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Visually guided ranging from observations of points, lines and curves via an identifier based nonlinear observer

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Abstract

In this paper we consider the problem of estimating the range information of features on an affine plane in $\mathbb{R}^3$ by observing its image with the aid of a CCD camera, wherein we assume that the camera is undergoing a known motion. The features considered are points, lines and planar curves located on planar surfaces of static objects. The dynamics of the moving projections of the features on the image plane have been described as a suitable differential equation on an appropriate feature space. This dynamics is used to estimate feature parameters from which the range information is readily available. In this paper the proposed identification has been carried out via a newly introduced identifier based observer. Performance of the observer has been studied via simulation.

Keywords: Range estimation; Feature correspondence; Observer design; Machine vision

1. Introduction

In this paper we consider the problem of range estimation from a sequence of images generated by a moving camera. We assume that the camera rotates and translates with a known, possibly time varying, angular and linear velocities, respectively. We assume that we have identified a suitable feature on the image plane and that we are able to establish a correspondence of this feature between various consecutive frames. The features considered in this paper are points, lines and planar curves. The main result that we show in this paper is that by tracking the motion of a feature on the image plane, one is able to derive a nonlinear filter that can estimate the true position of the feature in the 3D space.

The above problem of estimating the location of features can be posed as a problem of state estimation of a nonlinear system. In order to do this we first establish that the features (viz. points, lines and curves) can be parameterized as points in a feature space which can be given the structure of an analytic manifold (see [1, Ch. 12] for details). Once an appropriate parametrization has been obtained, we need to describe the dynamics of the feature as a consequence of the motion of the camera. Feature dynamics is described as a dynamical system on the feature space with state variables describing the location of the features. Finally we need to quantify a set of measured (observed) variables, i.e. variables that can be recovered from

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a sequence of images. In order to design an estimator, states of the dynamical system must be *observable* from the measured variables in order for the estimates to converge to the true values of parameters. In this paper we outline the procedure for such an estimator design.

The problem of range estimation, when the observed features are points, has already been considered in a number of papers including [10, 11, 17–19]. In all of these references except [11] the extended Kalman filter (EKF) has been used. The case of range estimation from observation of lines as features has been considered in [1]. In general line based estimation problems have been considered in [5, 9, 12, 16] and curve based problems have been considered in [4, 6, 7].

This paper contains a new result in the area of nonlinear state estimation for systems that have a particular special structure. We show that for such systems one can build an observer, which we shall call the identifier based observer (IBO). A sufficient condition for convergence of the estimates have been derived. This sufficient condition can be verified in advance or monitored on-line. The nonlinear state observer can now be applied to the problem of estimating the states of the feature dynamics enabling one to estimate the location of the features. The performance of the observer has been verified via simulation.

The proposed nonlinear filter (IBO) is based on a parameter identifier considered in model reference adaptive systems (see [14]). Morgan and Narendra [13] have proved the stability of a nonlinear system which arose in this approach to parameter identification. However, the problem considered here is much more difficult because the unknown portion of the states is not constant, but changes as a nonlinear function of the states. In other words, the result of [13] would apply to the systems of the form (2.1) only when $\dot{x}_2 \equiv 0$ while the estimator needed here must converge even when $\dot{x}_2 \neq 0$. The price that had to be paid for this extension is that a high-gain compensation is necessary, estimates must be kept bounded, and the observability condition under which estimates converge is stronger than the one used in [13].

The proposed IBO has not yet been implemented with real image data. Since the IBO is recursive, guaranteed to converge in an arbitrarily large (but bounded) set of initial conditions, and since the convergence is exponential (although the rate of convergence actually depends on the choice of the initial condition) we believe that the performance of IBO would be reliable, robust and would quickly compute the range on real data. For the EKF no such convergence guarantees are available.

In summary, this paper introduces a unified framework under which range estimation problems can be studied (assuming known motion) via observing points, lines and planar second-order curves utilizing a recursive (nonlinear) filter. No comparable treatment exists in the literature. Furthermore if the motion parameters are unknown, existence of a similar filter is presently unknown.

2. Identifier based observer

In this section we develop an observer for nonlinear systems based on a parameter identifier. A class of nonlinear systems for which this observer is applicable is given by

$$\dot{x}_1 = w^{\mathrm{T}}(x_1, u)x_2 + \phi(x_1, u), \qquad \dot{x}_2 = g(x_1, x_2, u), \qquad y = x_1, \tag{2.1}$$

where $x_1 \in X_1 \subset \mathbb{R}^{n_1}, x_2 \in X_2 \subset \mathbb{R}^{n_2}$ and $u \in U \subset \mathbb{R}^k$. Denote by $x = [x_1, x_2]^{\mathrm{T}}$. It follows that for $n = n_1 + n_2$, $x \in X \subset \mathbb{R}^n$ and $X = X_1 \oplus X_2$. Then $n_1 \times n_2$ matrix $w^{\mathrm{T}}(x_1, u)$ and the vector $g(x_1, x_2, u)$ are in general nonlinear functions of their arguments.

In order to design the observer we need the nonlinear vector g to satisfy Lipschitz condition. Since the global Lipschitz condition is too restrictive we shall assume boundedness of the states of system (2.1) and the local Lipschitz condition. If the global Lipschitz condition is satisfied we do not need the states to be bounded.

Assumption 2.1. Suppose that there exists a constant $M > 0$ such that $\|x(t)\| < M$ for every $t > 0$. Denote by Ω the following set: $\Omega = \{x \in \mathbb{R}^n\colon \|x(t)\| < M\}$. For a fixed constant $\gamma > 1$ define $\Omega_\gamma = \{x \in \mathbb{R}^n\colon$

$\|x(t)\| < \gamma M\}$. Assume that the function $g(x_1, x_2, u)$ satisfies Lipschitz condition in Ω_γ with respect to x_2, i.e. assume that there exists a positive constant α, such that

$$\|g(x_1, x_2, u) - g(x_1, z_2, u)\| < \alpha \|x_2 - z_2\| \tag{2.2}$$

for all $x_2, z_2 \in \Omega_\gamma \cap X_2$, uniformly in $x_1 \in \Omega_\gamma \cap X_1$ and $u \in U$.

The second assumption is an observability assumption. It resembles the persistence of excitation condition (see for example, [14, p. 247]) but is stronger.

Assumption 2.2. Suppose that the regressor matrix $w^{\mathrm{T}}(x_1, u)$ is piecewise smooth, uniformly bounded together with its first time derivative and suppose that there exist positive constants L_1, L_2, r, and β such that

$$\|w^{\mathrm{T}}(x_1, u)\| < L_1 \quad \text{and} \quad \left\|\frac{\mathrm{d}w^{\mathrm{T}}(x_1, u)}{\mathrm{d}t}\right\| < L_2, \tag{2.3}$$

$$\rho\beta I < \int_t^{t+\rho} w(x_1(\tau), u(\tau))\, w^{\mathrm{T}}(x_1(\tau), u(\tau))\,\mathrm{d}\tau \tag{2.4}$$

for all $0 < \rho < r$, for all $t \in \mathbb{R}^+$, for all trajectories that originate in X and for all applied inputs that are in U.

While the above observability assumption (2.4) is given in an integral form in order to simplify the proof of convergence of the estimation algorithm, it is not difficult to see that (2.4) is equivalent to the assumption that for some ε_1,

$$\lambda_{\min}\{w(x_1(t), u(t))\, w^{\mathrm{T}}(x_1(t), u(t))\} > \varepsilon_1 \tag{2.5}$$

for all $t \in \mathbb{R}^+$, where $\lambda_{\min}$ denotes the smallest eigenvalue of a matrix. Let us now consider the following discontinuous observer for the nonlinear system (2.1).

$$\begin{aligned} \dot{\hat{x}}_1 &= GA(\hat{x}_1 - x_1) + w^{\mathrm{T}}(x_1, u)\hat{x}_2 + \phi(x_1, u), \\ \dot{\hat{x}}_2 &= -G^2 w(x_1, u) P(\hat{x}_1 - x) + g(x_1, \hat{x}_2, u), \\ \hat{x}(t_i^+) &= M \frac{\hat{x}(t_i^-)}{\|\hat{x}(t_i^-)\|}, \end{aligned} \tag{2.6}$$

with the sequence of times t_i defined via $t_i = \min\{t: t > t_{i-1} \text{ and } \|\hat{x}(t)\| \geqslant \gamma M\}$, $t_0 = 0$ and where A is an $n \times n$ Hurwitz matrix. The matrix P is a positive-definite solution of the Liapunov matrix equation $A^{\mathrm{T}}P + PA = -Q$ with Q a positive-definite symmetric matrix. Note that if $g(x_1, x_2, u) = 0$ the problem becomes that of parameter identification and by choosing $G = 1$ and $M = \infty$ observer (2.6) becomes the observer based identifier considered in [13]. We shall refer to the above observer (2.6) as the IBO.

If we define estimation errors e_1 and e_2 by $e_1 \triangleq \hat{x}_1 - x_1$ and $e_2 \triangleq \hat{x}_2 - x_2$ the error dynamics between discontinuities takes on the following form:

$$\begin{aligned} \dot{e}_1 &= GAe_1 + w^{\mathrm{T}}(x_1, u)e_2, \\ \dot{e}_2 &= -G^2 w(x_1, u) Pe_1 + g(x_1, x_2, u) - g(x_1, \hat{x}_2, u). \end{aligned} \tag{2.7}$$

Theorem 2.3. *If Assumptions 2.1 and 2.2 are satisfied then there exists a positive constant G_0 such that the estimation errors e_1 and e_2 converge to zero exponentially if the constant G in (2.6) is chosen larger than G_0.*

The proof is given in Appendix A. The basic idea of the above observer structure is now described. The proposed observer is discontinuous and whenever the norm of the estimator state variable is equal to γM it is reset to have a norm equal to M. It is not difficult to verify that whenever we reset the estimates, the error always reduces. Moreover we show that in between two resetting, the error dynamics is exponentially stable.

Finally we show that the time interval between every two consecutive resetting is always greater than a fixed number so that there is no chattering. Thus the estimation error always approaches zero.

3. Feature based range estimation

In this section we consider the problem of range estimation by tracking images of moving features on the screen. The features that we consider are points, lines and planar curves. We propose appropriate parameterization of the space of features. As a result of the camera motion the features move and the motion is described by a dynamical system on the parameter space. The dynamical system has been described for all the three different features. Finally this dynamics has also been used to design estimators for the unknown part of the parameter vector.

3.1. Range estimation from feature points

We begin with the assumption that a reasonably accurate correspondence between points in consecutive images has been established. For further discussion on the problem of point correspondence we refer to [3, 10, 18].

The first problem in range estimation is to find an appropriate model which connects the location of the feature points on the screen and the range variables that we want to estimate. We start by assuming that the camera is rotating and translating with known or measured velocities that are possibly changing with time. Let $\omega_1, \omega_2, \omega_3$ and v_1, v_2, v_3 denote, respectively, the components of the angular and linear velocities in the coordinate system attached to the camera. It is well known that the coordinates X, Y, Z of a stationary point P in the moving camera coordinate frame satisfy

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \triangleq \Omega^{\mathrm{T}} \mathscr{X} - v, \tag{3.1}$$

with the obvious meaning of the symbols Ω, $\mathscr{X}$, and v. By accepting a pinhole model of the camera the coordinates of the image of the point P are given by $x = fX/Z$, $y = fY/Z$ where f stands for the focal length. To simplify the notation from now on we shall assume that $f = 1$. Note that the variables x and y represent our measurements.

In order to apply the IBO we must express the model in x, y and $d = 1/Z$ coordinates where d denotes inverse of the range data. This model is easily obtained from (3.1) and takes the following form.

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &= \begin{bmatrix} -v_1 + xv_3 \\ -v_2 + yv_3 \end{bmatrix} d + \begin{bmatrix} xy\omega_1 - (1 + x^2)\omega_2 + y\omega_3 \\ (1 + y^2)\omega_1 - xy\omega_2 - x\omega_3 \end{bmatrix}, \\ \dot{d} &= (\omega_1 y - \omega_2 x)d + v_3 d^2. \end{aligned} \tag{3.2}$$

A quick comparison of the above model (3.2) with the class of systems described by (2.1) suggests that the IBO might be applicable to estimate the inverse of the range of the point. In order to guarantee the convergence of the estimation errors to zero we must assure, however, that Assumptions 2.1 and 2.2 are satisfied. For the problem in hand it is reasonable to assume that the states of the system are bounded. First of all, in order for the point to appear on the screen, x and y must be bounded. Moreover the variable d, being the inverse of the depth Z, is positive and must be smaller than $1/f$ where f is the focal length of the camera (in this case $f = 1$). Therefore Assumption 2.1 is satisfied.

According to the observability condition (2.5) system (3.2) is observable, and Assumption 2.2 is satisfied, if the 2×1 matrix multiplying d is of uniform rank 1, or equivalently if $(v_1 - xv_3)^2 + (v_2 - yv_3)^2 > \varepsilon^2$ for some $\varepsilon > 0$. For $v_3 \neq 0$ the above expression defines a complement of a circle on the screen with the center at $(v_1/v_3, v_2/v_3)$ and the radius ε/v_3. The point $(v_1/v_3, v_2/v_3)$ is called the *focus of expansion* (FOE). It is a well known fact that the range of a feature point at the FOE cannot be determined. In addition, in the presence of measurement noise the range of points close to the FOE cannot be determined reliably. Therefore the above sufficient condition for observability is, for practical purposes, also necessary.

3.2. Range estimation from observing straight lines

In order to estimate a line in the 3D space we need to obtain a parameterization of lines in $\mathbb{R}^3$. Such a parameterization problem has been already considered in [1]. The parameterization obtained therein is however different from the one proposed here. Additionally in [1] the parameter estimation problem is based upon images from multiple cameras. This paper deals with the estimation problem with the aid of a single camera as has been the case in [17–19]. However [17–19] do not deal with line or curve based estimation.

In order to use the IBO machinery proposed in Section 2, we shall obtain a parameterization of the set of affine lines in $\mathbb{R}^3$ as points in Grass (2, 4) the Grassmannian manifold of homogeneous *2-dimensional* planes in $\mathbb{R}^4$. This parameterization of affine lines has been proposed in [8] in order to estimate the motion parameters of a moving object with the aid of a single camera. For the sake of completeness we shall repeat some of the basic results already outlined in [8].

We start with the parameterization of an affine line in $\mathbb{R}^3$ given as an intersection of two planes, i.e. the line is given by the set of points in $\mathbb{R}^3$ which satisfy

$$\begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \end{bmatrix} [X \quad Y \quad Z \quad 1]^{\mathrm{T}} = 0,$$

where obviously vectors $[m_1\ m_2\ m_3\ m_4]^{\mathrm{T}}$ and $[n_1\ n_2\ n_3\ n_4]^{\mathrm{T}}$ must be linearly independent. From the above parameterization of a line we define six Plücker coordinates as follows: $\eta_{ij} = m_i n_j - m_j n_i$, $i, j = 1, 2, 3$; $i \neq j$. It is not difficult to show that the variables η_{ij} satisfy $\eta_{12}^2 + \eta_{23}^2 + \eta_{31}^2 \neq 0$ (see [8]). Let η denote the vector of Plücker coordinates. Since η is defined up to a scale factor, one can normalize η as follows.

$$[u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6]^{\mathrm{T}} = \frac{1}{\sqrt{\eta_{12}^2 + \eta_{23}^2 + \eta_{31}^2}} [\eta_{14} \quad \eta_{24} \quad \eta_{34} \quad \eta_{12} \quad \eta_{23} \quad \eta_{31}]^{\mathrm{T}},$$

where $u_j, j = 1, \ldots, 6$, is a set of normalized Plücker coordinates. For each affine line in $\mathbb{R}^3$ there is a vector of 6 Plücker coordinates. The converse is however not true. For a vector Plücker coordinates to correspond to a line, it must satisfy

$$u_1 u_4 + u_2 u_5 + u_3 u_6 = 0. \tag{3.3}$$

According to this expression two vectors of dimension 3 defined via $\bar{u}_1 = [u_1\ u_2\ u_3]$ and $\bar{u}_2 = [u_4\ u_5\ u_6]$ are orthogonal. In fact these two vectors have a clear geometric meaning. The vector $\bar{u}_2$ is a unit vector in the direction of the line while the vector $\bar{u}_1$ is the cross-product between $\bar{u}_2$ and the vector from the origin to the closest point on the line.

The dynamics satisfied by the vector u has already been obtained in ([8, Theorem 1]) and is given by

$$\begin{bmatrix} \dot{\bar{u}}_1 \\ \dot{\bar{u}}_2 \end{bmatrix} = \begin{bmatrix} \Omega^{\mathrm{T}} & V \\ 0 & \Omega^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}, \tag{3.4}$$

where the matrix V is defined by

$$V = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}.$$

We now claim that among the Plücker coordinates $u_1, \ldots, u_6$ it is possible to measure u_1, u_2, u_3 upto a scale factor from the observation of projection of affine lines. Namely, the equation of the image of the line has been obtained in [8] and is given by

$$u_1 x + u_2 y + u_3 = 0. \tag{3.5}$$

Obviously, from the equation of the line on the image plane the vector $\bar{u}_1$ can be obtained up to a scale factor, which we shall treat as our measurement vector.

Since there is a one parameter ambiguity in $\bar{u}_1$, we propose to reparameterize (3.4). Assume that the affine line, that is being observed, does not pass through the origin, i.e. m_4 and m_8 are not both zero. In this case, then the vector (u_1, u_2, u_3) can never be the zero vector. Assume without any loss of generality that $u_3 \neq 0$. The situation is analogous if either $u_1 \neq 0$ or $u_2 \neq 0$. Geometrically $u_3 \neq 0$ implies that the projection of the line on the image plane does not pass through the origin of the image plane described by $(x, y) = (0, 0)$. Under this assumption we can divide Eq. (3.5) by u_3 to obtain $b_1 x + b_2 y + 1 = 0$ where obviously $b_1 = u_1/u_3$ and $b_2 = u_2/u_3$. If $u_3 = 0$, the system can be reparameterized by dividing (3.5) with either $u_1$or u_2. Define the variables p_1, p_2, and p_3 via $p_1 = u_4/u_3$, $p_2 = u_5/u_3$, $p_3 = u_6/u_3$. Then the following proposition gives the dynamics for the minimal set of line parameters b_i, $i = 1, 2$ and p_i, $i = 1, 2$.

Proposition 3.1. *If line* (3.5) *does not pass through the origin of the image plane, then the dynamics for the 4 parameters b_i, $i = 1, 2$ and p_i, $i = 1, 2$ is given by*

$$\begin{aligned} \begin{bmatrix} \dot{b}_1 \\ \dot{b}_2 \end{bmatrix} &= \begin{bmatrix} 0 & -b_1 v_1 - b_2 v_2 - v_3 \\ b_1 v_1 + b_2 v_2 + v_3 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} b_1 b_2 \omega_1 - (1 + b_1^2)\omega_2 + \omega_3 b_2 \\ (1 + b_2^2)\omega_1 - b_1 b_2 \omega_2 - b_1 \omega_3 \end{bmatrix}, \\ \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} &= \begin{bmatrix} b_1 \omega_1 & \omega_3 + b_2 \omega_2 \\ -\omega_3 - b_1 \omega_1 & -b_2 \omega_2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} v_2 p_1^2 - v_1 p_1 p_2 \\ v_2 p_1 p_2 - v_1 p_2^2 \end{bmatrix}. \end{aligned} \tag{3.6}$$

The proof of Proposition 3.1 follows by direct calculation of the derivatives of b and p using their definitions and using (3.4). The dependence of Eqs. (3.6) on p_3 is avoided by substituting $p_3 = -b_1 p_1 - b_2 p_2$ which follows directly from (3.3) and the definition of p and b. The measurement vector (b_1, b_2) is available from the equation of the line on the image plane given by $b_1 x + b_2 y + 1 = 0$.

Note that the system of equations (3.6) is in the form for which the IBO is applicable. In order to guarantee that the identifier indeed converges, we must verify that the two assumptions are satisfied. Assumption 2.1 is satisfied if the image of the line on the screen does not pass through a small neighborhood of the origin on the screen. Assumption 2.2, on the other hand, is satisfied if the components of the linear velocity are bounded and $|b_1 v_1 + b_2 v_2 + v_3| > \varepsilon$ for some positive constant ε. This condition is in agreement with the intuition. Obviously when $|b_1 v_1 + b_2 v_2 + v_3| = 0$ it follows that the velocity vector, the origin in $\mathbb{R}^3$ and the line are in the same plane; i.e. the camera is moving directly towards the line being observed. If this is the case the image of the line does not move on the screen and obviously the parameters p cannot be estimated. Finally note that once we have the estimates of b and p we can go back to the Plücker coordinates. The connection between b and p and the Plücker coordinates u is: $p_3 = -b_1 p_1 - b_2 p_2$, $u_3 = 1/\sqrt{p_1^2 + p_2^2 + p_3^2}$, $u_1 = b_1 u_3$, $u_2 = b_2 u_3$, $u_4 = u_3 p_1$, $u_5 = u_3 p_2$, $u_6 = u_3 p_3$.

3.3. Range estimation from observing planar curves

Analogous to what has been done in Section 3.2, in this section we estimate the equation of a planar curve in $\mathbb{R}^3$ from a sequence of projections on the image plane obtained from a moving camera. Our starting point is a very general parameterization of a second-order planar curve in 3D space given by the intersection of a plane and a second-order surface described as follows:

$$\begin{bmatrix} \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & \zeta_5 & \zeta_6 & \zeta_7 & \zeta_8 & \zeta_9 & \zeta_{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix} \rho = 0, \tag{3.7}$$

where

$$\rho = [X^2 \quad Y^2 \quad Z^2 \quad XY \quad XZ \quad YZ \quad X \quad Y \quad Z \quad 1]^{\mathrm{T}}.$$

It is not difficult to see that the above equation for a planar curve in the $\mathbb{R}^3$ is overparameterized. In order to obtain a minimal parameterization we proceed as follows.

Assume that the plane in which the second-order curve lies, does not pass through the origin of $\mathbb{R}^3$. By dividing the above equation (3.7) by Z^2 and substituting the expression for $1/Z$ from the second equation into the first, we obtain the following two equations that describe a planar second-order curve:

$$\eta_1 x^2 + \eta_2 y^2 + \eta_3 xy + \eta_4 x + \eta_5 y + \eta_6 = 0, \qquad a_1 x + a_2 y + a_3 + 1/Z = 0, \tag{3.8}$$

where, as before, $x = X/Z$, $y = Y/Z$, $a_i = \alpha_i/\alpha_4$, $i = 1,2,3$ and η's are appropriate expressions involving ζ's and α's. Note that we are allowed to divide by α_4 because, by our assumption the plane does not go through the origin, and therefore $\alpha_4 \neq 0$. Let us also define the vector $a = (a_1 \; a_2 \; a_3)$.

It may be remarked that the first equation in (3.8) is exactly the equation of the projection of the second-order curve on the image plane. The parameter vector η can therefore be observed (measured) up to a scale factor. The goal of the estimator is to estimate the vector a from these measurements of η.

In order to obtain an estimator, let us assume that $\eta_6 \neq 0$ which is equivalent to assuming that the projection of the curve does not pass through the origin on the image plane. We now propose to define new parameters for the curve on the image plane via $\xi_i = \eta_i/\eta_6$, $i = 1,\dots,5$. With these parameters the curve on the image plane is given by

$$\xi_1 x^2 + \xi_2 y^2 + \xi_3 xy + \xi_4 x + \xi_5 y + 1 = 0. \tag{3.9}$$

In the case when the image of the curve on the image plane pass through the origin a different reparameterization may be obtained by dividing the parameters η_i with a nonzero entry in η.

Let us define the vector ξ as $\xi = (\xi_1,\dots,\xi_5)$. Assuming that every point in $\mathbb{R}^3$ satisfies the differential equation (3.1) the dynamics is described in the following proposition.

Proposition 3.2. *If points in $\mathbb{R}^3$ satisfy (3.1) then parameters ξ and a of a planar second-order curve satisfy the following differential equations:*

$$\dot{\xi} = \begin{bmatrix} \alpha & 0 & -\xi_1\gamma \\ 0 & \beta & -\xi_2\gamma \\ \beta & \alpha & -\xi_3\gamma \\ \gamma & 0 & \alpha-\xi_4\gamma \\ 0 & \gamma & \beta-\xi_5\gamma \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} \xi_3\omega_3 - \xi_4\omega_2 - \xi_1\xi_4\omega_2 + \xi_1\xi_5\omega_1 \\ -\xi_3\omega_3 + \xi_5\omega_1 - \xi_2\xi_4\omega_2 + \xi_2\xi_5\omega_1 \\ -2\xi_1\omega_3 + 2\xi_2\omega_3 + \xi_4\omega_1 - \xi_5\omega_2 - \xi_3\xi_4\omega_2 + \xi_3\xi_5\omega_1 \\ 2\xi_1\omega_2 - \xi_3\omega_1 + \xi_5\omega_3 - 2\xi_6\omega_2 - \xi_4^2\omega_2 + \xi_4\xi_5\omega_1 \\ -2\xi_2\omega_1 + \xi_3\omega_2 - \xi_4\omega_3 + 2\xi_6\omega_1 - \xi_5\xi_4\omega_2 + \xi_5^2\omega_1 \end{bmatrix} \tag{3.10}$$

$$\hat{=}\; W^{\mathrm{T}}(\xi, v)a + \varphi(\xi,\omega),$$

$$\dot{a} = \Omega^{\mathrm{T}} a - (a^{\mathrm{T}} v)\, a, \tag{3.11}$$

where $\alpha = -2v_1\xi_1 - v_2\xi_3 - v_3\xi_4$, $\beta = -v_1\xi_3 - 2v_2\xi_2 - v_3\xi_5$, *and* $\gamma = -v_1\xi_4 - v_2\xi_5 - 2v_3$.

Proof. The proof follows by differentiating (3.9) with respect to time. The expressions for $\dot{x}$ and $\dot{y}$ are computed from (3.1) and the definition of x and y. Finally (3.8) is used to eliminate the quantity $1/Z$. A straightforward algebraic manipulation will lead to (3.10).

In order to show (3.11), note that we have $a_1 X + a_2 Y + a_3 Z + 1 = 0$. Once again by taking derivative with respect to time and using (3.1), the result follows. □

Note that the combined system (3.10) and (3.11) is in the form (2.1) for which the IBO is applicable. As before we shall have to verify that the two assumptions are satisfied. It is reasonable to assume that the boundedness Assumption 2.1 is satisfied because we can assume that the velocities of the camera are bounded and that the plane containing the curve does not pass too close to the origin in $\mathbb{R}^3$. We have to check whether the observability Assumption 2.2 is satisfied. This is completely determined by the rank of the matrix $W(\xi, v)\,W^{\mathrm{T}}(\xi, v)$. This is now described as follows.

Case 1: $\gamma = 0$. It is easy to see that the rank of $W^{\mathrm{T}}(\xi, v)$ drops below three if, in addition to $\gamma = 0$, we have $\alpha = 0$ and $\beta = 0$. This can be expressed as

$$\begin{bmatrix} 2\xi_1 & \xi_3 & \xi_4 \\ \xi_3 & 2\xi_2 & \xi_5 \\ \xi_4 & \xi_5 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0.$$

From the above condition it is obvious that for a generic second-order curve any nonzero value of linear velocities would give the rank of $W^{\mathrm{T}}(\xi, v)$ equal to 3.

Case 2: $\gamma \neq 0$. The rank of the matrix $W^{\mathrm{T}}(\xi, v)$ is less than 3 if the following three equations are satisfied simultaneously.

$$\xi_1\gamma^2 - \alpha^2 - \xi_4\alpha\gamma = 0, \quad \xi_2\gamma^2 - \beta^2 - \xi_5\beta\gamma = 0, \quad \xi_3\gamma^2 - 2\alpha\beta - \xi_5\alpha\gamma - \xi_4\beta\gamma = 0.$$

Note that when $\gamma = 0$ the above three equations are satisfied if and only if $\alpha = 0$ and $\beta = 0$ which is exactly the condition in Case 1. The above equalities are quadratic forms in v and can be rewritten as

$$v^{\mathrm{T}}Q_1(\xi)v = 0, \qquad v^{\mathrm{T}}Q_2(\xi)v = 0, \qquad v^{\mathrm{T}}Q_3(\xi)v = 0. \tag{3.12}$$

The matrices Q_i are cubic polynomial matrices in ξ. All three of them are nonsingular for generic values of ξ. From this we conclude that for a generic planar second-order curve and for nonzero vector of linear velocities the rank of $W^{\mathrm{T}}(\xi, v)$ is 3.

Remark 3.3. Outside a small neighborhood of points where the rank of $W^{\mathrm{T}}(\xi, v)$ drops, defined by (3.12), system (3.10) and (3.11) is observable and the estimate of the vector a converges to its true value. Finally note that the unobservability condition (3.12) depends only on known (measured) variables and can therefore be monitored on-line.

Remark 3.4. The above procedure can be repeated for third- or higher-order planar curves. The number of parameters would increase, but the number of unknown parameters is the same, i.e. 3. Moreover the structure of the parameter dynamics remains the same.

4. Simulation results

Simulations have been conducted to study the performance of the above range estimation algorithms, for each of the three different features. In the case of point correspondences we have compared the performance of the IBO to that of the EKF. For all the three cases a measurement noise has been added to the components of the velocity and to the observed parameters on the image plane. Components of the velocity are corrupted by zero mean Gaussian white noise sequence with the standard deviation 0.1 m/s, which is between 3% and 8% of the actual linear velocity. The sampling frequency is chosen to be 20 Hz.

When the observed features are points, the image plane parameters are two coordinates x and y. They are corrupted by 1% white Gaussian noise. Comparison between the IBO and the EKF is performed in two simulation runs corresponding to the cases when the lateral motion is dominant (Fig. 1(a)) and when the forward motion is dominant (Fig. 1(b)). The relative estimation errors (equal to the range estimation error divided by the range) for IBO (full line) and EKF (dotted line) are plotted versus time in seconds. It is obvious that both estimators give converging estimates with the performance of the EKF being somewhat better than that of the IBO. On the other hand IBO is simpler and has guaranteed performance.

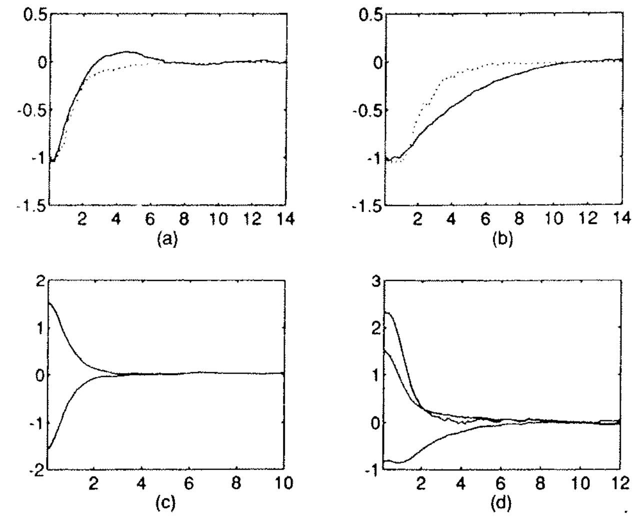

Fig. 1. Relative parameter estimation errors versus time in seconds for IBO (full line) and EKF (dotted line): (a) point observation, lateral motion; (b) point observation, forward motion; (c) line observation; (d) curve observation.

When the observed features are lines and curves, the observed parameters (b_1 and b_2 for the lines and ξ_i's for the curves) are assumed to be corrupted by a zero mean white Gaussian noise with the standard deviation 0.01. Fig. 1(c) shows the relative estimation errors for the unobserved line parameters p_1 and p_2. Fig. 1(d) shows the relative estimation errors for the unobserved curve parameters a_1, a_2, and a_3.

In conclusion we can say that the IBO provide accurate estimates of the feature parameters in the presence of measurement noise.

Appendix A

Proof of Theorem 2.3. The proof is based on the results of [13] and an approach for proving the convergence of high-gain observers proposed in [2]. Consider the differential equation for the estimation error given by (2.7). For a positive constant G define a linear change of coordinates $\xi = Te$ via

$$T = \begin{bmatrix} G^{-1}I & 0 \\ 0 & G^{-2}I \end{bmatrix},$$

where the identity matrices are of orders n_1 and n_2 corresponding to the dimensions of x_1 and x_2. It is easy to verify that the error dynamics in the new state ξ can be written in the following form:

$$\begin{aligned} \dot{\xi}_1 &= G(A\xi_1 + w^{\mathrm{T}}(t)\xi_2), \\ \dot{\xi}_2 &= Gw(t)P\xi_1 + G^{-2}(g(x_1, x_2, u) - g(x_1, \hat{x}_2, u)), \end{aligned} \tag{A.1}$$

where w is considered a function of time only (note that $x_1(t)$ and $u(t)$ are fixed functions of time known at every time instant). Assume now that $g(x_1, x_2, u) - g(x_1, \hat{x}_2, u) = 0$ and define the new time coordinate via $s = Gt$. The differential equation (A.1) becomes

$$\frac{\mathrm{d}\xi_1}{\mathrm{d}s} = A\xi_1 + \bar{w}^{\mathrm{T}}(s)\xi_2, \qquad \frac{\mathrm{d}\xi_2}{\mathrm{d}s} = \bar{w}(s)P\xi_1, \tag{A.2}$$

where $\bar{w}(s) \mathrel{\hat{=}} w(G^{-1}s)$.

The above system (A.2) is in the form satisfied by the error differential equation in parameter identification problem considered in [13]. It has been shown in Theorem 2 of [13] that the above system (A.2) is globally exponentially stable provided the regressor matrix $\bar{w}(s)$ satisfies

$$\left| \int_s^{s+\rho} \bar{w}^{\mathrm{T}}(\tau) v \, \mathrm{d}\tau \right| > \varepsilon_0 \tag{A.3}$$

for some $\rho > 0$, $\varepsilon_0 > 0$ and for all $v \in \mathbb{R}^{n_2}$. If the boundedness conditions (2.3) of Assumption 2.2 are satisfied, the "sufficient richness" condition (A.3) has been shown to be equivalent (see [14, p. 247]) to the standard persistence of excitation condition, with the constant ρ fixed, given by

$$\int_s^{s+\rho} \bar{w}(\tau) \bar{w}^{\mathrm{T}}(\tau) \, \mathrm{d}\tau > \varepsilon_1 I. \tag{A.4}$$

Now we want to show that condition (A.4) on $\bar{w}^{\mathrm{T}}(\tau)$ follows from condition (2.4) of Assumption 2.2. Note that

$$\int_s^{s+\rho} \bar{w}(\tau) \bar{w}^{\mathrm{T}}(\tau) \, \mathrm{d}\tau = \int_s^{s+\rho} w(G^{-1}\tau) w^{\mathrm{T}}(G^{-1}\tau) \, \mathrm{d}\tau. \tag{A.5}$$

By substituting $z = G^{-1}\tau$ the above integral (A.5) becomes

$$G \int_{G^{-1}s}^{G^{-1}(s+\rho)} w(z) w^{\mathrm{T}}(z) \, \mathrm{d}z > G\beta G^{-1} \rho I = \beta \rho I.$$

Therefore $\bar{w}(\tau)$ satisfy (A.3) and for the system given by (A.2) one can verify that the proof of Theorem 2 in [13] guarantees the existence of a Liapunov function $V_1(\xi_i)$ and three positive constants d_i, $i = 1, 2, 3$ such that

$$d_1 \| \xi(s) \|^2 < V_1(\xi) < d_2 \| \xi(s) \|^2,$$

$$\left\| \frac{\mathrm{d}}{\mathrm{d}s} V_1(\xi)|_{(\mathrm{A}.2)} \right\| \leqslant 0,$$

$$\int_s^{s+\rho} \frac{\mathrm{d}}{\mathrm{d}\tau} V_1(\xi)|_{(\mathrm{A}.2)} \, \mathrm{d}\tau \leqslant -d_3 \| \xi(s) \|^2.$$

Notation $\dot{V}_1(\xi)|_{(\mathrm{A}.2)}$ means that the time derivative of V is calculated along the trajectory of system (A.2). Note that constants d_i do not depend on G because neither the differential equation (A.2) nor the persistence of excitation condition depends on G.

It follows from Theorems 1.5.1 and 1.5.2 in [15] that there exist another Liapunov function $V_2(\xi, s)$ and four positive constants, c_i, $i = 1, 2, 3, 4$ such that

$$\begin{aligned} & c_1 \| \xi(s) \|^2 < V_2(\xi, s) < c_2 \| \xi(s) \|^2, \\ & \dot{V}_2(\xi, s)|_{(\mathrm{A}.2)} < -c_3 \| \xi(s) \|^2, \\ & \left\| \frac{\partial V_2(\xi, s)}{\partial \xi} \right\| < c_4 \| \xi(s) \|. \end{aligned} \tag{A.6}$$

Now we are ready to discard the assumption that $g(x_1, x_2, u) - g(x_1, \hat{x}_2, u) = 0$. In this case the error equation in ξ coordinates and in time s is given by

$$\begin{aligned} \frac{\mathrm{d}\xi_1}{\mathrm{d}s} &= A\xi_1 + \bar{w}^{\mathrm{T}}(s)\xi_2, \\ \frac{\mathrm{d}\xi_2}{\mathrm{d}s} &= \bar{w}(s)P\xi_1 + G^{-3}(g(x_1, x_2, u) - g(x_1, \hat{x}_2, u)). \end{aligned} \tag{A.7}$$

If we use the function $V_2(\xi, s)$ as a Liapunov function candidate for system (A.7) we obtain

$$\frac{d}{ds} V_2(\xi, s)|_{(A.7)} = \frac{d}{ds} V_2(\xi, s)|_{(A.2)} + \frac{\partial V_2(\xi, s)}{\partial \xi} G^{-3}((g(x_1, x_2, u) - g(x_1, \hat{x}_2, u))$$

$$\leqslant -c_3 \|\xi\|^2 + c_4 \|\xi\| G^{-3} \alpha \|x_2 - \hat{x}_2\| \leqslant -c_3 \|\xi\|^2 + c_4 \|\xi\| G^{-1} \alpha \|\xi\|$$

$$\leqslant (-c_3 + G^{-1} c_4 \alpha) \|\xi\|^2.$$

It is obvious that $-\dot{V}_2(\xi, s)|_{(A.7)}$ can be made a positive-definite function by choosing $G > c_4\alpha/c_3$ and for such a choice of G system (A.7) becomes globally exponentially stable. Because of the linear relationship between ξ and e we conclude that e converges to zero exponentially between discontinuities.

Since the states of system (2.1) are assumed bounded and the states of observer (2.6) are kept bounded it follows that the right-hand side of the differential equation in (2.6) is bounded. Therefore the length of time intervals between discontinuities $|t_i - t_{i-1}|$ is greater then a positive constant δ for all i. On every such a time interval the estimation error $e(t)$ converges to zero exponentially and, at a discontinuity, the estimation error satisfies $\|e(t_i^+)\| < \|e(t_i^-)\|$. Therefore we conclude that the estimation error is globally exponentially stable. □

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