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A robust reliable stabilization scheme for single input, single output systems using transcendental methods

Bijoy Kumar GHOSH

Dept. of Systems Science and Mathematics, Washington University, St. Louis, MO 63130, USA

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A stabilization scheme reliable against a single controller failure, originally proposed by Vidyasagar and Viswanadham, is analysed. A connection between the problem of robust-reliable stabilization and the problem of stabilization by a stable minimum phase compensator is established via interpolation theory.

Keywords: Reliable stabilization, Interpolation, Internal stabilization, Minimum-phase.

1. Introduction

The reliable stabilization problem is as follows.

Problem 1. Given a plant $g(s)$ and let $k(s)$ be a compensator such that

$$g(s)[1+k(s)g(s)]^{-1}$$

is internally stable. Does there exist a pair of compensators $k_1(s)$, $k_2(s)$ such that

$$k(s)=k_1(s)+k_2(s)$$

and $k_1(s)$, $k_2(s)$ each stabilize $g(s)$ internally?

A compensator $k(s)$ that satisfies the above criterion will be called *a reliable stabilizer* of $g(s)$. The motivation for the above problem is as follows. Assume that a plant and its compensator is implemented as shown in Fig. 1. This is done in order to increase the redundancy in the compensator implementation. During the normal mode, both $k_1(s)$ and $k_2(s)$ are 'good'. The compensator $k(s)$ is applied to $g(s)$ and can be chosen to have desirable properties such as optimality, low sensitivity, etc. However, if either $k_1(s)$ or $k_2(s)$, but not both, is 'bad', the overall system continues to be stable though other properties such as sensitivity might be affected adversely. Thus the stabilization scheme proposed in this paper is *reliable against a single controller failure*.

A weaker version of problem 1 is as follows.

Problem 2. Given a plant $g(s)$. Do there exist compensators $k(s)$, $k_1(s)$ and $k_2(s)$, with $k(s)=$

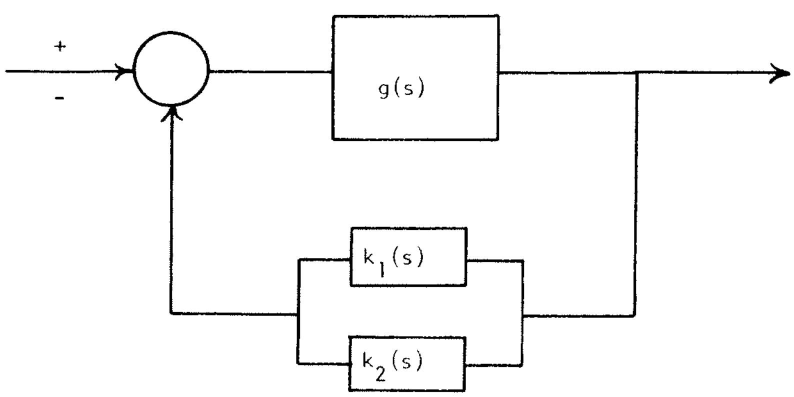

Fig. 1. A feedback scheme reliable against a single controller failure.

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$k_1(s)+k_2(s)$, such that each of the compensators $k(s)$, $k_1(s)$, $k_2(s)$ stabilizes $g(s)$ internally?

The above two problems were posed by Vidyasagar and Viswanadham, and we refer to [3] for a detailed discussion and motivation. In this paper we consider a robust version of problems 1 and 2. In order to describe the problem we need the following notations.

Let n be the McMillan degree of $g(s)$. We regard each $g(s)$ as a point in $\mathbb{R}^{2n+1}$, viz. if

$$g(s)=p(s)/q(s)$$

where

$$p(s)=a_0+\cdots+a_n s^n$$

and

$$q(s)=b_0+\cdots+b_{n-1}s^{n-1}+s^n$$

then $g(s)$ corresponds to the vector

$$(a_0,\ldots,a_n,b_0,\ldots,b_{n-1})\in\mathbb{R}^{2n+1}.$$

Moreover since $p(s)$ and $q(s)$ are coprime, this vector lies in the open dense set $\text{Rat}\, n\in\mathbb{R}^{2n+1}$ (see [1] for the strictly proper case).

Similarly we assume $k(s)\in\text{Rat}\, q$, $k_1(s)\in\text{Rat}\, q_1$ and $k_2(s)\in\text{Rat}\, q_2$. Define $N(k_i(s))$ to be an open neighborhood of $k_i(s)$ in $\text{Rat}\, q_i$, $i=1, 2$. We ask the following pair of questions.

Quetion 1. Given a plant $g(s)\in\text{Rat}\, n$. Do there exist three nonnegative integers q, q_1, q_2 and three stabilizing compensators $k(s)$, $k_1(s)$, $k_2(s)$ of the plant $g(s)$ where $k(s)\in\text{Rat}\, q$, $k_1(s)\in\text{Rat}\, q_1$, $k_2(s)\in\text{Rat}\, q_2$, $k(s)=k_1(s)+k_2(s)$, together with open neighborhoods $N(k_1(s))$, $N(k_2(s))$ with the property that if

$$k_1^*(s)\in N(k_1(s)),\qquad k_2^*(s)\in N(k_2(s))$$

then $k_1^*(s)$, $k_2^*(s)$ and $k_1^*(s)+k_2^*(s)$ each stabilize $g(s)$ internally.

Every compensator $k(s)\in\text{Rat}\, q$ which admits two other compensators $k_1(s)$, $k_2(s)$ satisfying the above property would be called *a robust reliable stabilizer of* $g(s)$. We therefore ask the following.

Question 2. Given a plant $g(s)\in\text{Rat}\, n$ and $k(s)\in\text{Rat}\, q$ such that $k(s)$ stabilizes $g(s)$ internally. Is $k(s)$ a robust reliable stabilizer of $g(s)$?

The motivation for the robust stabilization problem is quite clear. In this paper we answer question 1 and state the following:

Theorem 1.1. *A plant $g(s)$ is robust reliably stabilizable (i.e. admits a robust reliable stabilizer) iff $g(s)$ is stabilizable by a stable compensator.*

The problem of stabilization of a plant by a stable compensator has been completely resolved by Youla, Bongiorno and Lu [4]. Moreover the result of the above theorem may be contrasted with a result due to Vidyasagar and Viswanadham [3] that if robustness is not an issue, every multiinput multioutput plant may be reliably stabilized. As an answer to the robust reliable stabilization problem we state the following:

Theorem 1.2. *Let $g(s)=n_p(s)/d_p(s)$ be the coprime fraction representation of a plant and let $k(s)=n_c(s)/d_c(s)$ be the coprime fraction representation of a stabilizing compensator of $g(s)$ where*

$$n_c(s)n_p(s)+d_c(s)d_p(s)=1. \tag{1.1}$$

Then (a) *$k(s)$ is a reliable stabilizer of $g(s)$ iff there exists a stable minimum phase rational function $\Delta(s)$ which intersects $1/d_p(s)$ at those points in the closed right half plane where either $n_p(s)$ vanishes or $1+d_c(s)d_p(s)$ vanishes.*

(b) *$k(s)$ is a robust, reliable stabilizer of $g(s)$ iff $\Delta(s)$ satisfies the above condition and furthermore $\Delta(s)$ does not intersect $-d_c(s)$ at any other point in the closed right half plane except when $[1+d_c d_p](s)$ vanishes.*

In view of the interpolation lemma due to Youla, Bongiorno and Lu [4], the condition (a) of Theorem 1.2 is explicit, and may be restated as follows: "the total number of poles of $g(s)$ in between every pair of non-negative real zeros (including infinity) of $g(s)$ and $1+d_c(s)d_p(s)$ taken together is even". The condition (b), on the other hand, results in the following necessary conditions:

Corollary 1.3. *Let $g(s)=n_p/d_p$ be a given plant. The compensator $k(s)=n_c(s)/d_c(s)$ satisfying* (1.1) *is a robust reliable stabilizer of $g(s)$ only if $g(s)$ is stabilizable by a stable compensator and $d_c/[1+d_c d_p]$ is stabilizable by a stable, minimum phase compensator.*

Stabilization by a stable minimum phase compensator adds a new twist to the compensation problem and has not been previously addressed in the control systems literature.

2. Proof of Theorem 1.1

Let H be the ring of stable rational functions with real coefficients, i.e. the elements of H do not have poles in the closed right half plane including infinity. Let J be the set of multiplicative units of H.

Let $g(s)$ be written in the coprime fraction representation as

$$g(s) = n_p(s)/d_p(s), \quad n_p, d_p \in H. \tag{2.1}$$

(*If part*) Assume that $g(s)$ be stabilizable by the stable compensator

$$k_1(s) = n_1(s)/d_1(s), \quad n_1 \in H, d_1 \in J, \tag{2.2}$$

where

$$n_1(s)n_p(s) + d_1(s)d_p(s) = 1. \tag{2.3}$$

Consider the compensator

$$k(s) = \frac{n_1 - xd_pd_1}{d_1 + xn_pd_1} \tag{2.4}$$

where $x \in H$ which stabilizes $g(s)$. Moreover

$$(n_1 - xd_pd_1)n_p + (d_1 + xn_pd_1)d_p = 1 \tag{2.5}$$

so that (2.4) is indeed a coprime representation.

We want to show that for a suitable $X \in H$ the compensator

$$k_2(s) = k(s) - k_1(s) = -\frac{X}{d_1 + xn_pd_1} \tag{2.6}$$

internally stabilizes $g(s)$, i.e. there exists a $\Delta \in J$ such that

$$-xn_p(s) + (d_1 + xn_pd_1)d_p(s) = \Delta. \tag{2.7}$$

Equivalently we have the following from (2.3) and (2.7):

$$d_1d_p - xn_1n_pn_p = \Delta. \tag{2.8}$$

If $s^* \in C$ be such that $n_1n_p(s^*) = 0$ then clearly by (2.3)

$$d_1d_p(s^*) = 1. \tag{2.9}$$

By Youla, Bongiorno and Lu [4], there exists $\Delta \in J$, $X \in H$ such that (2.8) is satisfied, so that $k_2(s)$ indeed stabilizes $g(s)$ for a suitable $x(s) \in H$. In order to check robustness let

$$k_1'(s) = n_1'(s)/d_1'(s)$$

be arbitrary close to $k_1(s)$ and let

$$k_2'(s) = -x''\left[d_1'' + x''n_p''d_1''\right]^{-1}$$

be arbitrary close to $k_2(s)$ in the topology described in Section 1. The overall compensator implemented is

$$k'(s) = k_1'(s) + k_2'(s) = \frac{n_1'(s)}{d_1'(s)} - \frac{x''(s)}{d_1''(s) + x''(s)n_p''(s)d_1''(s)}. \tag{2.10}$$

It is easy to check that the closed loop characteristic polynomial of the system

$$g(s)[1 + k'(s)g(s)]^{-1} \tag{2.11}$$

vanishes arbitrary close to the zeros of the closed loop characteristic polynomial of

$$g(s)[1 + k(s)g(s)]^{-1} \tag{2.12}$$

and the zeros of $d_1(s)$. Thus for $k_1'(s)$ and $k_2'(s)$ sufficiently close to $k_1(s)$ and $k_2(s)$ respectively (since the zeros of $d_1(s)$ and the characteristic polynomial of (2.12) are in the left half plane), the system (2.11) is stable.

(*Only if*) By assumption, there exist stabilizing compensators

$$k_1(s) = n_1(s)/d_1(s), \quad k_2(s) = n_2(s)/d_2(s) \tag{2.13}$$

of the plant $g(s)$ where

$$n_1n_p + d_1d_p = 1, \tag{2.14}$$

$$n_2n_p + d_2d_p = 1 \tag{2.15}$$

with the property that

$$k_3(s) = k_1(s) + k_2(s) = (n_1d_2 + n_2d_1)/[d_1d_2] \tag{2.16}$$

also stabilizes $g(s)$. Moreover d_1 and d_2 do not have any common zeros in the closed right half plane for otherwise $k_3(s)$ is not a robust stabilizer, hence (2.16) is a coprime representation of $k_3(s)$. Thus there exists $\Delta \in J$ such that

$$(n_1 d_2 + n_2 d_1) n_p + d_1 d_2 d_p = \Delta, \tag{2.17}$$

or equivalently using (2.14)

$$d_2(s)\left[1 - d_p d_1\right] = \Delta - d_1. \tag{2.18}$$

Assume that n_p vanishes at $s_1, s_2 \in \mathbb{R}^+$ (where $\mathbb{R}^+$ is the non-negative real axis including infinity) for otherwise $g(s)$ is strongly stabilizable by [4]. Then clearly by (2.14) we have

$$(1 - d_1 d_p)(s_1) = (1 - d_1 d_p)(s_2) = 0. \tag{2.19}$$

Moreover using (2.18) we have

$$\Delta(s_1) = d_1(s_1) = d_p(s_1)^{-1}, \tag{2.20}$$

$$\Delta(s_2) = d_1(s_2) = d_p(s_2)^{-1}. \tag{2.21}$$

Since $\Delta(s)$ is a stable minimum phase rational function,

$$\text{sign } d_p(s_1) = \text{sign } d_p(s_2). \tag{2.22}$$

Following the above argument it is clear that in order to satisfy (2.18), the rational function $d_p(s)$ must have the same sign at those points in the closed non-negative real axis (including infinity) where n_p vanishes.

Following Youla, Bongiorno and Lu [4], $g(s)$ must be strongly stabilizable. □

3. Proof of Theorem 1.2

Every stabilizer of $g(s)$ can be written as

$$k_1(s) = \frac{n_c(s) - x(s) d_p(s)}{d_c(s) + x(s) n_p(s)} \tag{3.1}$$

where $x(s) \in H$. In order that $k(s)$ is a reliable stabilizer, a necessary and sufficient condition is to ensure that

$$\begin{aligned} k_2(s) &= k(s) - k_1(s) \\ &= x(s)\left[d_c(s)\left[d_c(s) + x(s) n_p(s)\right]\right]^{-1} \end{aligned} \tag{3.2}$$

is a stabilizer of $g(s)$. Equivalently

$$\begin{aligned} &x(s) n_p(s) + d_c(s)\left[d_c(s)\right. \\ &\quad \left. + x(s) n_p(s)\right] d_p(s) = \Delta(s) \end{aligned} \tag{3.3}$$

for some $x(s) \in H$, $\Delta(s) \in J$. Equivalently (3.3) may be written as

$$x(s) = \left[\Delta - d_c d_c d_p\right] / \left[\left[n_p\right]\left[1 + d_c d_p\right]\right]. \tag{3.4}$$

It is clear that $x(s)$ is a stable rational function iff $\Delta(s) \in J$ intersects $1/d_p(s)$ whenever either $n_p(s)$ or $[1 + d_c(s) d_p(s)]$ vanishes in the closed right half of the complex plane including infinity. This concludes the proof of Part (a) of Theorem 1.2.

In order to prove part (b) of the theorem we consider the following argument. By an analogous treatment similar to the proof of Theorem 1.1, $k(s) = k_1(s) + k_2(s)$ is a robust stabilizer provided $x(s)$ is chosen in such a way that

$$d_c(s) + x(s) n_p(s) \in J. \tag{3.5}$$

Equivalently a necessary and sufficient condition is given by the existence of $x(s) \in H$, $\Delta(s)$, $\Delta_1(s) \in J$ such that

$$d_c(s) + x(s) n_p(s) = \Delta_1(s), \tag{3.6}$$

$$d_c d_c d_p + x n_p (1 + d_c d_p) = \Delta(s), \tag{3.7}$$

where (3.7) is same as equation (3.4). The equations (3.6), (3.7) may be rewritten as

$$\Delta_1(s) = \frac{\Delta + d_c}{1 + d_c d_p}, \tag{3.8}$$

$$x(s) = \frac{\Delta - \Delta_1 d_c d_p}{n_p}. \tag{3.9}$$

Let s_1 be a point in the closed right half plane such that $n_p(s_1) = 0$ with multiplicity 1. Hence using (1.1) $d_c d_p(s_1) = 1$. Since $x(s) \in H$, using (3.9) we have $\Delta(s_1) = \Delta_1(s_1)$. Using (3.8) we have

$$\Delta(s_1) = d_c(s_1) = 1/d_p(s_1).$$

Hence $\Delta(s)$ intersects $1/d_p(s)$ at those points in the closed right half plane where $n_p(s)$ vanishes. On the other hand let s_2 be a point in the closed right half plane such that $1 + d_c d_p(s_2) = 0$. Since $\Delta_1(s) \in H$, using (3.8) it is clear that

$$\Delta(s_2) = -d_c(s_2) = 1/d_p(s_2).$$

Hence $\Delta(s)$ intersects $1/d_p(s)$ at those points in the closed right half plane where $1 + d_c d_p(s)$ vanishes. On the other hand since $\Delta_1(s)^{-1} \in H$, $\Delta(s)$ does not intersect $-d_c(s)$ at any other point in the closed right half plane. □

4. Proof of Corollary 1.3

The necessary condition that $g(s)$ be stabilizable by a stable compensator follows from Theorem 1.1, and is clear. On the other hand the necessary condition that $d_c/(1 + d_c d_p)$ is stabilizable by a stable, minimum phase compensator follows from (3.8). □

5. Conclusion

To conclude, we have posed and solved a robust reliable stabilization problem. Surprisingly this problem makes contact with interpolation theory [2]. Extension of our result to multiinput multioutput systems seems possible and is currently under investigation.

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