Motion and Shape Identification With Vision and Range

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***Abstract*—In this note, we consider the problem of motion and shape estimation using a camera and a laser range finder. The object considered is a plane which is undergoing a Riccati motion. The camera observes features on the moving plane perspectively. The range finder camera is capable of obtaining the range of the plane along a given "laser plane," which can either be kept fixed or can be altered in time. Finally, we assume that the identification is carried out as soon as the visual and range data is available or after a suitable temporal integration. In each of these various cases, we derive-to what extent motion and shape parameters are identifiable and characterize the results as orbit of a suitable group. The note does not emphasize any specific choice of algorithms.**

***Index Terms*—Affine motion, laser range finder, machine vision, parameter identification, perspective dynamical system, Riccati motion, rigid motion.**

I. Introduction

An important problem in machine vision, particularly in problems that relate to parameter estimation with a multiple sensor, it is important to be able to characterize to what extent motion and shape parameters can be identified. For example, in mobile robotics, one considers a mobile robotic platform carrying a set of sensors in the form of a charge-couple device (CCD) camera and laser range finder. The goal of the platform is to move around and observe stationary or moving objects in the environment, estimate the shape of the object and to estimate the associated motion parameters.

Earlier literature in this problem area specifically concentrated on solving the problem using CCD cameras alone [1], [3], [7], [8], [11]–[13], [15], [16]. In recent years, (see [14]), attention has gone back to using a single camera with parameter that can be identified possibly up to a depth ambiguity. Additionally, one uses a laser range finder to identify the unknown depth parameter and the hope is that a single CCD camera together with a range finder can identify parameters uniquely. This point of view is investigated in details in this note. We have considered three different motion models and two different configurations of the range finder, one in which the laser plane remains fixed and the other in which the plane can be altered in time. In each of these cases, we obtain the extent to which motion and shape parameters can be identified. Of course, our results are independent of the choice of specific algorithm that would be required to carry out the identification.

II. Background and Problem Formulation

In this section, we begin by describing the basic underlying motion and shape estimation problem that has already been considered in [4] and [5]. Let (X, Y, Z) be the three coordinates of $\mathbb{R}^3$ and let us consider a plane described as

$$\overline{p}X + \overline{q}Y + \overline{s}Z + \overline{w} = 0. \tag{2.1}$$

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The parameters that describe the plane are given by $p = \overline{p}/\overline{w}$, $q = \overline{q}/\overline{w}$, $s = \overline{s}/\overline{w}$. The vector $[\overline{p}\ \overline{q}\ \overline{s}\ \overline{w}]^T$ is the homogeneous coordinate vector of the plane (2.1) viewed as a point in $\mathbb{RP}^3$, the real projective space [9], [10] of homogeneous lines in $\mathbb{R}^4$.

We now assume that the plane (2.1) undergoes a homogeneous dynamics of the form

$$\frac{d}{dt}\begin{bmatrix}\overline{p}\\ \overline{q}\\ \overline{s}\\ \overline{w}\end{bmatrix} = \begin{pmatrix}-A^T & -f\\ -b^T & -d\end{pmatrix}\begin{bmatrix}\overline{p}\\ \overline{q}\\ \overline{s}\\ \overline{w}\end{bmatrix} \tag{2.2}$$

which would be our underlying motion model of the "shape dynamics" where

$$A = (a_{ij}) \;\; b = (\,b_1 \quad b_2 \quad b_3\,)^T \;\; f^T = (\,-f_1 \quad -f_2 \quad -f_3\,).$$

It is easy to see that the 4×4 matrix $\mathcal{A}$ in (2.2) is completely arbitrary except addition by a diagonal matrix of the form λI, for any scalar λ. Thus, without any loss of generality we assume that trace $\mathcal{A} = 0$. We assume that a CCD camera observes feature points (X, Y, Z) on the plane (2.1) by projecting onto the image plane of the camera using a "perspective" projection model. If the image plane has coordinates given by (x, y), we have

$$x = \frac{X}{Z} \quad y = \frac{Y}{Z}, \qquad Z \neq 0. \tag{2.3}$$

The dynamics of the projection of the feature point on the image plane is described by

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}d_1\\ d_2\end{pmatrix} + \begin{pmatrix}d_3 & d_4\\ d_5 & d_6\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}d_7x^2 + d_8xy\\ d_8y^2 + d_7xy\end{pmatrix} \tag{2.4}$$

which is also called the optical flow dynamics [6]. The optical flow is parameterized by eight parameters $d_1, \ldots, d_8$ which are time varying and are functions of the shape parameters p, q, s. If we define

$$d_1 = \frac{y_1}{y_9}, d_2 = \frac{y_2}{y_9}, \ldots, d_8 = \frac{y_8}{y_9}$$

where $[d_1\ \ldots\ d_9]^T$ are homogeneous coordinates of the coefficient vector of the optical flow dynamics (2.4), we have the following observation function for the perspective projection model (2.3):

$$\begin{bmatrix}y_1\\ y_2\\ \vdots\\ y_9\end{bmatrix} = \begin{pmatrix}0 & 0 & -b_1 & a_{13}\\ 0 & 0 & -b_2 & a_{23}\\ -b_1 & 0 & b_3 & a_{11}-a_{33}\\ 0 & -b_1 & 0 & a_{12}\\ -b_2 & 0 & 0 & a_{21}\\ 0 & -b_2 & b_3 & a_{22}-a_{33}\\ b_3 & 0 & 0 & -a_{31}\\ 0 & b_3 & 0 & -a_{32}\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{bmatrix}\overline{p}\\ \overline{q}\\ \overline{s}\\ \overline{w}\end{bmatrix}. \tag{2.5}$$

We now turn our attention to the laser range finder, which observes the range of the plane (2.1) along a laser plane and returns the parameters of the intersecting line

$$\begin{pmatrix}m_1 & m_2 & m_3 & m_4\\ n_1 & n_2 & n_3 & n_4\end{pmatrix}\begin{bmatrix}X\\ Y\\ Z\\ 1\end{bmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}. \tag{2.6}$$

Of course, the line (2.6) is always contained in (2.1), but it changes in time because the plane (2.1) moves in time. The line (2.6) is also

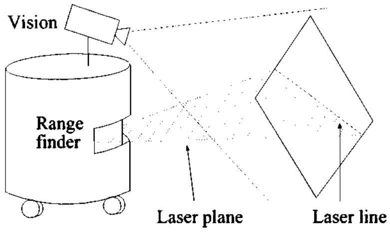

Fig. 1. Laser range finder and vision.

contained in a laser plane (see Fig. 1), and in this note, we consider two different cases, one in which laser plane is time invariant and the other case in which the laser plane can move in time.

If we define

$$\eta_{ij} = m_i n_j - m_j n_i \tag{2.7}$$

where $i,\ j = 1,2,3,4, i \neq j$, we have a homogeneous coordinate vector

$$[\eta_{12} \quad \eta_{13} \quad \eta_{14} \quad \eta_{23} \quad \eta_{24} \quad \eta_{34}]^T \tag{2.8}$$

of Plücker coordinates (see [2]) which satisfy the quadratic equation

$$\eta_{14}\eta_{23} + \eta_{24}\eta_{31} + \eta_{34}\eta_{12} = 0.$$

The Plücker coordinates (2.8) parameterize the line (2.6) observed by the "range finder," and we shall assume that these coordinates (2.8) are available as a function of time. Since the line (2.6) is contained in the plane (2.1), it follows that the matrix:

$$\begin{pmatrix} \overline{p} & \overline{q} & \overline{s} & \overline{w} \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \end{pmatrix}$$

has rank <3, constraining the shape parameters as follows:

$$\begin{pmatrix} \eta_{34} & 0 & \eta_{41} & \eta_{13} \\ 0 & \eta_{34} & \eta_{42} & \eta_{23} \\ \eta_{23} & \eta_{31} & \eta_{12} & 0 \\ \eta_{24} & \eta_{41} & 0 & \eta_{12} \end{pmatrix} \begin{bmatrix} \overline{p} \\ \overline{q} \\ \overline{s} \\ \overline{w} \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.9}$$

Note that the generic rank of the 4×4 matrix in (2.9) is 2, so that (2.9) imposes a pair of constraints on the shape parameters. The problem of motion and shape estimation with camera and range is described in Section III.

III. Estimation for the Generic Case Under Perspective Projection

We consider a plane (2.1) that moves according to a dynamics of the form (2.2). We assume that feature points on the plane are observed perspectively so that the coefficients of the optical flow (2.4) are assumed to be known. It follows that the state vector:

$$\mathcal{P} = (\overline{p}\ \overline{q}\ \overline{s}\ \overline{w})^T \tag{3.1}$$

satisfies (2.5). We also assume that at every instant of time a line (2.6) is observed by the laser range finder on the plane (2.1). It follows that the state vector (3.1) satisfies (2.9). Thus, we have a dynamical system

$$\frac{d}{dt}[\mathcal{P}] = \mathcal{A}[\mathcal{P}], \text{ trace } \mathcal{A} = 0 \tag{3.2}$$

$$[\mathcal{Y}] = \mathcal{C}[\mathcal{P}] \tag{3.3}$$

$$\mathcal{D}[\mathcal{P}] = 0 \tag{3.4}$$

where $\mathcal{A}$ is the 4×4 matrix in (2.2), $\mathcal{C}$ is the 9×4 matrix in (2.5) and $\mathcal{D}$ is the 4×4 matrix in (2.9). We consider the following problem.

Problem 3.1: Consider the homogeneous dynamical system (3.2) together with a CCD camera and laser-range-finder-based observations (3.3) and (3.4), respectively, where we assume that $[\mathcal{Y}(t)]$ is observed in a time interval $[0, T_1]$ and $\mathcal{D}(t)$ is observed in another time interval $[T_2, T_3]$. The problem is to estimate the motion parameters $\mathcal{A}$ and initial position of the plane $\mathcal{P}(0)$ to the extent possible.

Note that in the statement of Problem 3.1 we have deliberately chosen the time interval $[0, T_1]$, when the CCD camera makes its observations, and the time interval $[T_2, T_3]$, when the laser range finder collects the range information, to be different. In this note, we shall consider the following two distinct cases:

Case 1) when $T_1 = T_2 = T_3$, i.e., when the camera observes in a time interval $[0, T_1]$, $T_1 > 0$ and the laser range finder collections the range information at a specific time $t = T_1$;

Case 2) when $T_1 = T_2$ and $T_3 > T_2$, i.e., when the camera observes in a time interval $[0, T_1]$, $T_1 > 0$ and the laser range finder collects the range information in a time interval $[T_1, T_3]$, $T_3 > T_1$ after the camera has completed collecting its observation sequence.

Regarding Cases 1) and 2), we have the following important theorems.

Theorem 3.1: Consider the dynamical system (3.2)–(3.4). For a generic choice of the triplet $(\mathcal{A}, \mathcal{C}, [\mathcal{P}(0)])$, the set of all triplets that produce the same output (3.3) in a time interval $[0, T_1]$ and satisfy the constraint (3.4) at a specific time $t = T_1$ is described as

$$(\,Q\mathcal{A}Q^{-1}, \quad \mathcal{C}Q^{-1}, \quad [Q\mathcal{P}(0)]\,) \tag{3.5}$$

where Q is a nonsingular matrix of the form

$$Q^{-1} = \begin{pmatrix} 1 & 0 & 0 & \alpha\left(1 - \frac{q_{44}}{q_{11}}\right) + \frac{\beta q_{34}}{q_{11}} \\ 0 & 1 & 0 & \gamma\left(1 - \frac{q_{44}}{q_{11}}\right) + \frac{\delta q_{34}}{q_{11}} \\ 0 & 0 & 1 & \frac{q_{34}}{q_{11}} \\ 0 & 0 & 0 & \frac{q_{44}}{q_{11}} \end{pmatrix} \tag{3.6}$$

where

$$\alpha = \frac{\eta_{13}}{\eta_{34}} \quad \beta = \frac{\eta_{14}}{\eta_{34}} \quad \gamma = \frac{\eta_{23}}{\eta_{34}} \quad \delta = \frac{\eta_{24}}{\eta_{34}} \tag{3.7}$$

and where q_{34}/q_{11} and q_{44}/q_{11} are arbitrary.

Note, first of all, that the η_{ij}-s used in (3.7) are the Plücker coordinates of the line (2.6) observed by the laser range finder and are defined in (2.7). Note also that for a pair of real numbers μ, λ $(\lambda \neq 0)$ and for $Q \in GL(4)$ we define the following group action on the triplet $(\mathcal{A}, \mathcal{C}, [\mathcal{P}(0)])$:

$$\chi_1 \colon GL(n) \times Q \longrightarrow Q$$
$$(Q, \mathcal{A}, \mathcal{C}, [\mathcal{P}(0)]) \longmapsto (Q\mathcal{A}Q^{-1}, \mathcal{C}Q^{-1}, [Q\mathcal{P}(0)]) \tag{3.8}$$

$$\chi_2 \colon \mathbb{R} \times Q \longrightarrow Q$$
$$(\mu, \mathcal{A}, \mathcal{C}, [\mathcal{P}(0)]) \longmapsto (\mu I + \mathcal{A}, \mathcal{C}, [\mathcal{P}(0)]) \tag{3.9}$$

$$\chi_3 \colon \mathbb{R}^+ \times Q \longrightarrow Q$$
$$(\lambda, \mathcal{A}, \mathcal{C}, [\mathcal{P}(0)]) \longmapsto (\mathcal{A}, \lambda\mathcal{C}, [\mathcal{P}(0)]). \tag{3.10}$$

The combined action of the three groups in (3.8)–(3.10) would be called the perspective group action on the triplet $(\mathcal{A}, \mathcal{C}, [\mathcal{P}(0)])$. The main result of Theorem 3.1 is that the parameters $(\mathcal{A}, \mathcal{C}, [\mathcal{P}(0)])$ can be identified up to orbits of a subgroup of the perspective group (3.8)–(3.10), given by $\mu = 0$, $\lambda = 1$ and Q defined by (3.6). Note that (3.6) defines a two-parameter subgroup of $GL(4)$ parameterized by $\theta_1 = q_{44}/q_{11}$, $\theta_2 = q_{34}/q_{11}$ for fixed $\alpha, \beta, \gamma, \delta$. Note that if

$$P(\theta_1, \theta_2) = \begin{pmatrix} 1 & 0 & 0 & \alpha(1 - \theta_1) + \beta\theta_2 \\ 0 & 1 & 0 & \gamma(1 - \theta_1) + \delta\theta_2 \\ 0 & 0 & 1 & \theta_2 \\ 0 & 0 & 0 & \theta_1 \end{pmatrix}$$

then it is easy to check that

$$P(\theta_1, \theta_2) \cdot P(\theta_1', \theta_2') = P(\theta_1 \theta_1', \theta_2' + \theta_2 \theta_1')$$

and

$$P\left(\theta_1^{-1}, -\theta_2 \theta_1^{-1}\right) = P(\theta_1, \theta_2)^{-1}.$$

Thus, the two-parameter subset of $GL(4)$ described in (3.6) is indeed a subgroup of $GL(4)$.

Proof of Theorem 3.1: Our proof relies on a theorem proven in [4] that if the triple $(\mathcal{A}, \mathcal{C}, [\mathcal{P}(0)])$ is minimal as a homogeneous dynamical system (3.2), (3.3), then the set of all triplets that produce the same output as that of (3.3) is given by the orbits of the group described by (3.8)–(3.10). Additionally, since trace $\mathcal{A} = 0$ it follows that $\mu = 0$. Moreover, in order for $\lambda \mathcal{C} Q^{-1}$ to have the same structure as that of $\mathcal{C}$ it follows that $\lambda = 1$ and Q^{-1} must be of the form:

$$Q^{-1} = \begin{pmatrix} q_{11} & 0 & 0 & q_{14} \\ 0 & q_{11} & 0 & q_{24} \\ 0 & 0 & q_{11} & q_{34} \\ 0 & 0 & 0 & q_{44} \end{pmatrix} \tag{3.11}$$

under the following set of generic conditions:

$$b_1 a_{23} \neq b_2 a_{13} \quad b_1 a_{32} \neq b_3 a_{12} \quad b_2 a_{31} \neq b_3 a_{21}. \tag{3.12}$$

Thus, under the generic conditions (3.12), the set of all triplets $(\mathcal{A}, \mathcal{C}, [\mathcal{P}(0)])$ that would produce the same output as that of (3.3) is given by (3.5) where Q^{-1} is given by (3.11), indicating that these are the triplets that cannot be distinguished by the camera.

We now proceed to describe the points in the orbit (3.5), (3.11) that would satisfy the constraint (3.4) imposed by the laser range finder. Among all the Q^{-1} of the form (3.11) we propose to characterize those that are such that $[Q\mathcal{P}(0)] \in \mathrm{Ker}\ \mathcal{D}$ for all $\mathcal{P}(0) \in \mathrm{Ker}\ \mathcal{D}$. It is easy to see that

$$\mathrm{Ker}\ \mathcal{D} = \mathrm{span}\left\{ \begin{pmatrix} \frac{\eta_{14}}{\eta_{34}} \\ \frac{\eta_{24}}{\eta_{34}} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\eta_{31}}{\eta_{34}} \\ \frac{\eta_{32}}{\eta_{34}} \\ 0 \\ 1 \end{pmatrix} \right\}.$$

In order for $\mathrm{Ker}\mathcal{D}$ to be Q invariant where Q is of the form (3.11), it is easy to check that we must have

$$\begin{aligned} q_{14} &= \frac{\eta_{13}}{\eta_{34}}(q_{11} - q_{44}) + \frac{\eta_{14}}{\eta_{34}} q_{34} \\ q_{24} &= \frac{\eta_{23}}{\eta_{34}}(q_{11} - q_{44}) + \frac{\eta_{24}}{\eta_{34}} q_{34} \end{aligned}$$

which is precisely the structure prescribed in (3.6). (Q.E.D.)

Before we proceed, it may be useful to characterize the two-parameter orbit described by Theorem 3.1 on the space of motion parameters A, b, f, d and on the initial position of the plane (2.1) given by $p(0), q(0), s(0)$. The orbit on the motion parameters is described as follows:

$$\begin{aligned} -A^T &\longmapsto -A^T + \frac{\xi}{\zeta} b^T, \quad -b^T \longmapsto -\frac{1}{\zeta} b^T \\ -d &\longmapsto \left(-d - b^T \frac{\xi}{\zeta}\right) \\ -f &\longmapsto \delta\left(-f + \frac{\xi}{\zeta} d - A^T \frac{\xi}{\zeta} + \frac{\xi}{\zeta} b^T \frac{\xi}{\zeta}\right) \end{aligned} \tag{3.13}$$

where

$$\begin{pmatrix} I & \xi \\ 0 & \zeta \end{pmatrix}$$

is the matrix (3.6) so that $\zeta = q_{44}/q_{11}$ and ξ is appropriately defined. If we define $\pi(0) = (p(0)\ q(0)\ s(0))^T$ to be the initial position of the plane (2.1), the orbit on the initial position is described as follows:

$$\pi(0) \longmapsto \zeta \pi(0) - \xi. \tag{3.14}$$

An important point that is perhaps not very clear from the description so far is that the parameters $\alpha, \beta, \gamma, \delta$ in (3.6) can change with time because they are functions of the Plücker coordinates of the line (2.6) observed by the laser range finder, and that this line (2.6) moves in time. It follows that the orbit (3.13), (3.14) is not the same at every instance of time and in Theorem 3.1 we chose $t = T_1$. On the other hand, since the motion parameters A, b, d, f and the initial position $\pi(0)$ of the plane (2.1) are time invariant, in principal we can intersect the orbits (3.13), (3.14) at various instance of time for $t \geq T_1$, one for each measurement of the laser range finder. This is precisely what is done in the next theorem.

Theorem 3.2: Consider the dynamical system (3.2)–(3.4). For a generic choice of the triplets $(\mathcal{A}, \mathcal{C}, [\mathcal{P}(0)])$, there is an unique triplet that produces the same output (3.3) in a time interval $[0, T_1]$ and satisfy the constraint (3.4) in a time interval $[T_1, T_3]$ where $T_3 > T_1 > 0$, provided that the lines observed by the laser range finder satisfy the generic constraint

$$(\delta(t_2) - \delta(t_1))(\alpha(t_2) - \alpha(t_1)) \neq (\beta(t_2) - \beta(t_1))(\gamma(t_2) - \gamma(t_1)) \tag{3.15}$$

for $t_1, t_2 \in [T_1, T_3]$ and where $\alpha, \beta, \gamma, \mathrm{and} \delta$ are defined in (3.7).

Proof of Theorem 3.2: Let us consider two distinct time instants $t_1, t_2 \in [T_1, T_3]$ at which time we assume that the laser range finder has been used to make measurements. It follows that we have two distinct orbits in the parameter space one for each time t_1 and t_2 and our goal is to construct the intersection between the two orbits. Assume that A^*, b^*, d^*, f^* are the true motion parameters and that $\pi(0)^*$ is the true initial position of the plane. Let us also assume that $A, b, d, f, \pi(0)$ be another value of the parameters that are contained in orbits passing through the true value for each time t_1 and t_2. It follows from (3.13) that in particular we have

$$-\frac{1}{\zeta(t_1)} b^{*T} = -\frac{1}{\zeta(t_2)} b^{*T}.$$

Thus, if $b^* \neq 0$, we must have

$$\zeta(t_1) = \zeta(t_2). \tag{3.16}$$

It also follows from (3.13) that

$$\frac{\xi(t_1)}{\zeta(t_1)} b^{*T} = \frac{\xi(t_2)}{\zeta(t_2)} b^{*T}$$

implying that if $b^* \neq 0$ we have

$$\xi(t_1) = \xi(t_2). \tag{3.17}$$

Recalling the structure of the vector ξ from (3.6), it follows that

$$\begin{pmatrix} \alpha(t_1) - \alpha(t_2) & \beta(t_1) - \beta(t_2) \\ \gamma(t_1) - \gamma(t_2) & \delta(t_1) - \delta(t_2) \end{pmatrix} \begin{pmatrix} 1 - \frac{q_{44}}{q_{11}} \\ \frac{q_{34}}{q_{11}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.18}$$

Thus, if the functions $\alpha, \beta, \gamma, \delta$ measured by the laser range finder satisfy the generic condition (3.15), we conclude from (3.18) that $q_{34}/q_{11} = 0$, $q_{44}/q_{11} = 1$. Hence the matrix Q is an identity matrix. (Q.E.D.)

For the generic constraint (3.15) to be satisfied, the line (2.6) observed by the laser range finder has to be suitably located. In a specific experimental setup, the laser range finder is permanently fixed to a mobile platform and collects the range information along a fixed plane

$$X = h \tag{3.19}$$

at a given constant height h. The equation of the observed line (2.6) in this case reduces to

$$\begin{pmatrix} 1 & 0 & 0 & -h \\ 0 & \overline{q} & \overline{s} & \overline{w} \end{pmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.20}$$

The Plücker coordinates (2.7) corresponding to the line (3.20) are given by $\eta_{12} = \overline{q}$, $\eta_{13} = \overline{s}$, $\eta_{14} = \overline{w}$, $\eta_{23} = 0$, $\eta_{24} = h\overline{q}$ and $\eta_{34} = h\overline{s}$. Thus, we have the parameters $\alpha, \beta, \gamma, \delta$ given by (3.7) as follows:

$$\alpha = \frac{1}{h} \quad \beta = \frac{\overline{w}}{h\overline{s}} \quad \gamma = 0 \quad \delta = \frac{\overline{q}}{\overline{s}} \tag{3.21}$$

where h is assumed to be a constant and $\overline{q}, \overline{s}, \mathrm{and} \overline{w}$ are functions of time. Clearly, (3.15) is not satisfied and we have the following theorems.

Theorem 3.3: Consider the dynamical system (3.2)–(3.4). With assumptions as in Theorem 3.1 except that the laser range finder collects the range information along a fixed plane (3.19) so that (3.21) is satisfied. The triplets $(\mathcal{A}, \mathcal{C}, [\mathcal{P}(0)])$ are identifiable up to orbits of the following subgroup of $GL(4)$ given by

$$Q^{-1} = \begin{pmatrix} 1 & 0 & 0 & \frac{1 - \frac{q_{44}}{q_{11}} + \frac{\overline{w} q_{34}}{\overline{s} q_{11}}}{h} \\ 0 & 1 & 0 & \frac{\overline{q} q_{34}}{\overline{s} q_{11}} \\ 0 & 0 & 1 & \frac{q_{34}}{q_{11}} \\ 0 & 0 & 0 & \frac{q_{44}}{q_{11}} \end{pmatrix}. \tag{3.22}$$

Theorem 3.4: Consider the dynamical system (3.2)–(3.4). With assumptions as in Theorem 3.2 except that the laser range finder collects the range information along a fixed plane (3.19) so that (3.15) is not satisfied. The triplets $(\mathcal{A}, \mathcal{C}, [\mathcal{P}(0)])$ are identifiable up to orbits of the following subgroup of $GL(4)$:

$$Q^{-1} = \begin{pmatrix} 1 & 0 & 0 & \frac{1-\theta}{h} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \theta \end{pmatrix} \tag{3.23}$$

where the orbit is described by (3.5) provided that the vectors $(\overline{q}(t_1)/\overline{s}(t_1) \;\; \overline{w}(t_1)/\overline{s}(t_1))$ and $(\overline{q}(t_2)/\overline{s}(t_2) \;\; \overline{w}(t_2)/\overline{s}(t_2))$ are linearly independent at two different time instances $t_1, t_2 \in [T_1, T_3]$, where $T_3 > T_1 > 0$.

Proof of Theorem 3.3: Follows easily from Theorem 3.1 by substituting (3.21) in (3.6) to obtain (3.22). (Q.E.D.)

Proof of Theorem 3.4: From the proof of Theorem 3.2, we conclude that if $b^* \neq 0$, we must have (3.16) and (3.17). Recalling the structure of ξ from (3.22) and recalling that h is time invariant it follows that:

$$\begin{aligned} \left(\frac{\overline{w}(t_1)}{\overline{s}(t_1)} - \frac{\overline{w}(t_2)}{\overline{s}(t_2)}\right) \frac{q_{34}}{q_{11}} &= 0 \\ \left(\frac{\overline{q}(t_1)}{\overline{s}(t_1)} - \frac{\overline{q}(t_2)}{\overline{s}(t_2)}\right) \frac{q_{34}}{q_{11}} &= 0. \end{aligned} \tag{3.24}$$

Thus, if

$$\begin{pmatrix} \frac{\overline{w}(t_1)}{\overline{s}(t_1)} \\ \frac{\overline{q}(t_1)}{\overline{s}(t_1)} \end{pmatrix} \neq \begin{pmatrix} \frac{\overline{w}(t_2)}{\overline{s}(t_2)} \\ \frac{\overline{q}(t_2)}{\overline{s}(t_2)} \end{pmatrix} \tag{3.25}$$

it follows from (3.24) and (3.25) that:

$$q_{34} = 0. \tag{3.26}$$

Combining (3.22) and (3.26) we obtain Q^{-1} in the form (3.23) where $\theta = q_{44}/q_{11}$ is a free parameter. (Q.E.D.)

The one-parameter orbit prescribed by the Theorem 3.4 is described as follows:

$$\begin{aligned} a_{i1} &\longmapsto a_{i1} + \frac{b_i}{h}\left(1 - \frac{1}{\theta}\right), \quad i = 1,2,3 \\ a_{ij} &\longmapsto a_{ij}, \quad i = 1,2,3, \quad j = 2,3 \\ b_i &\longmapsto \frac{b_i}{\theta}, \quad i = 1,2,3 \\ d &\longmapsto d + \frac{b_1}{h}\left(\frac{1}{\theta} - 1\right) \\ f_1 &\longmapsto \theta\left\{f_1 + \left(\frac{a_{11} - d}{h}\right)\left(1 - \frac{1}{\theta}\right) \right. \\ &\qquad \left. + \frac{b_1}{h^2}\left(\frac{1-\theta}{\theta}\right)^2\right\} \\ f_j &\longmapsto \theta\left\{f_j + \frac{a_{1j}}{h}\left(1 - \frac{1}{\theta}\right)\right\}, \quad j = 2,3. \end{aligned}$$

IV. Estimation for Affine and Rigid Motion Under Perspective Projection

In this section, we consider two important special cases of the motion dynamics (2.2) that are described as follows:

$$\frac{d}{dt}\begin{bmatrix} \overline{p} \\ \overline{q} \\ \overline{s} \\ \overline{w} \end{bmatrix} = \begin{pmatrix} -A^T & 0 \\ -b^T & -d \end{pmatrix} \begin{bmatrix} \overline{p} \\ \overline{q} \\ \overline{s} \\ \overline{w} \end{bmatrix} \tag{4.1}$$

and

$$\frac{d}{dt}\begin{bmatrix} \overline{p} \\ \overline{q} \\ \overline{s} \\ \overline{w} \end{bmatrix} = \begin{pmatrix} \Omega & 0 \\ -b^T & 0 \end{pmatrix} \begin{bmatrix} \overline{p} \\ \overline{q} \\ \overline{s} \\ \overline{w} \end{bmatrix}. \tag{4.2}$$

Dynamics described by (4.1) is called "affine motion" and the dynamics described by (4.2) is called "rigid motion," (see [5] for details). Let us first concentrate on the affine dynamics (4.1). We assume that the CCD camera observes via perspective projection in a time interval $[0, T_1]$. We also assume that the laser range finder provides the range information at a given time $t = T_1$. It would follow from Theorem 3.1, and from the orbit described in (3.13), that in this case parameters can be identified up to orbit of a subgroup of (3.6) for which f remains 0. Substituting $f = 0$ in (3.13) we obtain

$$A^T\xi = \xi\left(d + \frac{b^T}{\zeta}\xi\right) \tag{4.3}$$

as an additional restriction on (3.6) imposed by the affine structure of the motion dynamics in (4.1).

From (4.3), we infer that the vector "ξ is either 0" or "ξ is an eigenvector of A^T for a real eigenvalue." We now look at these two cases separately.

When $\xi = 0$, if we assume generically that

$$(\alpha(T_1)\,\gamma(T_1)) \neq 0 \tag{4.4}$$

it follows that $\zeta = 1$ and we conclude that $Q^{-1} = I$. The orbit (3.5) for $Q^{-1} = I$ is a single point.

When $\xi \neq 0$, it follows from (4.3) that $\xi = \xi^*$ is an eigenvector of A^T for a real eigenvalue say λ^*. We need to solve the equation

$$\xi^* = \begin{pmatrix} \alpha \\ \gamma \\ 0 \end{pmatrix}\left(1 - \frac{q_{44}}{q_{11}}\right) + \begin{pmatrix} \beta \\ \delta \\ 1 \end{pmatrix}\frac{q_{34}}{q_{11}}$$

for ξ^* to be an admissible vector. Let us now assume generically that No eigenvector of A^T belongs to the linear span of

$$\text{the two vectors } (\alpha\ \gamma\ 0)^T \text{ and } (\beta\ \delta\ 1)^T. \tag{4.5}$$

It would follow that, generically, (4.5) would never be satisfied.

TABLE I
ORBIT DIMENSIONS OF THE UNOBSERVABLE PARAMETERS FOR VARIOUS CHOICES OF THE SENSORS. THE LASER PLANE CHANGES IN TIME

	Perspective Projection		
	Vision	Vision and Range at a given time	Vision and Range in a time interval
Riccati motion	4	2	No ambiguity
Affine motion	1	No ambiguity	No ambiguity
Rigid motion	1	No ambiguity	No ambiguity

Hence, the only solution to our problem is when $\xi = 0$ and we have the following theorem.

Theorem 4.1: Consider the homogeneous dynamical system (4.1), (2.5), (2.9) where we assume that $f = 0$. For a generic choice of the triplet $(A, \mathcal{C}, [\mathcal{P}(0)])$, there is an unique triplet that produce the same output (2.5) in a time interval $[0, T_1]$ and satisfy the constraint (2.9) at $t = T_1$ provided that the generic conditions (4.4) and (4.5) are satisfied.

Remark 4.1: Many special cases of Theorem 4.1 can be obtained. In particular, when the laser range finder observes the range along a given fixed plane (3.19), $\alpha, \beta, \gamma, \text{and} \delta$ take special values described in (3.21). The generic condition (4.4) is automatically satisfied since $h \neq 0$. The two vector in (4.5) reduces to $(1\,0\,0)^T$ and $(\overline{w}\; h\overline{q}\; h\overline{s})^T$. Another special case arises when the motion dynamics is rigid as described by (4.2). In each of these cases, parameters are identified uniquely and the details are left to the readers.

V. CONCLUSION

In this note, we use a CCD camera and a laser range finder to estimate motion and shape parameters. Often in order to get the parameter estimates, only the image data is utilized but in this note we have shown that in order to get good estimates, we should utilize the laser range finder data as well. If we compare the dimension of the parameter ambiguity, measured in terms of orbit of the subgroup, between the case wherein only the CCD camera is used (see [4] and [5]) with the case in which additionally the laser data is also used, we can obtain Table I.

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On Nonlinear Controllability and Series Expansions for Lagrangian Systems With Dissipative Forces

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Abstract—**This note presents series expansions and nonlinear controllability results for Lagrangian systems subject to dissipative forces. The treatment relies on the assumption of dissipative forces of linear isotropic nature. The approach is based on the affine connection formalism for Lagrangian control systems, and on the homogeneity property of all relevant vector fields.**

Index Terms—**Mechanical control systems, nonlinear controllability, series expansions.**

I. INTRODUCTION

This note presents novel controllability and perturbation analysis results for control systems with Lagrangian structure. The work belongs to a growing body of research devoted to the geometric control of mechanical systems. The objective is the development of coordinate-free analysis and design tools applicable in a unified manner to robotic

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