + b_k) = \bar{d}_k \quad \forall k \geq 0 \tag{4.14}$$

and

$$f_k(a_k \bar{d}_{k-1} + c_k) = e_k \bar{d}_{k-1} \quad \forall k \geq 1. \tag{4.15}$$

Eliminating $\bar{d}_k$ in (4.14) and (4.15), we obtain

$$f_{k+1} = \frac{f_k b_k e_{k+1}}{c_{k+1} - f_k[c_{k+1} - a_{k+1} b_k]} \quad \forall k \geq 0 \tag{4.16}$$

and

$$\bar{d}_k = \frac{f_k b_k}{1 - f_k} \quad \forall k \geq 0. \tag{4.17}$$

Given the plants parameters a_k, b_k, and c_k, and given e_k, the coefficient of the closed-loop system to be assigned, (4.16) describes a nonlinear recursion in f_k. Equation (4.17), on the other hand, is a nonlinear function that computes the feedback gain in real time.

Among several queations that we might ask about (4.16) and (4.17), an important one in terms of understanding the properties of the coefficient assignment algorithm is the following.

Question 4.5. If a_k, b_k, c_k, and e_k are time invariant and given, respectively, by a, b, c, and e, what is the asymptotic behavior of (4.16) and (4.17)?

Defining $\alpha = bc$, $\beta = c$, and $\gamma = -[c - ab]$, we obtain the following recursion from (4.16):

$$f_{k+1} = \frac{\alpha f_k}{\beta + f_k \gamma}. \tag{4.18}$$

Note that (4.18) has two stationary points 0 and $(\alpha - \beta)/\gamma$. It is easy to show that the second stationary point corresponds to the time-invariant solution; i.e., the corresponding value of the gain $\bar{d}$ equals the value of the gain, which assigns the parameter e in the closed-loop system if $a_\cdot, b_\cdot, c_\cdot$, and $e_\cdot$ were time invariant and known. The stationary point 0, on the other hand, corresponds to an infinite gain.

To examine the trajectory of f_{k+1} as defined by (4.18), we write

$$f_k = \frac{g_k}{h_k} \tag{4.19}$$

Substituting (4.19) into (4.18) yields

$$\frac{g_{k+1}}{h_{k+1}} = \frac{\alpha g_k}{\beta h_k + \gamma g_k},$$

which may be rewritten as

$$\begin{bmatrix} g_{k+1} \\ h_{k+1} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ \gamma & \beta \end{bmatrix} \begin{bmatrix} g_k \\ h_k \end{bmatrix}.$$

With an initial (nonzero) estimate f_0 of the true value of $f(=(\alpha - \beta)/\gamma)$ we may write

$$\begin{bmatrix} g_k \\ h_k \end{bmatrix} = -f_0 \alpha^k \begin{bmatrix} \alpha - \beta \\ \gamma \end{bmatrix} + [(\beta - \alpha) + \gamma f_0]\beta^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, if $|\alpha| > |\beta|$ (i.e., $|be| > |c|$)

$$\lim_{k \to \infty} f_k = \lim_{k \to \infty} \frac{g_k}{h_k} = \frac{\alpha - \beta}{\gamma}.$$

On the other hand, if $|\alpha| < |\beta|$,

$$\lim_{k \to \infty} f_k = 0.$$

We may summarize the above results as follows: for choices of a, b, c, and e for which $|be| < |c|$, the adaptive gain $\bar{d}_\cdot$ tends toward 0, the infinite gain. For choices of a, b, c, and e for which $|be| > |c|$, the adaptive gain converges to the unique time-invariant solution.

Remark 4.6. Example 4.4 illustrates that under suitable conditions on the coefficients of a time-invariant plant, the coefficient assignment algorithm may be viewed as a globally convergent adaptive controller. It may also be noted that when $\alpha = \beta = 1$ or when $\alpha = \beta = -1$, $|f_k|$ converges to 0 as k tends to infinity. On the other hand, when $\alpha = -\beta = 1$, f_k is periodic of period 2.

Example 4.7. Consider the closed-loop system defined by Fig. 1.1 (assume $j = 1$), where the plant G is given by

$$y_k + d_k(1)y_{k-1} + d_k(2)y_{k-2} = n_k(1)u_{k-1} + n_k(2)u_{k-2} \tag{4.20}$$

and the compensator C is given by

$$u_k^* + \bar{d}_k(1)u_{k-1}^* = \bar{n}_k(0)y_k + \bar{n}_k(1)y_{k-1}. \tag{4.21}$$

Define $\mathcal{X}_k(z^{-1})$ given by (4.1)–(4.3) as follows:

$$\mathcal{X}_k(z^{-1}) = 1 + X_k z^{-1}.$$

It can be shown that by writing the parameters of the compensator (4.21) in terms of the plant parameters and the $\Delta_j(i)$ it is possible to derive the following recursive equation for X_k:

$$X_{k+4} = \frac{f(X_{k+3}, X_{k+2}, X_{k+1})}{g(X_{k+3}, X_{k+2}, X_{k+1})}, \tag{4.22}$$

where

$$\begin{aligned} f(X_{k+3}, X_{k+2}, X_{k+1}) = {} & \phi_k(1)X_{k+3} + \phi_k(2)X_{k+2} + \phi_k(3)X_{k+1} + \phi_k(4)X_{k+3}X_{k+2} \\ & + \phi_k(5)X_{k+2}X_{k+1} + \phi_k(6)X_{k+3}X_{k+2}X_{k+1} + \phi_k(7) \end{aligned}$$

and

$$\begin{aligned} g(X_{k+3}, X_{k+2}, X_{k+1}) = {} & \phi_k(8)X_{k+3} + \phi_k(9)X_{k+2} + \phi_k(10)X_{k+1} + \phi_k(11)X_{k+3}X_{k+2} \\ & + \phi_k(12)X_{k+3}X_{k+1} + \phi_k(13)X_{k+2}X_{k+1} + \phi_k(14)X_{k+3}X_{k+2}X_{k+1} + \phi_k(15). \end{aligned}$$

In the above equation, $\phi_k(i) i = 1, \ldots, 15$ are nonlinear functions of the plant parameters and the parameters to be assigned at times $k+1, k+2, k+3$.

A complete analysis of recursions of the type (4.22) is a subject of future research. We would analyze (4.22) under certain special cases. If we denote X_k by y_k/ζ_k, we can rewrite the recursion (4.22) as follows.

$$\begin{aligned} y_{k+4} = {} & \phi_k(1)y_{k+3}\zeta_{k+2}\zeta_{k+1} + \phi_k(2)\,\zeta_{k+3}y_{k+2}\zeta_{k+1} \\ & + \phi_k(3)\,\zeta_{k+3}\zeta_{k+2}y_{k+1} + \phi_k(4)\,y_{k+3}y_{k+2}\zeta_{k+1} \\ & \phi_k(5)\,\zeta_{k+3}y_{k+2}y_{k+1} + \phi_k(6)\,y_{k+3}y_{k+2}y_{k+1} + \phi_k(7)\,\zeta_{k+3}\zeta_{k+2}\zeta_{k+1}. \\ \zeta_{k+4} = {} & \phi_k(8)\,y_{k+3}\zeta_{k+2}\zeta_{k+1} + \phi_k(9)\,\zeta_{k+3}y_{k+2}\zeta_{k+1} + \phi_k(10)\,\zeta_{k+3}\zeta_{k+2}y_{k+1} \\ & + \phi_k(11)\,y_{k+3}y_{k+2}\zeta_{k+1} + \phi_k(12)\,\zeta_{k+3}y_{k+2}y_{k+1} \\ & + \phi_k(13)\,y_{k+3}\zeta_{k+2}y_{k+1} + \phi_k(14)\,\zeta_{k+3}\zeta_{k+2}\zeta_{k+1} + \phi_k(15)\,y_{k+3}y_{k+2}y_{k+1}. \end{aligned}$$

If we assume without any loss of generality that $y_k^2 + \zeta_k^2 = 1$, we can reparameterize $y_k = \cos\theta_k, \zeta_k = \sin\theta_k$. Furthermore, if we choose

$$\begin{aligned} & \phi_k(1) = \phi_k(2) = \phi_k(3) = -1, \phi_k(6) = \phi_k(4) = \phi_k(5) = \phi_k(7) = 0, \\ & \phi_k(8) = \phi_k(9) = \phi_k(10) = \phi_k(15) = 0, \\ & \phi_k(11) = \phi_k(12) = \phi_k(13) = 1, \phi_k(14) = -1, \end{aligned} \tag{4.23}$$

we have

$$\cos \theta_{k+4} = \cos(\theta_{k+3} + \theta_{k+2} + \theta_{k+1}),$$
$$\sin \theta_{k+4} = \sin(\theta_{k+3} + \theta_{k+2} + \theta_{k+1}).$$

Thus, for the special choices of $\phi_k(\cdot)$ given by (4.23), the recursion (4.22) reduces to

$$\theta_{k+4} = \theta_{k+3} + \theta_{k+2} + \theta_{k+1}. \tag{4.24}$$

We now claim that (4.24) describes an Anosov flow [32] on T^3, the three-dimensional torus. Consider the system

$$\begin{bmatrix} \alpha_{k+1} \\ \beta_{k+1} \\ \nu_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_k \\ \beta_k \\ \nu_k \end{bmatrix} \tag{4.25}$$

on $\mathbb{F}^3$, where $\alpha_0 = \theta_0, \beta_0 = \theta_1$, and $\nu_0 = \theta_2$. It follows that $\nu_k = \theta_{k+2}$. Let the 3×3 matrix in (4.25) be denoted by A. Since all entries of A are integers, det $A = 1$ and A is hyperbolic, it follows that the map induced on T^3 by A is a hyperbolic toral automorphism, which we denote by L_A. It follows from [32, Thm. 4.8] that periodic points of L_A are dense in T^3, L_A is topologically transitive, and L_A has sensitive dependence on initial conditions. Thus, the hyperbolic toral automorphism is chaotic on all of T^3 (see [32, p. 197]).

What we conclude, therefore, is that for choices of $\phi_k(\cdot)$ given by (4.23), the time-varying coefficient assignment problem is chaotic. Thus, if we use adpative coefficient assignment as a strategy for compensation, we must carefully avoid chaotic dynamics.

5. Conclusion. In this paper, we have given conditions under which an input–output map for a time-varying system admits left and/or right matrix fraction representations. Using these representations, we have described procedures for the simultaneous coefficient assignment of a family of time-varying systems using nonrecursive algebraic techniques. It is important to note that these techniques are generalizations of well-known design methodologies in time-invariant system theory [1]–[3] to the time-varying case. When the number of input and output channels is such that the conditions of the nonrecursive coefficient assignment scheme is not satisfied, recursive procedures to coefficient assign time-varying systems are given in the form of a recursive algorithm. These recursive procedures have no analogues in the time-invariant case, and thus, represent a new design procedure. For certain special cases, solution of the proposed recursive algorithm is shown to be chaotic. This fact indicates that further work needs to be done and a complete analysis of the algorithms given in §4 is a subject of future research.

6. Appendix I. The purpose of this appendix is to prove Theorem 2.13. Let us consider a right factorization of lag $\bar{\ell}$ given by (2.24). Let us also consider a left factorization of the form (2.12) of lag ℓ. The two representations have the same input–output properties provided:

$$\left[\sum_{i=0}^{\bar{\ell}} \bar{N}_k(i)z^{-i}\right] \circ \left[I + \sum_{i=1}^{\bar{\ell}} \bar{D}_k(i)z^{-i}\right]^{-1} = \left[I + \sum_{i=1}^{\ell} D_k(i)z^{-i}\right]^{-1} \circ \left[\sum_{i=0}^{\ell} N_k(i)z^{-i}\right] \tag{6.1}$$

or, equivalently,

$$\left[I + \sum_{i=1}^{\ell} D_k(i) z^{-i}\right] \circ \left[\sum_{i=0}^{\bar{\ell}} \bar{N}_k(i) z^{-i}\right] - \left[\sum_{i=0}^{\ell} N_k(i) z^{-i}\right] f \circ \left[I + \sum_{i=1}^{\bar{\ell}} \bar{D}_k(i) z^{-i}\right] = 0. \tag{6.2}$$

Expanding (6.2) and equating like powers of z^{-1}, we obtain

$$N_k(0) = \bar{N}_k(0) \text{ for } k = 0, 1, 2, \ldots$$

and

$$\nu_k M_k = \phi_k \text{ for } k = 0, 1, 2, \ldots, \tag{6.3}$$

where

$$\nu_k = [D_k(1), D_k(2), \ldots, D_k(\ell), N_k(1), N_k(2), \ldots, N_k(\ell)],$$
$$\phi_k = [\bar{N}_k(0)\bar{D}_k(1) - \bar{N}_k(1), \ldots, \bar{N}_k(0)\bar{D}_k(\bar{\ell}) - \bar{N}_k(\bar{\ell}); 0, \ldots, 0],$$

$M(k) =$

$$\begin{bmatrix} \bar{N}_{k-1}(0) & \bar{N}_{k-1}(1) & \bar{N}_{k-1}(2) & \cdots & \bar{N}_{k-1}(\bar{\ell}) & O & O & O \\ O & \bar{N}_{k-2}(0) & \bar{N}_{k-2}(1) & & \cdots & \bar{N}_{k-2}(\bar{\ell}) & O & O \\ & & \vdots & & & & & \\ O & O & \bar{N}_{k-\ell}(0) & & \cdots & & & \bar{N}_{k-\bar{\ell}}(\bar{\ell}) \\ -I & -\bar{D}_{k-1}(1) & -\bar{D}_{k-1}(2) & \cdots & -\bar{D}_{k-1}(\bar{\ell}) & O & O & O \\ O & -I & -\bar{D}_{k-2}(1) & & \cdots & -\bar{D}_{k-2}(\bar{\ell}) & O & O \\ & & \vdots & & & & & \\ O & O & \cdots & -I & -\bar{D}_{k-\ell}(1) & \cdots & & -\bar{D}_{k-\ell}(\bar{\ell}) \end{bmatrix}. \tag{6.4}$$

As ν_k is a $p \times \ell(p+m)$ matrix, ϕ_k is a $p \times (\ell+\bar{\ell})m$ matrix and M_k is a $\ell(p+m) \times (\ell+\bar{\ell})m$ matrix; it follows that a sufficient condition for (6.3) to have a solution ν_k is that M_k is of full row rank and $\ell(p+m) \geq (\ell+\bar{\ell})m$, i.e., if $\ell p \geq \bar{\ell} m$.

It is not too hard to check (see [27] for details) that the condition that M_k is not of full row rank is given by a proper algebraic set in $\mathcal{P}$ (in the topology described in §2 (25)). In fact, for $k = \ell + \tau$, the condition that M_k is not of full row rank is obtained as proper algebraic set in the restriction of $\mathcal{P}$ to

$$\prod_{j=\tau}^{\ell-1+\tau} \mathbb{F}^N$$

for $\tau = 0, 1, 2, \ldots$. Thus, there is a countable intersection of open and dense set in $\mathcal{P}$ for which M_k is of full row rank for all $k \geq \ell$.

Remark. In general, it is not entirely obvious why the associated algebraic sets in $\mathcal{P}$ that makes M_k singular is proper. The proof consists of picking a minor of M_k with nonidentically vanishing determinant. The details being technical are relegated to [27].

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