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AN APPROACH TO SIMULTANEOUS SYSTEM DESIGN, I. SEMIALGEBRAIC GEOMETRIC METHODS*

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Abstract. This paper introduces semialgebraic parameterization as an approach to analyze simultaneous stabilization and pole placement problems. Rational families of plants of a given McMillan degree, that are simultaneously stabilizable by a fixed family of compensators, are parameterized. For a discrete family of plants, the parameterization problem reduces to the simultaneous stabilization or the pole placement problem of a r-tuple of multi input multi output plants by a nonswitching compensator. It is shown that by removing a semialgebraic subset of a proper algebraic set, the "space of plants" can be decomposed into components that are either simultaneously stabilizable or simultaneously unstabilizable. Under special cases, explicit parameterization of the semialgebraic set is obtained. Finally a necessary condition for the simultaneous stabilization of single input or single output plants is obtained.

Key words. plant, compensator, semialgebraic-set, decision-theory

AMS(MOS) subject classifications. 93, 14

1. Introduction. Classically, in control theory one considers a lumped, linear, time-invariant, proper or strictly proper plant and one of the design objectives is to construct an output feedback scheme that would stabilize the plant. Not all plants have a compensator that satisfies the specified design constraints and it is of interest to parameterize those plants that do. In this paper, the semialgebraic properties of the parameterization problem is studied. In a related part II we study this problem via algebraic geometric methods.

In order to introduce the class of problems to be investigated, we consider a $p\times m$ rational, transfer function matrix $G(s)$ modelling a m input p output plant and address the following problem:

Problem 1.1. Let S be a topological space. Consider a family of plants $G_\lambda(s)$ parameterized by $\lambda\in S$. Does there exist a compensator $K_{f(\lambda)}(s)$ where

$$f: S\to S_1\subset S \tag{1.1}$$

such that the closed loop systems $G_\lambda[I+K_{f(\lambda)}(s)G_\lambda(s)]^{-1}$ are stable for all $\lambda\in S$.

If the answer to Problem 1.1 is "yes", one asks the following parameterization problem.

Problem 1.2. Let f be fixed. Describe the family of plants $G_\lambda(s)$ for which there exists a family of compensators $K_{f(\lambda)}(s)$ such that the closed loop system $G_\lambda(s)[I+K_{f(\lambda)}(s)G_\lambda(s)]^{-1}$ is stable for all $\lambda\in S$.

If $S=S_1$ and f is the identity map, an adaptive control problem, called the "switching compensator problem" is obtained. If f is a constant map, the so-called "nonswitching compensator problem" also known as the "blending problem" is obtained. An important class of the noswitching compensator problem, called the "simultaneous stabilization problem" arises when the set S is discrete. The problem is to describe the set of r-tuples of plants $G_1(s),\cdots,G_r(s)$ that admit a stabilizing compensator. Let us now consider the following two examples.

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Example 1.1 (*a switching compensator parobelm*). Let S be the set of real numbers $\mathbb{R}$. Consider $G_\lambda(s)$ to be a family of plants of degree 1 given by $G_\lambda(s) = 1/(s+\lambda^2)$, where $\lambda \in \mathbb{R}$. Let K_λ be a family of feedback gains given by $K_\lambda = k_1\lambda + k_2$ where $k_1, k_2, \lambda \in \mathbb{R}$. Let us now ask the following question: Does there exist some values of $k_1, k_2 \in \mathbb{R}^2$ such that the closed loop system $G_\lambda(s)[1+K_\lambda G_\lambda(s)]^{-1}$ is stable for all $\lambda \in \mathbb{R}$?

Of course it is easy to see that K_λ stabilizes $G_\lambda(s)$ for all $\lambda \in \mathbb{R}$ iff

$$\lambda^2 + k_1\lambda + k_2 > 0. \tag{1.2}$$

Clearly, for

$$k_1^2 < 4k_2, \tag{1.3}$$

the inequality (1.2) is satisfied for all $\lambda \in \mathbb{R}$. Therefore, the family of plants $G_\lambda(s)$ is stabilizable by the family of switching compensators $K_\lambda(s)$ provided (1.3) is satisfied.

Example 1.2 (*a parameterization problem*). As a continuation of Example 1.1, let $G_\lambda(s) = a/(bs+\lambda^2)$, $K_\lambda = k_1\lambda + k_2$. Let us now ask the following question: For which $a, b \in \mathbb{R}^2$ does there exist $k_1, k_2 \in \mathbb{R}^2$ such that the closed loop system $G_\lambda(s)[1+K_\lambda G_\lambda(s)]^{-1}$ is stable for all $\lambda \in \mathbb{R}$?

Once again, K_λ stabilizes $G_\lambda(s)$ for all $\lambda \in \mathbb{R}$ iff

$$b(\lambda^2 + ak_1\lambda + ak_2) > 0. \tag{1.4}$$

Eliminating the variables $k_1, k_2 \in \mathbb{R}^2$ from the above inequation, we may check that the set of $a, b, \lambda \in \mathbb{R}^3$ for which there exists some $k_1, k_2 \in \mathbb{R}^2$ satisfying (1.4) is given by

$$(b>0 \text{ and } a=0 \text{ and } \lambda \neq 0) \quad \text{or} \quad (a \neq 0). \tag{1.5}$$

The set of $(a, b) \in \mathbb{R}^2$ for which (1.5) is satisfied for all $\lambda \in \mathbb{R}$ is given by

$$\{(a, b) | a \neq 0\}, \tag{1.6}$$

which is also the required solution to the parameterization problem.

As pointed out in [15], the compensator problem 1.1 and the parameterization Problem 1.2 is encountered in reliability studies. One frequently encounters situations when it is desirable to stabilize a plant with multiple modes of operation. For example, if $G_1(s)$ models a plant in its nominal mode, one might consider $G_2(s), \cdots, G_r(s)$ as the models of the plant in the faulted mode. It might be desirable to construct a feedback nonswitching compensator that simultaneously stabilizes $G_1(s), \cdots, G_r(s)$. In another situation, for example in considering an adaptive control problem, one considers a parameterized family of plants $G_\lambda(s)$ and wishes to construct a stabilizing compensator $K_\lambda(s)$ such that $(G_\lambda(s), K_\lambda(s))$ is stable for all λ in a parameter set. If $K_\lambda(s)$ is independent of λ, then the problem reduces to the blending problem, i.e. of constructing a fixed compensator for a family of plants. For details on the motivation and other references we refer to [1], [12].

The main idea of this paper is now summarized. First of all, the space of proper plants of McMillan degree n and the space of proper compensators of McMillan degree q have been described as a quasi-affine variety. In particular the plants and the compensators under consideration are parameterized as semialgebraic subsets of the affine spaces $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively. By using the Routh–Hurwitz criterion [11], the set of plants and the set of compensators that correspond to a stable system in the closed loop is described by a set of semialgebraic conditions in the product space $\mathbb{R}^N \times \mathbb{R}^M$. The stabilizable plants are now described by the application of the decision method [2] which utilizes a rational procedure of eliminating the compensator variables from the above semialgebraic sets of conditions. By Tarski [26] and Seidenberg [22] (see also Cohen [7]) this results in a semi-algebraic parameterization of the set of stabilizable plants in $\mathbb{R}^N$.

Of course the above argument continues to hold if one considers the pole-placement problem instead of the stabilization problem and a r-tuple of plants or more generally a rational family of plants of a fixed McMillan degree instead of a single plant. It may be noted, however, that the above parameterization may not be obtained by an efficient algorithm since it is known [10] that the Tarski-Seidenberg algorithms are computationally inefficient. However, a recent improvement by Collins [8] and by Arnon [3] have considerably improved the efficiency.

The organization of this paper is as follows. In § 2 a parameterization of the space of input output systems of a fixed McMillan degree has been described. In § 3 the set of stabilizable/pole assignable plants have been parameterized by the application of the Routh-Hurwitz condition [11] and the decision theory [2]. The parameterization problem of § 3 is generalized in § 4 to a family of plants, rather than one. Specifically, a simultaneously stabilizable family of plants has been parameterized. In § 5, the path component properties of the pole placement problem is described. In §§ 6 and 7 we restrict our attention to the case of a r-tuple of single input or single output plants (in particular $1\times m$ plants). In § 6, an explicit solution to the parameterization problem is obtained under the hypothesis that $(q+1)(m+1)=\sum n_i+rq$ where n_i, $i=1,\cdots,r$ and q are the McMillan degrees of the plants and the compensator, respectively. Especially when the above hypothesis is not satisfied, in § 7 we parameterize a set of unstabilizable r-tuples of plants, thereby obtaining a necessary condition to the simultaneous stabilization problem. The paper concludes in § 8 with a discussion on the possibility of parameterizing the set of stabilizable/pole assignable r-tuples of plants by a dynamic compensator of finite but a priori unbounded McMillan degree. This new problem serves to give some measure of the relative depth of the parameterization questions posed and solved in this paper.

2. A parameterization of the space of systems. Let k be either $\mathbb{R}$ or $\mathbb{C}$. We now parameterize the set of $p\times m$ proper, rational, matrix valued transfer functions over k as a subset of k^N where $N=n(m+p)+mp$. Let (A, B, C, D) be a 4-tuple of matrices in k^N of orders $n\times n$, $n\times m$, $p\times n$ and $p\times m$, respectively. Let us consider the following.

DEFINITION. A tuple of matrices (A, B, C, D) is defined to be a minimal system of degree n if the proper $p\times m$ transfer function

$$G(s)=\sum_{i=1}^{\infty} CA^{i-1}B/s^i+D \tag{2.1}$$

is of McMillan degree n.

It is well known that a tuple (A, B, C, D) is minimal iff it is observable and reachable. The space of minimal systems of degree n is denoted by $\tilde{S}^n_{m,p}$. Let us consider the following proposition.

LEMMA 2.1. *$\tilde{S}^n_{m,p}$ is an irreducible subset of k^N.*

Proof. The affine space k^N is irreducible. $\tilde{S}^n_{m,p}$ is a nonempty Zariski open subset of k^N since it contains observable and reachable 4-tuples of matrices. Hence $\tilde{S}^n_{m,p}$ is irreducible. (See Hartshorne [16].) Q.E.D.

It is quite possible that two 4-tuples of matrices (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) correspond via (2.1) to the same transfer function $G(s)$. The above is indeed the case iff there exist a nonsingular $n\times n$ matrix $g\in Gl(n, k)$ such that

$$A_2=gA_1g^{-1},\quad B_2=gB_1,\quad C_2=C_1g^{-1},\quad D_2=D_1. \tag{2.2}$$

Thus there exists an action of $Gl(n, k)$ on the space $\tilde{S}^n_{m,p}$ and the problem is to parameterize the moduli space $\tilde{S}^n_{m,p}/Gl(n, k)$. This is done as follows.

LEMMA 2.2. *$\tilde{S}^n_{m,p}/Gl(n, k)$ is an analytic manifold of dimension $n(m+p)+mp$. Moreover, there exists a set of local co-ordinate charts with rational co-ordinate functions.*

Remark. The proof of Lemma 2.2 is an adaptation from Clark [6], Hazewinkel and Kalman [18], Byres and Hurt [4] and Hazewinkel [17]. We choose to restate this well-known fact since the algebraic structure of the moduli space is important in what we derive later on in this paper.

Proof. Let $\tilde{\Sigma}_0$ denote the set of reachable pairs of matrices (A, B) and denote

$$\Sigma_0 = \tilde{\Sigma}_0/Gl(n, k). \tag{2.3}$$

It is well known (see [4], [18]) that Σ_0 is an analytic manifold of dimension nm which admits a rational atlas. Let $\tilde{\Sigma}^n_{m,p}$ be the observable and reachable triples (A, B, C) and define

$$\Sigma^n_{m,p} = \tilde{\Sigma}^n_{m,p}/Gl(n, k). \tag{2.4}$$

Using a result due to Byrnes and Hurt [4], $\Sigma^n_{m,p} \to \Sigma_0$ is canonically an algebraic vector bundle of rank pn. Hence $\Sigma^n_{m,p}$ is a $n(m+p)$-dimensional analytic manifold having rational coordinate functions. Finally the proof of this lemma follows from the observation that

$$\tilde{S}^n_{m,p}/Gl(n, k) = \Sigma^n_{m,p} \times k^{mp}. \tag{2.5}$$

Q.E.D.

Alternatively, it is also possible to parameterize $\tilde{S}^n_{m,p}/Gl(n, k)$ in the following way.

Let $H^n_{m,p}$ be the affine $(2n-1)mp$-dimensional space of $p \times m$ block Hankel matrices of the type

$$H = \begin{bmatrix} H_1 & H_2 & \cdots & H_n \\ H_2 & H_3 & \cdots & H_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ H_n & H_{n+1} & & H_{2n-1} \end{bmatrix} \tag{2.6}$$

where H_i, $i = 1, \cdots, 2n-1$ are $p \times m$ matrices. Consider the affine space

$$H^n_{m,p} \times k^{mp} \times k^{mp}, \tag{2.7}$$

and define the subset

$$S^n_{m,p} \subset H^n_{m,p} \times k^{mp} \times k^{mp} \tag{2.8}$$

given by

$$\begin{aligned} S^n_{m,p} = \{(H, H_{2n}, H_0) | \text{rank } H = n \text{ and col}\,(H_{n+1}, \cdots, H_{2n}) \\ = \sum_{i=1}^{n} [\alpha_i I_p] \text{ col}\,(H_i, \cdots, H_{n+i-1}) \text{ for some } \alpha_1, \cdots, \alpha_n \in k\}. \end{aligned} \tag{2.9}$$

There is an algebraic map

$$\Phi: \tilde{S}^n_{m,p} \to S^n_{mp} \tag{2.10}$$

defined by

$$\Phi(A, B, C, D) = \left(\begin{bmatrix} CB & \cdots & CA^{n-1}B \\ CA^{n-1}B & \cdots & CA^{2n-2}B \end{bmatrix}, CA^{2n-1}B, D \right). \tag{2.11}$$

We now consider the following lemma.

LEMMA 2.3. *$S^n_{m,p}$ is a quasi affine algebraic variety in the affine space $k^{(2n+1)mp}$.*

Proof. By (2.9), $S^n_{m,p}$ is an open subset of a closed algebraic subset in $k^{(2n+1)mp}$. Moreover, $S^n_{m,p}$ is irreducible, since $\tilde{S}^n_{m,p}$ is, and the map Φ is algebraic (see Shaferavich [23]). Q.E.D.

A topology on $S^n_{m,p}$ is induced from the Zariski topology [16] on $k^{(2n+1)mp}$. It is well known, by realization theory [19], that every element of $S^n_{m,p}$ corresponds with a $p\times m$ proper plant of McMillan degree n. Hence $S^n_{m,p}$ is isomorphic to the moduli space $\tilde{S}^n_{m,p}/Gl(n,k)$.

3. A parameterization of the space of stabilizable/pole assignable systems. Let k be the real field $\mathbb{R}$. To begin with, we consider the product space of $p\times m$ proper plants of degree n and $m\times p$ proper compensators of degree q, given by

$$S^n_{m,p}\times S^q_{p,m}, \tag{3.1}$$

which is of course a quasi affine variety. (See [12]). We now consider the following two problems

Problem 3.1. Describe the set of plants $G(s)$ in $S^n_{m,p}$ and compensators $K(s)$ in $S^q_{p,m}$ such that the pair $(G(s), K(s))$ is stable in the closed loop.

Problem 3.2. Describe the set of plants $G(s)$ in $S^n_{m,p}$ and compensators $K(s)$ in $S^q_{p,m}$ such that the closed loop system $G(s)[I+K(s)G(s)]^{-1}$ has poles in a prescribed $n+q$ tuple of self-conjugate complex points $s_1,\cdots,s_{n+q}$.

We now show that there exists a semialgebraic parameterization to the set of plants and compensators satisfying the properties stipulated in Problems 3.1 and 3.2.

THEOREM 3.1. *The subset U_1 of (3.1) given by*

$$U_1=\{(G(s),K(s))\,|\,K(s) \text{ stabilizes } G(s)\}, \tag{3.2}$$

is semialgebraic.

In order to prove Theorem 3.1 we need the following lemma.

LEMMA 3.1. *Let Φ be the map*

$$\Phi: S^n_{m,p}\times S^q_{p,m}\to\mathbb{R}^{n+q} \tag{3.3}$$

defined by

$$\Phi(G(s),K(s)) = \text{coefficients of the monic characteristic polynomial } \pi(s) \text{ of the closed loop system } G(s)[I+K(s)G(s)]^{-1}. \tag{3.4}$$

Then Φ is rational.

Proof. Consider the affine spaces

$$H_n=H^n_{m,p}\times\mathbb{R}^{mp}\times\mathbb{R}^{mp} \tag{3.5}$$

and

$$H_q=H^q_{p,m}\times\mathbb{R}^{mp}\times\mathbb{R}^{mp} \tag{3.6}$$

associated with $S^n_{m,p}$ and $S^q_{p,m}$, respectively. By realization theory [19], every point in the quasi-affine algebraic variety $S^n_{m,p}\times S^q_{p,m}$ of $H_n\times H_q$ corresponds to a plant-compensator pair $(G(s),K(s))$, and therefore corresponds to the closed loop plant $\underset{\sim}{G}(s)=G(s)[I+K(s)G(s)]^{-1}$. Moreover, the coefficients of the characteristic polynomial of $\underset{\sim}{G}(s)$ is rational in the parameters of $H_n\times H_q$ since $G(s)$ and $K(s)$ are of fixed degrees n and q, respectively. By restriction to $S^n_{m,p}\times S^q_{p,m}$, Φ is clearly rational. Q.E.D.

LEMMA 3.2. *Consider the real $n+q(th)$ degree polynomial*

$$p(s)=s^{n+q}+a_{n+q-1}s^{n+q-1}+\cdots+a_0 \tag{3.7}$$

parameterized as points in $\mathbb{R}^{n+q}$. *Then the set*

$$S = \{(a_0, \cdots, a_{n+q-1}) | p(s) \text{ is stable}\} \tag{3.8}$$

is semialgebraic in $\mathbb{R}^{n+q}$.

Proof. The proof is omitted as it is the well-known Routh–Hurwitz condition [11]. Q.E.D.

Theorem 3.1 now follows trivially from the Lemmas 3.1 and 3.2. Note that the compensator $K(s)$ stabilizes the plant $G(s)$ just in case the associated characteristic polynomial $\pi(s)$ is stable. Note also that the coefficients of $\pi(s)$ are rational in the plant and the compensator parameters. Q.E.D.

The proof of the following theorem is analogous and is omitted.

THEOREM 3.2. *The subset* U_2 *of* (3.1) *given by*

$$U_2 = \{(G(s), K(s)) | \text{the poles of } G(s)(I+K(s)G(s))^{-1} \text{ are at a given set of self conjugate complex points } s_1, \cdots, s_{n+q}\} \tag{3.9}$$

is semialgebraic.

We now state the stabilizability problem as follows.

Problem 3.3. Describe the set of plants in $S^n_{m,p}$ for which there exists a compensator $K(s)$ in $S^q_{p,m}$ such that the closed loop system $G(s)[I+K(s)G(s)]^{-1}$ is stable.

We also state the pole assignability problem as follows.

Problem 3.4. Describe the set of plants in $S^n_{m,p}$ for which there exists a compensator $K(s)$ in $S^q_{p,m}$ such that the closed loop system $G(s)[I+K(s)G(s)]^{-1}$ has poles in an arbitrary $n+q$ set of self-conjugate complex points.

The following theorem reveals the semialgebraic nature of the sets described in Problems 3.3 and 3.4.

THEOREM 3.3. *The subset* U_s *of* U_1 *given by*

$$U_s = \{G(s) | \exists K(s) \in S^q_{p,m} \text{ and } G(s)[I+K(s)G(s)]^{-1} \text{ is stable}\} \tag{3.10}$$

is semialgebraic. The subset U_p *of* U_2 *given by*

$$U_p = \{G(s) | \text{for all self conjugate set } s_1, \cdots, s_{n+q} \text{ of complex numbers, } \exists\ K(s) \in S^q_{p,m} \text{ and } G(s)[I+K(s)G(s)]^{-1} \text{ has poles at } s_1, \cdots, s_{n+q}\} \tag{3.11}$$

is semialgebraic.

Proof. Consider the product space (3.1) and consider the projection

$$\text{proj}: S^n_{m,p} \times S^q_{p,m} \to S^n_{m,p}. \tag{3.12}$$

It is clear that

$$U_s = \text{proj } U_1. \tag{3.13}$$

By Theorem 3.1, U_1 is semialgebraic and proj is a rational map. Thus by the Tarski [26], Seidenberg [22] theory of elimination over $\mathbb{R}$, U_s is semialgebraic. In order to show that U_p is semialgebraic consider the product space

$$S^n_{m,p} \times S^q_{p,m} \times \mathbb{R}^{n+q}, \tag{3.14}$$

and its subset U_2' given by

$$U_2' = \{(G(s), K(s), (c_0, \cdots, c_{n+q-1})) | \text{ the poles of } G(s)[I+K(s)G(s)]^{-1} \text{ are at the zeros of } \pi(s) = s^n + c_{n+q-1}s^{n+q-1} + \cdots + c_0\}. \tag{3.15}$$

Consider now the following two projections:

$$\text{proj}_1: S^n_{m,p} \times S^q_{p,m} \times \mathbb{R}^{n+q} \to S^n_{m,p} \times \mathbb{R}^{n+q} \tag{3.16}$$

and

$$\text{proj}_2 \colon S^n_{m,p} \times \mathbb{R}^{n+q} \to S^n_{m,p}. \tag{3.17}$$

It is easy to see that

$$U_p = \overline{\text{proj}_2 [\overline{\text{proj}_1 (U_2')}]}, \tag{3.18}$$

where $\bar{\Omega}$ denotes the complement of Ω in the respective ambient space. Since the complement of a semialgebraic set is semialgebraic, by Tarski-Seidenberg [26], [22], U_p is semialgebraic. Q.E.D.

Example 3.1. Consider the single input single output plant $g(s)$ and the compensator $k(s)$ given by

$$g(s) = 1/(s^2 + bs + c), \tag{3.19}$$

$$k(s) = k/(s + \alpha). \tag{3.20}$$

The characteristic polynomial is given by

$$(s^2 + bs + c)(s + \alpha) + k, \tag{3.21}$$

which vanishes in the left half-plane iff

$$\alpha + b > 0 \text{ and } \alpha c + k > 0 \text{ and } (\alpha + b)(\alpha b + c) - (\alpha c + k) > 0. \tag{3.22}$$

The above inequalities (3.22) have been obtained by the application of the Routh–Hurwitz condition to the characteristic polynomial (3.21). In order to describe the set of stabilizable plants, we need to eliminate the variables k, α from (3.22) and obtain

$$b > 0 \text{ or } c - b^2 > 0. \tag{3.23}$$

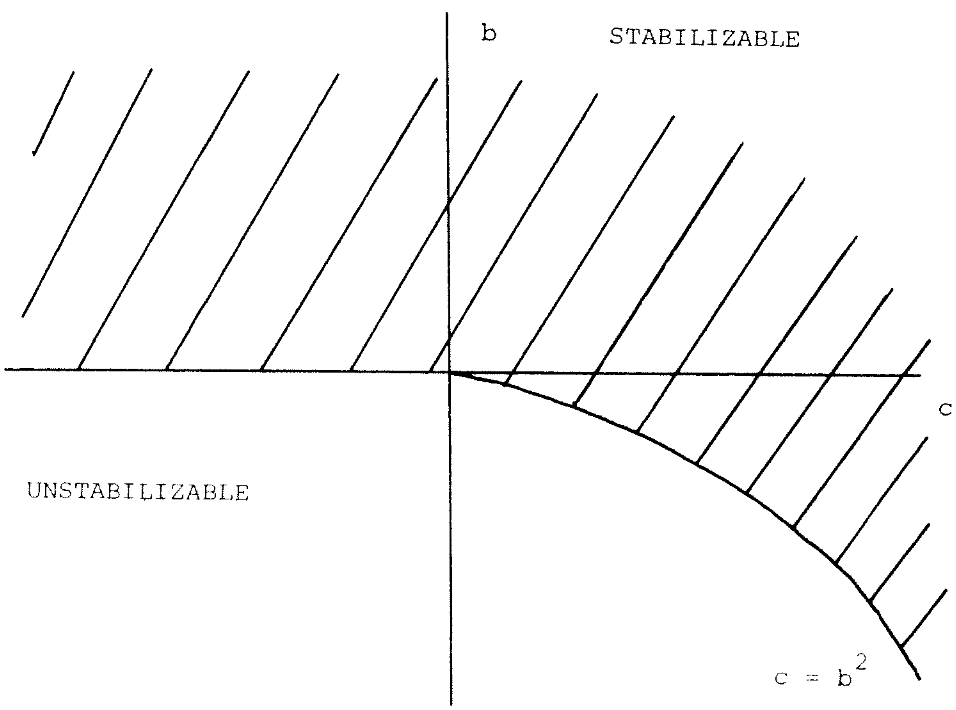

FIG. 3.1. *The set of stabilizable and unstabilizable plants of Example* 3.1.

In other words, for every b, c satisfying (3.23) there exists $k, \alpha \in \mathbb{R}$ such that (3.22) is satisfied. Thus the set of stabilizable plants are given by the shaded region in Fig. 3.1.

4. Semialgebraic path component properties of families of systems. Consider the space of r-tuples of plants

$$W = S^{n_1}_{m,p} \times \cdots \times S^{n_r}_{m,p} \tag{4.1}$$

of McMillan degrees of $n_1, \cdots, n_r$ respectively. We now define a family of r-tuples of plants and a family of compensators as follows. Let $\Lambda_1 \subset \mathbb{R}^{s_1}$, $\Lambda_2 \subset \mathbb{R}^{s_2}$ be a pair of semialgebraic subsets. A collection of families of r-tuple of plants in W, parameterized by a semialgebraic subset W_1 of W is given by an algebraic function

$$\phi_1 : \Lambda_1 \times W_1 \to W. \tag{4.2}$$

Similarly, a collection of families of compensators in $S^q_{p,m}$ parameterized by a semialgebraic subset W_2 of W is given by the algebraic function

$$\phi_2 : \Lambda_2 \times W_2 \to S^q_{p,m}. \tag{4.3}$$

Note that a given element $w_1 \in W_1$ indeed defines a family of r-tuples of plants given by

$$\phi_1(\cdot, w_1) : \Lambda_1 \to W. \tag{4.4}$$

Similarly,

$$\phi_2(\cdot, w_2) : \Lambda_2 \to S^q_{p,m} \tag{4.5}$$

for $w_2 \in W_2$, defines a family ofcompensators. For notational simplicity, we would denote by $w_1 \in W_1$, the family of r-tuples of plants given by $\phi_1(\cdot, w_1)$. Similarly, $w_2 \in W_2$ will be used to denote the family of compensators $\phi_2(\cdot, w_2)$. We now define the notion of σ-stabilizability. Let h be a fixed algebraic function

$$h : \Lambda_1 \to \Lambda_2. \tag{4.6}$$

Let f_i, $i = 1, \cdots, r$ respectively, be the projections

$$f_i : W \to S^{n_i}_{m,p}. \tag{4.7}$$

DEFINITION (*σ-stabilizability*). The family of r-tuples of plants $w_1 \in W_1$, is said to be σ-stabilizable by a family of compensators $w_2 \in W_2$, if the closed loop systems

$$f_i\phi_1(\lambda, w_1)[I + \phi_2(h(\lambda), w_2) f_i\phi_1(\lambda, w_1)]^{-1} \tag{4.8}$$

have poles with real part less than $-\sigma$, for all $\lambda \in \Lambda_1$ and for all $i = 1, \cdots, r$.

In order to introduce the main result of this section we consider the following problem.

Problem. Describe the subset W_1' of W_1 given by

$$W_1' = \{w_1 \in W_1 | \exists \sigma > 0, w_2 \in W_2 \text{ and } w_1 \text{ is } \sigma\text{-stabilizable by } w_2\}. \tag{4.9}$$

The main result is now summarized as follows.

THEOREM 4.1. *For a fixed algebraic function h described by (4.6), the set W_1' of σ-stabilizable families of r-tuple of plants for $\sigma > 0$, is open, semialgebraic. There exists a semialgebraic subset X of a proper algebraic set, where X consists of non-σ stabilizable families of r-tuples of plants in W_1, such that $W_1 - X$ has finitely many path components with the property that—if w_1 and w_1' are in the same path component of $W_1 - X$ then the* *family w_1 is σ-stabilizable iff the family w_1' is σ-stabilizable by some family of compensators $w_2 \in W_2$.*

Proof. To see that W_1' is open, let $w_1 \in W_1'$. By assumption, there exists a $\sigma_0 > 0$ such that w_1 is σ_0-stabilizable by some $w_2 \in W_2$. Equivalently, the closed loop systems $f_i\phi_1(\lambda, w_1)[I + \phi_2(h(\lambda), w_2), f_i(\phi_1(\lambda, w_1))]^{-1}$ have poles with real part less than $-\sigma_0$, for all $\lambda \in \Lambda_1$ and for all $i = 1, \cdots, r$. Since $\sigma_0 > 0$, every element w_1' sufficiently close to w_1 in W_1', is σ_0'-stabilizable for σ_0' sufficiently close to σ_0. Hence $\sigma_0' > 0$. Hence W_1' is open.

To see that W_1' is semialgebraic, consider the space

$$W_1 \times W_2 \times \Lambda_1 \times \mathbb{R}^-, \tag{4.10}$$

where $\mathbb{R}^-$ is the open negative real axis. Consider the subset U in (4.10) given by

$$U = \{(w_1, w_2, \lambda_1, \sigma) | f_i(\phi_1(\lambda_1, w_1)) \text{ is } \sigma\text{-stabilizable by } \phi_2(h(\lambda_1), w_2)\}. \tag{4.11}$$

By the Routh-Hurwitz condition [11], U is semialgebraic. Let p be the projection

$$\text{proj}_1\colon W_1 \times W_2 \times \Lambda_1 \times \mathbb{R}^- \to W_1 \times \Lambda_1 \tag{4.12}$$

by the Tarski-Seidenberg [26], [22] theory of elimination over R, $U_1 = \text{proj}_1 U$ is semialgebraic in $W_1 \times \Lambda_1$. Moreover $\bar{U}_1$, the complement of U_1 in $W_1 \times \Lambda_1$ is also semialgebraic. Consider the projection

$$\text{proj}_2\colon W_1 \times \Lambda_1 \to W_1. \tag{4.13}$$

The set $U_2 = \overline{\text{proj}_2(\bar{U}_1)}$ is semialgebraic in W_1. Moreover, U_2 precisely corresponds to those families of r tuples of plants that are σ-stabilizable. Hence $W_1' = U_2$ is semialgebraic.

Thus W_1' is open, semialgebraic and by Delzell's theorem [9], is described by conjunction and disjunction of strict inequalities.

$$h_j(\cdot) > 0 \tag{4.14}$$

for $j = 1, \cdots, t$, where t is some positive integer. Let us now define a proper, algebraic subset X_1 of W_1 given by

$$X_1 = \bigcup_j \{h_j(\cdot) = 0\} \tag{4.15}$$

for $j = 1, \cdots, t$. Since X_1 is algebraic, by Whitney [20], $W_1 - X_1$ has finitely many path components. Let X be defined by

$$\begin{aligned} X &= \bigcup_j [\{h_j(\cdot) = 0 \cap \bar{W}_1'] \\ &= X_1 \cap \bar{W}_1'. \end{aligned} \tag{4.16}$$

Clearly X is semialgebraic and is a subset of the proper algebraic set X_1. Moreover

$$W_1 - X = (W_1 - X_1) \cup (X_1 - \bar{W}_1'). \tag{4.17}$$

Since $W_1 - X_1$ has finitely many path components with the property that every component either contains simultaneously σ-stabilizable family of plants for some $\sigma > 0$ or simultaneously σ-unstabilizable family of plants, it is clear $W_1 - X$ also has the same property. This is because a σ-stabilizable component and a σ-unstabilizable component cannot be separated by a boundary of σ-stabilizable family of plants, σ-stabilizability being an open condition. Q.E.D.

Let us now consider a refinement of Theorem 4.1 under the hypothesis that Λ_1 is compact.

THEOREM 4.2. *Let the parameter space Λ_1 be compact. For a fixed algebraic function h described by (4.6), the set W_1' of stabilizable families of r-tuple of plants is open, semialgebraic. There exists a semialgebraic subset X of a proper algebraic set, where X consists of nonstabilizable families of r-tuples of plants in W_1, such that $W_1 - X$ has finitely many path components with the property that—if w_1 and w_1' are in the same path components of $W_1 - X$, then the family w_1 is stabilizable iff the family w_1' is stabilizable by some family of compensators $w_2 \in W_2$.*

Remark. Note first of all that in Theorem 4.2 we are discussing stabilizability (or zero stabilizability) as opposed to Theorem 4.1 where we have σ-stabilizability, for $\sigma > 0$.

An important corollary of Theorem 4.2, follows in the special case when $\Lambda_1 = \{1, 2, \cdots, r\}$ and h is a constant function.

COROLLARY 4.1. *There exists a semi-algebraic subset of a proper algebraic set X, consisting of nonstabilizable r-tuples of plants such that $W_1 - X$ has finitely many components with the property that—if $(G_1(s), \cdots, G_r(s))$ and $(G_1'(s), \cdots, G_r'(s))$ are in the same path component then $(G_1(s), \cdots, G_r(s))$ is simultaneously stabilizable by a fixed nonswitching compensator iff $(G_1'(s), \cdots, G_r'(s))$ is so.*

If on the other hand Λ_2 the compensator parameter space is a single point, so that the algebraic function h described by (4.6) is a constant function, we have the simultaneous version of the "blending problem" originally addressed by Tannenbaum [24], [25].

Finally we come to the proof of Theorem 4.2.

Proof. The proof of this theorem is analogous to Theorem 4.1. However, one has to show that the set W_1' given by (4.18) is open.

$$W_1' = \{w_1 \in W_1 \mid \exists w_2 \in W_2 \text{ and } w_1 \text{ is stabilizable by } w_2\}. \tag{4.18}$$

Let $w_1 \in W_1'$. By hypothesis, there exists a $w_2 \in W_2$ with the following property—if $\pi_i(s, \lambda_1)$ is the characteristic polynomial of the closed loop system for $\lambda_1 \in \Lambda_1$ and $i \in \{1, \cdots, r\}$, then $\pi_i(s, \lambda_1)$ has roots in the open left half of the complex plane for all $\lambda_1 \in \Lambda_1$ and $i \in \{1, \cdots, r\}$. Let $s_{i,j}^{(\lambda_1)}$, $j = 1, \cdots, n_i + q$ be the roots of $\pi_i(s, \lambda_1)$, for a fixed $i \in \{1, \cdots, r\}$, $\lambda_1 \in \Lambda_1$. Define

$$r^{(\lambda_1)} = \max_{i,j} \operatorname{Re} s_{i,j}^{(\lambda_1)}, \tag{4.19}$$

where $r^{(\lambda_1)}$ is clearly a continuous function of λ_1. Since Λ_1 is compact, $r^{(\lambda_1)}$ attains a maximum value r where $r < 0$, since $w_1 \in W_1'$. Consider w_1' in a sufficiently small open neighborhood N_{w_1} of w_1 in W_1, together with w_2 in W_2. By repeating the above argument, there exists $r' < 0$ sufficiently close to r, such that the real parts of the roots of the associated characteristic polynomials $\pi_1'(s, \lambda_1)$ for all $i \in \{1, \cdots, r\}$ and $\lambda_1 \in \Lambda_1$ is less than or equal to r'. Hence $N_{w_1} \subset W_1'$. Q.E.D.

Remark. A semialgebraic subset of a proper algebraic set is automatically of Lebesgue measure zero. Thus, roughly speaking, one is deleting a "thin" set of nonstabilizable plants (family of plants) so that the remaining set has a "component" property. This, we remark, is conceptually significant since we now have to analyze big pieces of components rather than an individual family of plants.

5. Path component properties of the simultaneous pole placement problem. In this section we want to generalize the pole assignability Problem 3.4 to the simultaneous pole placement problem described as follows.

Problem 5.1. Parameterize the set of r-tuples of m input p output plants, $G_1(s), \cdots, G_r(s)$ each of a given McMillan degree n_i, $i=1, \cdots, r$, respectively, for which there exist a nonswitching p input m output proper compensator $K(s)$ of McMillan degree q that arbitrarily assigns all the poles of the closed loop systems $G_i(s)[I+K(s)G_i(s)]^{-1}$; $i=1, \cdots, r$.

Generalizing the technique used in the proof of Theorem 3.3, we now show the following.

THEOREM 5.1. *The set W_p of simultaneously pole assignable (by a proper compensator of Mcmillan degree q) r-tuples of plants $G_1(s), \cdots, G_r(s)$, in $W=S^{n_1}_{m,p}\times\cdots\times S^{n_r}_{m,p}$ is semialgebraic. There exists a proper algebraic set X in W such that $W-X$ has finitely many path components with the property that—if $(G_1(s), \cdots, G_r(s))$ and $(G'_1(s), \cdots, G'_r(s))$ are in the same path component, then $(G_1(s), \cdots, G_2(s))$ is simultaneously pole assignable by a proper compensator of McMillan degree q iff $(G'_1(s), \cdots, G'_r(s))$ is so.*

Remark. We do not claim in Theorem 5.1 that the set W_p is open.

Proof. The fact that W_p is semialgebraic is an immediate generalization of Theorem 3.3 (see [12] for details). Thus W_p is described by conjunction and disjunction of inequalities and equalities of the type

$$h_j(\cdot)>0, \qquad g_k(\cdot)=0 \tag{5.1}$$

for $j=1, \cdots, t_1$, $k=1, \cdots, t_2$ where t_1, t_2 are some positive integers. Define X to be the proper algebraic set given by

$$X=\bigcup_j \{h_j(\cdot)=0\} \bigcup_k \{g_k(\cdot)=\}. \tag{5.2}$$

It is clear that X is the proper algebraic set with the required path component property. Q.E.D.

For the rest of this section, our aim is to refine Theorem 5.1. First of all we claim that the set W_p, of pole assignable r-tuple of plants is not open in general. This we show by considering the following example.

Example 5.1. Consider a single input single output plant of McMillan degree 1 given by $(p_1 s+p_2)/(s+p_3)$ parameterized in $\mathbb{R}^3$ by the point (p_1, p_2, p_3). Consider the gain feedback c_1/c_2. To say that the gain places the poles of the system at $s=-\alpha$ is to say that the following matrix equation

$$[c_1 \quad c_2]\begin{bmatrix} p_1 & p_2 \\ 1 & p_3 \end{bmatrix}=[1 \quad \alpha] \tag{5.3}$$

has a solution. The set of all p_1, p_2, p_3 for which (5.3) has a solution is given by $\mathscr{S}_1 \cup \mathscr{S}_2$ where

$$\mathscr{S}_1=\{(p_1, p_2, p_3) | p_2 \neq p_1 p_3\},$$
$$\mathscr{S}_2=\{(p_1, p_2, p_3) | p_2=p_1 p_3, p_3=\alpha\}.$$

Note that although $\mathscr{S}_1$ is an open set, $\mathscr{S}_2$ is a proper Zariski closed subset of $\mathbb{R}^3$. Note also that the set

$$\mathbb{R}^2-\{(p_1, p_2, p_3) | p_2=p_1 p_3\} \tag{5.4}$$

has finitely many components that are pole assignable.

6. An explicit solution of the parameterization problem. In this section, we consider an r-tuple of $p\times m$, min $(m, p)=1$ strictly proper plants of McMillan degrees n_i, $i=$

$1, \cdots, r$, respectively. The problem is to parameterize explicitly the set of r-tuples of plants that can be stabilized by a proper compensator of McMillan degree q. In this section we obtain the parameterization under the assumption $(q+1)(m+1) = \Sigma n_i + rq$. Without any loss of generality we can assume that $m \geqq p$, for if $K(s)$ stabilizes $G_i^T(s)$, then $K^T(s)$ stabilizes $G_i(s)$. A given set of r, m input 1 output plants of McMillan degree $\leqq n_i$ may be represented as

$$\left[\sum_{j=0}^{n_i} p^i_{m+1,j}s^j\right]^{-1}\left[\sum_{j=0}^{n_i} p^i_{1,j}s^j, \cdots, \sum_{j=0}^{n_i} p^i_{m,j}s^j\right] \tag{6.1}$$

for $i = 1, 2, \cdots, r$, where $p^i_{m+1,n_i} = 1$, $i = 1, \cdots, r$ and $p^i_{k,n_i} = 0$, $k = 1, \cdots, m$, $i = 1, \cdots, r$. Similarly, a 1 input m output proper compensator of McMillan degree $\leqq q$ is represented as

$$\left[\sum_{j=0}^{q} a_{1,j}s^j, \cdots, \sum_{j=0}^{q} a_{m,j}s^j\right]^T\left[\sum_{j=0}^{q} a_{m+1,j}s^j\right]^{-1}. \tag{6.2}$$

The associated return difference equation, $\det[I + K(s)G_p(s)] = 0$ is given by

$$\begin{aligned}\pi_i(s) &= \sum_{k=1}^{m+p}\left[\sum_{j=0}^{n_i} p^i_{k,j}s^j\right]\left[\sum_{j=0}^{q} a_{k,j}s^j\right]\\ &\triangleq \sum_{j=0}^{n_i+q} c_{i,j}s^j \quad \forall i = 1, \cdots, r,\end{aligned} \tag{6.3}$$

where $c_{i,n_i+q} = 1\ \forall i = 1, \cdots, r$. A generic r-tuple of plants defines a linear mapping χ, via (6.3), between the compensator parameters and the coefficient of the return difference polynomials given by

$$\chi: \mathbb{R}^{q(m+1)+m+1} \to \mathbb{R}^{\Sigma n_i + rq+1}, \tag{6.4}$$

where χ may be defined as

$$\chi(\underline{a}) = \underline{a}S = (c_{1,0}, \cdots, c_{1,n_1+q-1}, \cdots, c_{r,n_r+q-1}, c_{1,n_1+q}), \tag{6.5}$$

where $\underline{a}$ is the $q(m+1)+m+1$ compensator parameters $a_{i,j}$ and S is the associated Sylvestor's matrix (see [12], [15] for details). It is known [12] that for a generic r-tuple of plants, the rows of S are independent. Thus the image of χ given by (6.4) is a subspace of codimension 1 in $\mathbb{R}^{\Sigma n_i + rq+1}$. Let us now define

$$\mathcal{D} = \left\{(c_{i,j}, j = 0, \cdots, n_i + q; i = 1, \cdots, r) \,\middle|\, \sum_{j=0}^{n_i+q} c_{i,j}s^j \text{ has roots in the open left half of the complex plane, for } i = 1, \cdots, r\right\} \tag{6.6}$$

and its convex hull $\Omega(\mathcal{D}) \subset R^{n_1+q} \times \cdots \times \mathbb{R}^{n_r+q}$. It was shown by Chen [5] that

LEMMA 6.1 (Chen [5]). *The convex hull of $\mathcal{D}$ is given by*

$$\Omega(\mathcal{D}) = \{(c_{i,j}) | c_{i,j} > 0\}. \tag{6.7}$$

Moreover if image χ is an affine hyperplane then

$$\text{image}(\chi) \cap \Omega(\mathcal{D}) \neq \varnothing \text{ iff image}(\chi) \cap \mathcal{D} \neq \varnothing. \tag{6.8}$$

Thus the stabilizability of the r-tuple of plants (6.1) is equivalent to solving (6.5) for some $c_{i,j} > 0\ \forall i, j$. We now prove the following lemma.

LEMMA 6.2. *Assume $(q+1)(m+1)=\sum n_i+rq$. For a given r-tuple of plants (chosen generically) let $\underline{\alpha}$ be a vector orthogonal to image χ. Then the r-tuple of plants is simultaneously unstabilizable iff $\underline{\alpha}\in\Omega(\mathscr{D})$.*

Proof. (if) By assumption $\underline{\alpha}\in\Omega(\mathscr{D})$ so that

$$\text{image}(\chi)\cap\Omega(\mathscr{D})=\emptyset. \tag{6.9}$$

Thus by Lemma 6.1,

$$\text{image }\chi\cap\mathscr{D}=\emptyset. \tag{6.10}$$

(only if) Assume that $\underline{\alpha}\notin\Omega(\mathscr{D})$. Then clearly there exists $\underline{\alpha}_1\in\Omega(\mathscr{D})$ such that $\underline{\alpha}\perp\underline{\alpha}_1$. Hence $\underline{\alpha}_1\in$ image χ so that

$$\text{image }\chi\cap\Omega(\mathscr{D})\neq\emptyset$$

or equivalently by Lemma 6.1,

$$\text{image }\chi\cap\mathscr{D}\neq\emptyset.$$

Q.E.D.

We now use Lemma 6.2 to obtain a parameterization of the set of unstabilizable r-tuples of plants. Let us represent the Sylvestor matrix S in (6.5) as

$$S=\begin{bmatrix} s_{11} & \cdots & s_{1,t} \\ s_{t-1,1} & \cdots & s_{t-1,t}\end{bmatrix}, \tag{6.11}$$

where $t=\sum n_i+rq+1$. Let

$$v_1\bar{i}_1+v_2\bar{i}_2+\cdots+v_t\bar{i}_t \tag{6.12}$$

(6.12)

be a vector orthogonal to image χ, where $\bar{i}_j, j=1,\cdots,t$ is the standard basis of $\mathbb{R}^t$. Then clearly

$$v_i=(-1)^{i+1}\det S_i,\ i=1,\cdots,t, \tag{6.13}$$

where S_i is obtained from S by deleting its ith column. The set of stabilizable r-tuples of plants is now obtained by the semialgebraic condition

$$(v_1<0 \text{ or } \cdots \text{ or } v_t<0) \tag{6.14}$$

and

$$(v_1>0 \text{ or } \cdots \text{ or } v_t>0).$$

Example 6.1. Assume $q=0$, $r=2$, $m=2$, $p=1$, $n_1=1$, $n_2=2$. The Sylvestor matrix S is given by

$$S=\begin{bmatrix} p^1_{1,0} & p^2_{1,0} & p^2_{1,1} & 0 \\ p^1_{2,0} & p^2_{2,0} & p^2_{2,1} & 0 \\ p^1_{3,0} & p^2_{3,0} & p^2_{3,1} & 1 \end{bmatrix}. \tag{6.15}$$

Thus,

$$\begin{aligned} v_1 &= p^2_{1,0}p^2_{2,1}-p^2_{2,0}p^2_{1,1} \\ v_2 &= p^1_{2,0}p^2_{1,1}-p^1_{1,0}p^2_{2,1}, \\ v_3 &= p^1_{1,0}p^2_{2,0}-p^1_{2,0}p^2_{1,0}, \\ v_4 &= \det\begin{bmatrix} p^1_{1,0} & p^2_{1,0} & p^2_{1,1} \\ p^1_{2,0} & p^2_{2,0} & p^2_{2,1} \\ p^1_{3,0} & p^2_{3,0} & p^2_{3,1}\end{bmatrix}. \end{aligned} \tag{6.16}$$

The simultaneously unstabilizable plants are given by

$$v_i v_j > 0 \quad \text{for } i, j \in \{1, 2, 3, 4\}. \tag{6.17}$$

7. A necessary condition for simultaneous stabilization of min $(m, p) = 1$ plants. We begin this section with the remark that the process of parameterizing the simultaneously stabilizable/unstabilizable r-tuples of plants (as described in §§ 2, 3 and 4) is computationally inefficient since it involves eliminating the set of compensator coefficients using decision algebra [2]. Thus we view §§ 2, 3 and 4 to be qualitative.

In [15], Ghosh and Byrnes have obtained sufficient conditions for the generic simultaneous stabilization of a r-tuple of plants. Furthermore the techniques described in [15] can be used to construct a simultaneously stabilizing compensator. In this section we obtain a necessary condition to the following simultaneous stabilization problem.

Problem 7.1. Given an r-tuple of $1 \times m$ proper plants $G_1(s), \cdots, G_r(s)$ of McMillan degrees n_i, $i = 1, \cdots, r$, respectively, does there exist a compensator $K(s)$ of McMillan degree q such that the closed-loop systems $G_i(s)[I + K(s)G_i(s)]^{-1}$, $i = 1, \cdots, r$ are (internally) stable?

Note that in the above Problem 7.1, the McMillan degree q of the compensator is held fixed. In order to illustrate the main idea of this section we consider the following example.

Example 7.1. Consider a triplet of single input single output plants $p_i(s)/q_i(s)$, $i = 1, 2, 3$ of McMillan degrees n_1, n_2, n_3 respectively. Let $c(s)/d(s)$ be the corresponding stabilizing compensators. Thus there exist Hurwitz polynomials $\Delta_1(s)$, $\Delta_2(s)$, $\Delta_3(s)$ such that

$$p_i(s)c(s) + q_i(s)d(s) = \Delta_i(s), \qquad i = 1, 2, 3, \tag{7.1}$$

or equivalently,

$$\begin{bmatrix} p_1(s) & q_1(s) \\ p_2(s) & q_2(s) \\ p_3(s) & q_3(s) \end{bmatrix} \begin{bmatrix} c(s) \\ d(s) \end{bmatrix} = \begin{bmatrix} \Delta_1(s) \\ \Delta_2(s) \\ \Delta_3(s) \end{bmatrix}. \tag{7.2}$$

A necessary condition for (7.2) to be satisfied is given by

$$\Delta_1(s)\eta_{32}(s) + \Delta_2(s)\eta_{13}(s) + \Delta_3(s)\eta_{21}(s) = 0, \tag{7.3}$$

where $\eta_{ij}(s) = p_i(s)q_j(s) - q_i(s)p_j(s)$.

In fact if the triplet of plants is chosen generically, then (7.3) is also a sufficient condition for the existence of $c(s)$, $d(s)$ satisfying 7.2 (see [13] for details). Now if we assume $n_1 = n_2 = n_3 = 1$ and $q = 0$, then generically we may write

$$\Delta_i(s) = \Delta_{i0} + \Delta_{i1}s, \tag{7.4}$$

and

$$\begin{aligned} \eta_{13}(s) &= \xi_{10} + \xi_{11}s + \xi_{12}s^2, \\ \eta_{13}(s) &= \xi_{20} + \xi_{21}s + \xi_{22}s^2, \\ \eta_{21}(s) &= \xi_{30} + \xi_{31}s + \xi_{32}s^2, \end{aligned} \tag{7.5}$$

so that (7.3) can be written as

$$[\Delta_{10}\,\Delta_{11}\,\Delta_{20}\,\Delta_{21}\,\Delta_{30}\,\Delta_{31}]\begin{bmatrix}\xi_{10} & \xi_{11} & \xi_{12} & 0\\ 0 & \xi_{10} & \xi_{11} & \xi_{12}\\ \xi_{20} & \xi_{21} & \xi_{22} & 0\\ 0 & \xi_{20} & \xi_{21} & \xi_{22}\\ \xi_{30} & \xi_{31} & \xi_{32} & 0\\ 0 & \xi_{30} & \xi_{31} & \xi_{32}\end{bmatrix}=[0\,0\,0\,0]. \tag{7.6}$$

A necessary and sufficient condition that $\Delta_i(s)$ defined in (7.4) is stable is given by the following:

$$\Delta_{i0} \text{ and } \Delta_{i1} \text{ have the same sign,} \quad \text{for all } i=1,2,3.$$

Assume for the purpose of illustration that every entry of the vector

$$[\Delta_{10}\,\Delta_{11}\,\Delta_{20}\,\Delta_{21}\,\Delta_{30}\,\Delta_{31}] \tag{7.7}$$

is positive. Thus to say that (7.6) is satisfied is to say that the polytope generated by the 6 points $p_1, p_2, p_3, p_4, p_5, p_6$ contain the origin, where

$$\begin{aligned} p_1 &= (\xi_{10}\,\xi_{11}\,\xi_{12}\,0), & p_2 &= (0\,\xi_{10}\,\xi_{11}\,\xi_{12}),\\ p_3 &= (\xi_{20}\,\xi_{21}\,\xi_{22}\,0), & p_4 &= (0\,\xi_{20}\,\xi_{21}\,\xi_{22}),\\ p_5 &= (\xi_{30}\,\xi_{31}\,\xi_{32}\,0), & p_6 &= (0\,\xi_{30}\,\xi_{31}\,\xi_{32}). \end{aligned} \tag{7.8}$$

Thus a necessary and sufficient condition that the triplet of single input single output plants of McMillan degree 1 is stabilizable by a feedback gain is that the polytope generated by

$$\begin{aligned} &p_1, p_2, p_3, p_4, p_5, p_6 \quad \text{or} \quad p_1, p_2, p_3, p_4, -p_5, -p_6 \quad \text{or}\\ &p_1, p_2, -p_3, -p_4, p_5, p_6 \quad \text{or} \quad p_1, p_2, -p_3, -p_4, -p_5, -p_6 \end{aligned} \tag{7.9}$$

contains the origin.

We would now like to generalize the argument presented in Example 7.1. Note first of all that for $n_1=n_2=n_3=q=1$ or for $n_1=n_2=n_3=2$, $q=0$ one can mimic the argument of Example 7.1 to obtain a necessary and sufficient condition for the simultaneous stabilization of a triplet of plants.

The proof relies on the fact that a quadratic polynomial $\Delta_i(s)$, $i=1,2,3$ is stable iff every coefficient of $\Delta_i(s)$ has the same sign. However, in general, a polynomial $\Delta(s)$ of degree n is stable only if every coefficient of $\Delta(s)$ is of the same sign. This idea is now applied as follows.

Consider a $m+2$ tuple of $1\times m$ proper plants represented as

$$G_i(s)=\left[\frac{n_{i1}(s)}{n_{i,m+1}(s)}\ \frac{n_{i2}(s)}{n_{i,m+1}(s)}\ \cdots\ \frac{n_{im}(s)}{n_{i,m+1}(s)}\right] \tag{7.10}$$

for $i=1,2,\cdots,m+2$. Consider a $m\times 1$ proper compensator represented as

$$k(s)=\left[\frac{c_1(s)}{c_{m+1}(s)}\ \cdots\ \frac{c_m(s)}{c_{m+1}(s)}\right]^T. \tag{7.11}$$

The compensator (7.11) stabilizes each one of the plants (7.10) iff there exists Hurwitz polynomials $\Delta_1(s),\cdots,\Delta_{m+2}(s)$ such that

$$n_{i1}(s)c_1(s)+\cdots+n_{i,m+1}(s)c_{m+1}(s)=\Delta_i(s) \tag{7.12}$$

for $i=1,\cdots,m+2$. Writing (7.12) in the matrix notation as

$$\begin{bmatrix} n_{11}(s) & \cdots & n_{1,m+1}(s) \\ n_{m+1,1}(s) & \cdots & n_{m+1,m+1}(s) \\ n_{m+2,1}(s) & \cdots & n_{m+2,m+1}(s) \end{bmatrix} \begin{bmatrix} c_1(s) \\ c_{m+1}(s) \end{bmatrix} = \begin{bmatrix} \Delta_1(s) \\ \Delta_{m+1}(s) \\ \Delta_{m+2}(s) \end{bmatrix}, \tag{7.13}$$

a necessary condition for the existence of $c_1(s),\cdots,c_{m+1}(s)$ that satisfy (7.13) is given by the condition

$$\det \begin{bmatrix} n_{11}(s) & \cdots & n_{1,m+1}(s) & \Delta_1(s) \\ n_{m+2,1}(s) & \cdots & n_{m+2,m+1}(s) & \Delta_{m+2}(s) \end{bmatrix} = 0. \tag{7.14}$$

The equation (7.14) can be viewed as a generalization of (7.3). Using the necessary condition for the stability of a polynomial, viz that every coefficient is of the same sign, one obtains the required necessary condition.

8. Conclusion and future developments. In this paper we describe an approach to simultaneous system design using semialgebraic geometric methods. The procedure described in this paper relies heavily on the fact that the McMillan degree of the compensator under consideration is bounded. Although this might be a reasonable assumption under most practical situations, it is also of interest to consider Problems 1.1 and 1.2 with the assumption that the McMillan degree of the compensator is not a priori bounded. We remark that the "space of compensators" under the above hypothesis is not finite-dimensional and semialgebraic geometric methods using decision algebra [2] are therefore not applicable. Using transcendental and interpolation methods, we have obtained (see [14]) semialgebraic parameterization of the simultaneously stabilizable and pole assignable collections of plants. Complete solution to the parameterization Problems 1.1 and 1.2 where the compensators are of unbounded McMillan degree is a subject of future research.

As an additional remark, it may be interesting to point out a recent research work by Richter and DeCarlo [21] on eigenvalue assignment by decentralized feedback and the fact the simultaneous system design problem may be viewed as a decentralized feedback problem with special structure.

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REFERENCES

[1] J. ACKERMANN AND S. TURK, *A common controller for a family of plant models*, 21st IEEE Conference on Decision and Control, 1982, pp. 240-244.

[2] B. D. O. ANDERSON, N. K. BOSE AND E. I. JURY, *Output feedback stabilization and related problems-solution via decision methods*, IEEE Trans. Automat. Control, AC-20 (1975), pp. 53-66.

[3] D. S. ARNON, *Algorithms for the geometry of semialgebraic sets*, Technical Report Number 436, Computer Science Dept., Univ. Wisconsin, Madison, 1981.

[4] C. I. BYRNES AND N. E. HURT, *On the moduli of linear dynamical systems*, Adv. in Math., Suppl. Series, 4 (1978), pp. 83-122; also in Modern Mathematical Systems Theory, MIR Press, Moscow, 1978. (In Russian.)

[5] R. CHEN, Ph.D. Dissertation, Univ. Florida, Gainesville, 1979.

[6] J. M. C. CLARK, *The consistent selection of local co-ordinates in linear system identification*, Proc. J.A.C.C., Purdue, 1976, pp. 576-580.

[7] P. COHEN, *Decision procedures for real and p-adic fields*, Comm. Pure Appl. Math., 22 (1969), pp. 131-151.

[8] G. E. COLLINS, *Quantifier elimination for real closed fields by cylindrical algebraic decomposition*, Second G.I. Conference, Kaiserslautern, 1975, Lecture Notes in Computer Science 33, Springer-Verlag, New York (1975), pp. 134–183.

[9] C. M. DELZELL, *A constructive, continuous solution to Hilbert's 17th problem, and other results in semialgebraic geometry*, Ph.D. Dissertation, Stanford Univ., Stanford, CA, 1980.

[10] M. J. FISCHER AND M. O. RABIN, *Superexponential complexity of presburger arithmetic*, MIT, MAC Tech. Memo. 43, Feb. 1974, also in Complexity of Computation. Proc. Sympos., New York, 1973, pp. 27–41. SIAM–AMS Proceedings, Vol. VII, American Mathematical Society, Providence, RI, 1974.

[11] F. R. GANMACHER, *The Theory of Matrices*, Chelsea, New York.

[12] B. K. GHOSH, *Simultaneous stabilization and pole placement of a multimode linear dynamic system*, Ph.D. Dissertation, Harvard Univ., Cambridge, MA, 1983.

[13] ———, *Simultaneous partial pole placement—a new approach to multimode system design*, IEEE Trans. Automat. Control, submitted.

[14] ———, *Transcendental and interpolation methods in simultaneous stabilization and simultaneous partial pole placement problems*, this Journal, submitted.

[15] B. K. GHOSH AND C. I. BYRNES, *Simultaneous stabilization and simultaneous pole placement by non-switching dynamic compensation*, IEEE Trans. Automat. Control, AC-28 (1983), pp. 735–741.

[16] R. HARTSHORNE, *Algebraic Geometry*, Springer-Verlag, New York, 1977.

[17] M. HAZEWINKEL, *Moduli and canonical forms for linear dynamical systems* II: *The topological case*, Math. System Theory, 10 (1977), pp. 363–385.

[18] M. HAZEWINKEL AND R. E. KALMAN, *On invariants, canonical forms and moduli for linear constant finite-dimensional dynamical systems*, in Lecture Notes in Economic and Mathematical System Theory 131, Springer-Verlag, Berlin, 1976, pp. 48–60.

[19] R. E. KALMAN, M. ARBIB AND P. FALB, *Topics in Mathematical System Theory*, McGraw-Hill, New York, 1965.

[20] J. W. MILNOR, *Singular points of complex hypersurfaces*, Ann. Mathematics Studies 61, Princeton Univ. Press, Princeton, NJ, 1974.

[21] S. RICHTER AND R. DECARLO, *A homotopy method for eigenvalue assignment using decentralized state feedback*, IEEE Trans. Automat. Control, Vol. AC-29 (1984), pp. 148–158.

[22] A. SEIDENBERG, *A new decision method for elementary algebra*, Ann. Math., 60 (1954), pp. 365–374.

[23] I. R. SHAFERAVICH, *Basic Algebraic Geometry*, Springer-Verlag, New York, 1974.

[24] A. TANNENBAUM, *Feedback stabilization of linear dynamical plants with uncertainty in the gain factor*, Int. J. Control, 32 (1980), pp. 1–16.

[25] ———, *Invariance and System Theory: Algebraic and Geometric Aspects*, Springer-Verlag, Berlin, 1981.

[26] A. TARSKI, *A Decision Method for Elementary Algebra and Geometry*, Univ. California Press, Berkeley, 1951.