Homogeneous Dynamical Systems Theory

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***Abstract*—In this paper, we study controlled homogeneous dynamical systems and derive conditions under which the system is perspective controllable. We also derive conditions under which the system is observable in the presence of a control over the complex base field. In the absence of any control input, we derive a necessary and sufficient condition for observability of a homogeneous dynamical system over the real base field. The observability criterion obtained generalizes a well known Popov–Belevitch–Hautus (PBH) rank criterion to check the observability of a linear dynamical system. Finally, we introduce rational, exponential, interpolation problems as an important step toward solving the problem of realizing homogeneous dynamical systems with minimum state dimensions.**

***Index Terms*—Author, please supply your own keywords or send a blank e-mail to keywords@ieee.org to receive a list of suggested keywords.**

I. Introduction

THE class of problems we study in this paper has to do with the problem of controlling and observing the orientation of a state vector in $\mathbb{R}^n$. Of course, the problem of "orientation-control" has been a subject of study in the nonlinear control literature for at least the last two decades [1]. A typical example of such a control problem is "satellite orientation control" by active means such as gas jets, magnetic torquing etc. see [2]. In recent years, an important example of the orientation control problem arises in Biology and is known as the "gaze-control" problem. The "gaze-control" problem has close connection with the problem of "eye-movement" (see [3]–[5]), where the problem is to orient the eye so that a target is in the field of view. An example of the dynamical system we study in this paper is described as follows:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} + \begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix} u \qquad (1.1)$$

$$y = \frac{c_{11}x_1 + c_{12}x_2 + c_{13}x_3}{c_{21}x_1 + c_{22}x_2 + c_{23}x_3} \qquad (1.2)$$

where the scalar output $y(t)$ may be considered to be the slope of the line spanned by the vector $(y_1, y_2)^T$ and where

$$\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}. \qquad (1.3)$$

For a dynamical system of the form (1.1), (1.2), one is interested in controlling only the direction of the state vector $(x_1, x_2, x_3)^T$ and such problems are, therefore, of interest in gaze control. To generalize the control problem, we consider a dynamical system with state variable $x \in \mathbb{R}^n$, control variable $u \in \mathbb{R}^m$ and observation function $y \in \mathbb{RP}^{p-1}$, the projective space of homogeneous lines in $\mathbb{R}^p$, where we assume that $p > 1$. The dynamical system is described as

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= [Cx]\end{aligned} \qquad (1.4)$$

and the projective valued observation function is defined as

$$\begin{aligned}y: \mathbb{R}^n - S &\rightarrow \mathbb{RP}^{p-1}\\ x &\longmapsto [Cx]\end{aligned} \qquad (1.5)$$

where $[Cx]$ is the homogeneous line spanned by the nonzero vector $Cx \in \mathbb{R}^p$. The set S is defined as

$$S = \{x: Cx = 0\}.$$

The pair (1.4) and (1.5) is a linear dynamical system with a homogeneous observation function and has been introduced in [6]–[9] as an example of a perspective dynamical system. The following two problems would initiate two of the important questions discussed in this paper.

Problem 1.1 (Perspective Control Problem): Let $[\xi_1]$ and $[\xi_2]$ be two distinct elements of $\mathbb{RP}^{n-1}$, does there exist a $T > 0$ and $u(t)$, $t \in [0, T]$ such that

$$[\xi_2] = \left[e^{AT}\xi_1^* + \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau\right] \qquad (1.6)$$

where ξ_1^* is a nonzero vector in $\mathbb{R}^n$ such that $[\xi_1^*] = [\xi_1]$?

Problem 1.2 (Perspective Observation Problem): Let $[\xi_1]$ and $[\xi_2]$ be two distinct elements of $\mathbb{RP}^{n-1}$, does there exist $T > 0$ and $u(t)$, $t \in [0, T]$ such that

$$\left[Ce^{At}\xi_1^* + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau\right] \neq \left[Ce^{At}\xi_2^* + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau\right] \qquad (1.7)$$

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for all $t \in [T_1, T_2]$ where $0 \leq T_1 < T_2 \leq T$, and where ξ_1^* and ξ_2^* are two nonzero vectors in $\mathbb{R}^n$ such that $[\xi_1^*] = [\xi_1]$, $[\xi_2^*] = [\xi_2]$?

The two problems 1.1 and 1.2 refer to the extent the state vector $x(t)$ can be controlled up to its direction using a control input $u(t)$ and can be observed up to its direction using a projective valued observation $y(t)$. The proposed class of problems can be motivated from machine vision and robotics where the goal is to guide a robot, possibly a mobile robot, with *charge coupled device* cameras as sensors. These tasks are typically known as *visual servoing* and has been of interest for at least the last two decades (see [10], [11] for an early and more recent reference). The problem 1.2 arises in computer vision as the *observability problem* [12], [13] from visual motion, where the goal is to estimate the location of a moving object from the image data and may be the structure of the object and motion parameters as well. Often, the image data is produced by a line [14].

The problem of *visual servoing* typically deals with the problem of controlling the movement of a robot arm or a mobile platform guided only by a visual sensor, such as a *charged coupled device* camera. In the last decade, this problem has been studied in great details and the associated literature is large. A major difference in the servoing scheme has been *passive servoing* wherein the camera is assumed to be held permanently fixed to the ceiling and *active servoing* [15], [16], where the camera moves with the manipulator. The problem of active camera manipulation is of particular importance in mobile and walking robots, see [17] and [18].

The *observability problem* or perhaps the *identifiability problem,* if the parameters are changing in time, deals with the problem of ascertaining the location of a target at the very least and subsequently estimating the structure and motion parameters, if possible (see [9], [19], and [20] for some new references on this problem). What makes the *perspective observation problem* interesting is that typically a single camera is unable to observe the precise position of a target exactly. Thus it becomes imperative to observe the targets up to their directions with the hope that eventually multiple cameras can precisely locate the position. This point of view is in sharp contrast with stereo based algorithms, wherein multiple cameras are also used, but one requires feature correspondence between various cameras.

In the last ten years, many excellent books, tutorials, and surveys have been written on the topic of *vision based control and observation* (see [21]–[27]). To summarize the main content of this paper, we analyze problem 1.1, and show that the dynamical system (1.1) is *perspective controllable* in between two states iff the two states are in the same orbit of a Riccati flow. We also analyze problem 1.2, and show that the *perspective observability* of the dynamical system (1.1), (1.2) can be ascertained via a suitable generalization of the Popov–Belevitch–Hautus (PBH) rank test, (see [28]). Such rank tests have already been derived in [29]–[31] assuming that the control input $u(t)$ is not present. We introduce *perspective realizability* problems and establish connection between these realization problems and the well-known *exponential interpolation* problems [32].

II. Perspective Control Problem and the Riccati Flow

We start this section with the following definition of *perspective controllability*.

Definition 2: We shall say that the dynamical system (1.4) is perspective controllable if problem 1.1(1.1) has an affirmative answer for almost every pair of points $[\xi_1]$, $[\xi_2] \in \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}$ in the usual product topology.

Remark 2.2: A property is said to be *generically satisfied,* if it is satisfied for an open and dense set of points in a topological space. The phrase *almost every pair* is used to mean *every pair that belongs to a generic subset of* $\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}$. One choice of the generic set would be to choose all pairs of vectors ξ_1, ξ_2 in $\mathbb{R}^n$ such that $\xi_i \notin H$, $i = 1, 2$, 2 where H is defined in (2.1) as follows.

For a pair of nonzero vectors ξ_1^*, ξ_2^* in $\mathbb{R}^n$, let us define

$$H = \text{span}\{B, AB, \ldots, A^{n-1}B\} \tag{2.1}$$

and

$$H_i = \text{column span}\{\xi_i^*, B, AB, \ldots, A^{n-1}B\} \tag{2.2}$$

for $i = 1, 2$. The following theorem characterizes the main result in perspective control.

Theorem 2.3: The dynamical system (1.4) is perspective controllable iff for almost every pair of points $[\xi_1^*]$, $[\xi_2^*]$ in $\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}$, the subspaces H_1 and H_2 have the same dimension and there exists a real number $T > 0$ such that

$$e^{AT} H_1 = H_2. \tag{2.3}$$

Remark 2.4: Note that the notion of controllability in the perspective setting is slightly different from the concept of controllability of a linear dynamical system. A linear system is controllable if every pair of vectors in $\mathbb{R}^n \times \mathbb{R}^n$ is controllable, whereas for perspective controllability, it is enough to have almost every pair in $\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}$ to be controllable. This difference is important, indeed if $\xi_1^* \in H$ and $\dim H = n - 1$, then there is no way to reach a state outside of H despite the fact that the system may still be perspective controllable.

Proof of Theorem 2.3: Initializing the dynamical system (1.4) at $x(0) = x_0$ we have

$$e^{-At}x(t) - x_0 = \int_0^t e^{-A\tau} Bu(\tau)\, d\tau. \tag{2.4}$$

It is well known (see [33]) that the set of all vectors in the right hand side of (2.4) for various choice of $u(\tau)$, is given by all vectors in the subspace H. Assume that (1.4) is perspective controllable, it follows that for almost every pair $[\xi_1^*]$, $[\xi_2^*]$:

$$\alpha e^{-AT}\xi_2^* - \beta\xi_1^* = \int_0^T e^{-A\tau} Bu(\tau)\, d\tau \tag{2.5}$$

for some scalars $\alpha \neq 0$, $\beta \neq 0$ and $T > 0$ and for some $u(\tau)$, $\tau \in [0, T]$. Thus, we have

$$e^{-AT}\xi_2^* \subset \text{span}\{\xi_1^*, B, AB, \ldots, A^{n-1}B\} \tag{2.6}$$

from which we infer that $e^{-AT}H_2 \subset H_1$. Multiplying (2.5) by e^{AT}, we also have

$$e^{AT}\xi_1^* \subset \text{span}\,\{\xi_2^*, e^{AT}H\}. \tag{2.7}$$

Hence, we infer that $e^{AT}\xi_1^* \subset H_2$ and we have $e^{AT}H_1 \subset H_2$. Thus, (2.3) is satisfied.

Conversely, because H_1 and H_2 have the same dimension it follows that either ξ_1 and ξ_2 both belong to H or they both do not. In the former case, it is trivial to find α, β and $T > 0$ such that (2.5) is satisfied. In the latter case it follows from (2.3) that there exists a $T > 0$ such that $e^{-AT}\xi_2^* \subset H_1$, i.e.,

$$e^{-AT}\xi_2^* = \gamma\xi_1^* + v \tag{2.8}$$

where $v \in H$. Note also that $\gamma \neq 0$ for otherwise it would follow that $\xi_2^* \in H$ violating the assumption that it is not. It now follows easily from (2.8) that one can satisfy (2.5) by choosing $\alpha = 1$, $\beta = \gamma$ and an appropriate choice of $u(\tau)$. (Q.E.D.)

Note that Theorem 2.3 may be viewed as a criterion for checking perspective controllability of a homogeneous system (1.4), in between the two directions $[\xi_1]$ and $[\xi_2]$. The following is an important corollary for perspective control.

Corollary 2.5: If $n \geq 3$, the dynamical system (1.4) is perspective controllable iff

$$\dim H \geq n - 1. \tag{2.9}$$

Proof of Corollary 2.5: Assume (2.9), it follows that for all ξ_1^*, ξ_2^* that are not in H, we have $H_1 = H_2$ and hence (2.3) is trivially satisfied. Thus from Theorem 2.3 it follows that the dynamical system is perspective controllable for every pair of vectors ξ_1^*, ξ_2^* not in H. These vectors would give rise to a generic pair of points in $\mathbb{RP}^{n-1}$, hence, the dynamical system (1.4) is perspective-controllable.

Conversely, assume that $\dim H = n - 2$. For $n \geq 3$ there exists two vectors ξ_1^*, ξ_2^* not in H and are such that

$$e^{At}\xi_1^* \not\subset H_2 \tag{2.10}$$

for all $t \in [0, \infty)$. Hence (1.4) is not perspective controllable in between ξ_1^*, and ξ_2^*. Moreover there exists an open neighborhood of $[\xi_1^*]$ and $[\xi_2^*]$, such that (2.10) is satisfied. Hence, a generic pair is not perspective controllable. (Q.E.D.)

Remark 2.6: For $n = 2$, the above corollary 2.5 is not true. If $\dim H = 0$ it would follow that for two nonzero vectors ξ_1^* and ξ_2^* in $\mathbb{R}^2$, we would have $\dim H_i = 1$ for $i = 1, 2$. Perspective controllability would depend upon, whether or not (2.3) is satisfiable for some $T \geq 0$. For certain choice of A and B (2.4) is satisfiable for every pair of nonzero vectors. One such choice is when $B = 0$ and when

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Remark 2.7: In this remark we point out that the perspective controllability problem has connection with a flow on an associated Grassmannian which is described by a Riccati equation. Let us assume that

$$\dim H_1 = \dim H_2 = n_1 \tag{2.11}$$

and let us represent by Grass (n_1, n), the space of all n_1 dimensional homogeneous planes in $\mathbb{R}^n$. Clearly the equation

$$\dot{x} = Ax \tag{2.12}$$

defines a flow in Grass (n_1, n) described by

$$\begin{aligned} \chi\colon \text{Grass}\,(n_1, n) \times \mathbb{R} &\to \text{Grass}\,(n_1, n) \\ (H, T) &\longmapsto e^{AT}H. \end{aligned} \tag{2.13}$$

The flow (2.12) can also be described via a Riccati equation as follows. Let us denote

$$H = \text{span of } \begin{bmatrix} \Theta_0 \\ \Theta_1 \end{bmatrix} \tag{2.14}$$

where Θ_0 is a $n_1 \times n_1$ nonsingular matrix and where Θ_1 is a $(n - n_1) \times n_1$ matrix. Writing

$$X = \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{2.15}$$

where X is a $n \times n_1$ matrix, and defining $W = X_1 X_0^{-1}$, we have

$$\begin{aligned} \dot{W} &= A_{21} + A_{22}W - WA_{11} - WA_{12}W \\ W(0) &= \Theta_1\Theta_0^{-1}. \end{aligned} \tag{2.16}$$

It is trivial to verify that the Riccati flow (2.16) is an equivalent representation of the homogeneous flow (2.13) up to time t when $X_0(t)$ is singular.

Remark 2.8: Regarding the proof of Corollary 2.5, for all the vectors ξ_1^*, ξ_2^* that are not in H, dimensions of H_1 and H_2 are equal and is one greater than the dimension of H. So if the dimension of H is $n - 1$ or n, it would follow that H_1 and H_2 are the same subspace and is equal to the entire space $\mathbb{R}^n$. If the dimension of H is less than $n-1$, the subspaces H_1 and H_2 still have the same dimension but they are proper subspaces of $\mathbb{R}^n$. If we consider the flow described by (2.13), and initialize the flow, say on H_1, the problem is to ascertain if the other subspace H_2 lies on the orbit of the flow. Except for the special case $n = 2$, as noted in Remark 2.6, it is always possible to choose a generic pair of vectors ξ_1^*, ξ_2^*, such that (2.10) is satisfied.

For many other properties of the phase portrait of a Riccati equation we refer to [34]. We now have the following restatement of Theorem 2.3 which we state without proof.

Theorem 2.9: The dynamical system (1.4) is perspective controllable in between the two directions $[\xi_1]$ and $[\xi_2]$ in $\mathbb{RP}^{n-1}$ iff the subspaces H_1 and H_2 have the same dimension, say n_1, and are in the same orbit of the homogeneous flow (2.13).

III. Perspective Observability in the Absence of a Control

As a special case of the perspective observability problem 1.2 considered in the introduction, we now consider the dynamical system (1.4) under the assumption that $B = 0$, i.e., there is no influence of the control. We continue to assume that the observation function $y(t)$ is projective valued and is given by (1.5). The perspective observation problem is described as follows.

Problem 3.1: Let ξ_1^* and ξ_2^* be two linearly independent vectors in $\mathbb{R}^n$, does there exist $T_2 > T_1 \geq 0$ such that

$$[Ce^{At}\xi_1^*] \neq [Ce^{At}\xi_2^*]$$

for all $t \in [T_1, T_2]$?

Note that the problem 3.1 is equivalent to problem 1.2 when the control input is set to zero. The notion of perspective observability in this section would always refer to problem 3.1. Problem 3.1 already has a satisfactory answer over $\mathbb{C}^n$ and has been studied in [29] and [30]. It has been shown that a necessary and sufficient condition for perspective observability over $\mathbb{C}^n$ is that the rank of the matrix

$$\text{rank}\begin{pmatrix}(A-\lambda_1 I)(A-\lambda_2 I)\\ C\end{pmatrix} = n \tag{3.1}$$

for all pairs of eigenvalues λ_1, λ_2 (may be the same) of the matrix A. Over $\mathbb{R}^n$ the rank condition (3.1) is only sufficient. In order to state the main observability result of this section, we define two-fold wedge product of $\mathbb{R}^n$ as follows.

Definition 3.2: Define $\mathbb{R}^n \wedge \mathbb{R}^n$ as the vector space whose vectors are linearly generated by the vectors

$$\{x_1 \wedge x_2 \mid x_1, x_2 \in \mathbb{R}^n\}.$$

Addition in $\mathbb{R}^n \wedge \mathbb{R}^n$ is multilinear and alternating in the components. If $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$, then it follows from the multilinearity and the alternating property of the wedge product that:

$$B := \{e_i \wedge e_j | 1 \leq i < j \leq n\}$$

is a basis of $\mathbb{R}^n \wedge \mathbb{R}^n$. If a vector v in $\mathbb{R}^n \wedge \mathbb{R}^n$ has a representation $v = x_1 \wedge x_2$ for some particular vectors $x_1, x_2 \in \mathbb{R}^n$, one says that v is a decomposable vector.

We now introduce the following *hat system* that has already been considered in [29]. Define the vector space $\mathbb{R}^n \wedge \mathbb{R}^n$, [36] and consider the linear map

$$\hat{A}\colon \mathbb{R}^n \wedge \mathbb{R}^n \rightarrow \mathbb{R}^n \wedge \mathbb{R}^n$$

given by

$$x \wedge y \longmapsto Ax \wedge y + x \wedge Ay.$$

We also consider the linear map

$$\hat{C}\colon \mathbb{R}^n \wedge \mathbb{R}^n \rightarrow \mathbb{R}^m \wedge \mathbb{R}^m$$

given by

$$x \wedge y \longmapsto Cx \wedge Cy.$$

We now define the promised hat system as follows:

$$\dot{\hat{x}} = \hat{A}\hat{x} \qquad \hat{z} = \hat{C}\hat{x} \tag{3.2}$$

where $\hat{x} \in \mathbb{R}^n \wedge \mathbb{R}^n$ and $\hat{z} \in \mathbb{R}^m \wedge \mathbb{R}^m$. The main result of this section is summarized in the following theorem.

Theorem 3.3: The following three conditions are equivalent over $\mathbb{R}$.

1) For $B = 0$, the dynamical system (1.4) is perspective unobservable.
2) There exists a real number $\hat{\lambda}$ and a real decomposable vector $\theta_1 \wedge \theta_2$, $\theta_1, \theta_2 \in \mathbb{R}^n$ such that
$$\begin{pmatrix}\hat{A} - \hat{\lambda} I\\ \hat{C}\end{pmatrix}\theta_1 \wedge \theta_2 = 0. \tag{3.3}$$
3) There exist two numbers λ_1, λ_2 which are either both real (may be the same) or complex conjugates of each other such that
$$\text{rank}\begin{pmatrix}(A-\lambda_1 I)(A-\lambda_2 I)\\ C\end{pmatrix} < n. \tag{3.4}$$

Remark 3.4: The first two equivalent conditions of the theorem 3.3 basically says that the perspective unobservability of (1.4), assuming $B = 0$, is equivalent to the regular unobservability (in the sense of a linear system) of the hat system (3.3) with the additional requirement that the unobseravbility subspace of the hat system must contain a decomposable vector. Decomposability of a vector in $\mathbb{R}^n \wedge \mathbb{R}^n$ is hard to check. The third condition of the Theorem 3.3 provides a computationally feasible solution, which involves checking the rank of a matrix for every real or complex conjugate pairs of eigenvalues of the matrix A.

Remark 3.5: Theorem 3.3 essentially obtains a necessary and sufficient condition for perspective unobservability over $\mathbb{R}$ and the condition is described as a generalization of the well known PBH rank condition.

Proof of Theorem 3.3 $(1 \iff 2)$*:* $(2 \Rightarrow 1)$ Assume that there exists a vector $x \wedge y$ such that (3.3) is satisfied. It follows that:

$$\begin{aligned}\hat{A}(x \wedge y) &= \hat{\lambda}(x \wedge y)\\ \hat{C}(x \wedge y) &= 0.\end{aligned} \tag{3.5}$$

From (3.5), we conclude the following:

$$\begin{aligned}Ce^{At}x \wedge Ce^{At}y &= \hat{C}(e^{At}x \wedge e^{At}y)\\ &= \hat{C}e^{\hat{A}t}(x \wedge y)\\ &= \hat{C}e^{\hat{\lambda}t}(x \wedge y)\\ &= e^{\hat{\lambda}t}\hat{C}(x \wedge y)\\ &= 0.\end{aligned} \tag{3.6}$$

Thus, as a vector in $\mathbb{R}^m$, $Ce^{At}x$ and $Ce^{At}y$ are linearly independent implying that (1.4) is perspective unobservable for $B = 0$.

$(1 \Rightarrow 2)$ Assume that (1.4) is perspective unobservable for $B = 0$. It follows that there exists two vectors $\theta_1, \theta_2 \in \mathbb{R}^n$ such that $\theta_1 \wedge \theta_2 \neq 0$ and:

$$\begin{aligned}&Ce^{At}\theta_1 \wedge Ce^{At}\theta_2 = 0\\ \iff\ &\hat{C}e^{\hat{A}t}(\theta_1 \wedge \theta_2) = 0.\end{aligned} \tag{3.7}$$

In order to show (3.3), we need to show that there is a real decomposable eigenvector of $\hat{A}$ in the kernel of $\hat{C}$.

Define two complex conjugate vectors $\overline{\theta}_1$ and $\overline{\theta}_2$ as follows:

$$\begin{aligned}\overline{\theta}_1 &= \theta_1 + i\theta_2\\ \overline{\theta}_2 &= \theta_1 - i\theta_2.\end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that:

$$\hat{C}e^{\hat{A}t}(\overline{\theta}_1 \wedge \overline{\theta}_2) = 0. \tag{3.9}$$

The relation (3.9) follows easily from the fact that:

$$\overline{\theta}_1 \wedge \overline{\theta}_2 = 2i\theta_2 \wedge \theta_1.$$

We now proceed by expanding the vectors $\overline{\theta}_1$ and $\overline{\theta}_2$ in terms of the eigenvectors (generalized eigenvectors) v_i of A. Let

$$\{v_1, v_2, \ldots, v_r\} \tag{3.10}$$

be the self conjugate set of eigenvectors (generalized eigenvectors) of the matrix A. Expanding $\overline{\theta}_1 \wedge \overline{\theta}_2$ in terms of the eigenvectors (generalized eigenvectors) $v_i \wedge v_j$ of $\hat{A}$ for $i, j = 1, \ldots, r, i \neq j$ we write

$$\overline{\theta}_1 \wedge \overline{\theta}_2 = \sum_{i, j=1, i\neq j}^{r} \alpha_{ij} v_i \wedge v_j$$

where we can assume that $\alpha_{12} \neq 0$ and either v_1, v_2 are both real or are complex conjugates of each other. We can also assume that the vectors in the set (3.10) have been ordered so that we claim that

$$\lambda_i + \lambda_j = \lambda_1 + \lambda_2 \quad \Longleftrightarrow \quad \lambda_i = \lambda_1, \lambda_j = \lambda_2$$

and when $\lambda_i + \lambda_j = \lambda_1 + \lambda_2$, we have

$$p_i + p_j = p_1 + p_2, \quad \Longleftrightarrow \quad p_i = p_1, p_j = p_2$$

where p_i is the multiplicity of the eigenvalue λ_i corresponding to v_i. Such an ordering is always possible and has been detailed in [29]. It now follows from (3.9) that:

$$\hat{C}e^{\hat{A}t}(v_1 \wedge v_2) = 0 \tag{3.11}$$

where $v_1 \wedge v_2$ is an eigenvector of $\hat{A}$ corresponding to the eigenvalue $\lambda_1 + \lambda_2$.

If v_1 and v_2 are both real, then the proof is over. If they are complex conjugates of each other, we can write

$$\begin{cases} v_1 = \eta_1 + i\eta_2 \\ v_2 = \eta_1 - i\eta_2 \end{cases} \quad \eta_1, \eta_2 \in \mathbb{R}^n. \tag{3.12}$$

It is easy to check that $v_1 \wedge v_2 = 2i\eta_2 \wedge \eta_1$. It follows from (3.11) that:

$$\begin{pmatrix} \hat{A} - (\lambda_1 + \lambda_2)I \\ \hat{C} \end{pmatrix} v_1 \wedge v_2 = 0. \tag{3.13}$$

Thus, we have

$$\begin{pmatrix} \hat{A} - (\lambda_1 + \lambda_2)I \\ \hat{C} \end{pmatrix} \eta_1 \wedge \eta_2 = 0. \tag{3.14}$$

Since $\lambda_1 + \lambda_2$ is always real, this completes the proof.

Proof of Theorem 3.3 $(2 \iff 3)$*:* $(3 \Rightarrow 2)$ When λ_1 and λ_2 are both real, there is nothing else to prove. If v is a vector such that $(A - \lambda_1 I)(A - \lambda_2 I)v = 0$ and $Cv = 0$, we define $\theta_1 = v$ and $\theta_2 = (A - \lambda_2 I)v$. It is easy to see that $(\hat{A} - (\lambda_1 + \lambda_2)I)(\theta_1 \wedge \theta_2) = 0$ and $\hat{C}(\theta_1 \wedge \theta_2) = 0$ which implies (3.3).

When λ_1 and λ_2 are complex conjugates of each other we write

$$\begin{aligned} \lambda_1 &= \lambda_1^* + i\lambda_2^* \\ \lambda_2 &= \lambda_1^* - i\lambda_2^*. \end{aligned} \tag{3.15}$$

Let v be as before. We then define

$$\begin{aligned} x &= (A - \lambda_1 I)v \\ y &= (A - \lambda_2 I)v. \end{aligned} \tag{3.16}$$

It is easy to verify that

$$\begin{aligned} &(\hat{A} - (\lambda_1 + \lambda_2)I)(x \wedge y) \\ &\quad = Ax \wedge y + x \wedge Ay - \lambda_1 x \wedge y - \lambda_2 x \wedge y \\ &\quad = ((A - \lambda_2 I)x) \wedge y + x \wedge ((A - \lambda_1 I)y \\ &\quad = 0 \wedge y + x \wedge 0 = 0. \end{aligned} \tag{3.17}$$

Moreover

$$\begin{aligned} \hat{C}(x \wedge y) &= Cx \wedge Cy \\ &= (CAv - \lambda_1 Cv) \wedge (CAv - \overline{\lambda}_1 Cv) \\ &= CAv \wedge CAv \text{ since } Cv = 0 \\ &= 0. \end{aligned} \tag{3.18}$$

It is an easy calculation to show that

$$\frac{i}{2\lambda_2^*} x \wedge y = v \wedge Av.$$

It would therefore follow that:

$$(\hat{A} - (\lambda_1 + \lambda_2)I)(v \wedge Av) = 0$$

and

$$\hat{C}(v \wedge Av) = 0.$$

We define $\theta_1 = v$ and $\theta_2 = Av$ and the proof is complete.

$(2 \Rightarrow 3)$ By assumption, we have $\theta_1, \theta_2 \in \mathbb{R}^n$ such that (3.3) is satisfied. By decomposing the vector $\theta_1 \wedge \theta_2$ in terms of the eigenvectors (generalized eigenvectors) of the matrix $\hat{A}$, we can repeat the necessity proof of the first part of this theorem to show that there exist two eigenvectors (or generalized eigenvectors) v_1 and v_2 of A such that $\hat{C}e^{\hat{A}t}(v_1 \wedge v_2) = 0$ for all $t \geq 0$. In particular, we have $Cv_1 \wedge Cv_2 = 0$. Define a scalar β such that

$$C(v_1 + \beta v_2) = 0.$$

For $v = v_1 + \beta v_2$ it would follow that $Cv = 0$ and $(A - \lambda_1 I)(A - \lambda_2 I)v = 0$ completing the proof. (Q.E.D.)

IV. Perspective Observability in the Presence of a Control

In this section, we would consider the *perspective observation problem* 1.2. However, the base field is assumed to be the complex field $\mathbb{C}$, therefore appropriate changes have to be made to the statement of problem 1.2. The essential question is that if the state $x(t)$ of the dynamical system (1.4), (1.5) is not observable from the output function $y(t)$ in (1.5), with $u(t) = 0$, can it be made observable by a proper choice of a control signal

$u(t)$? It is perhaps clear that if we have two nonzero vectors $\xi_1^*, \xi_2^* \in \mathbb{C}^n$ such that $Ce^{A\sigma}\xi_1^* = Ce^{A\sigma}\xi_2^*$ for all $\sigma \geq 0$, (1.7) can never be satisfied for any control input $u(t)$. In order to state the main result, we assume that these two nonzero vectors satisfy

$$Ce^{A\sigma}\xi_1^* \neq Ce^{A\sigma}\xi_2^* \tag{4.1}$$

for $\sigma \in [T_1, T_2]$, where $T_2 > T_1 \geq 0$.

Definition 4.1: A pair of points $[\xi_1]$, $[\xi_2]$ in $\mathbb{CP}^{n-1}$ are said to be perspective observable if problem 1.2 has an affirmative answer.

We now prove the following lemma.

Lemma 4.2: Over the base field $\mathbb{C}$, consider the dynamical system (1.4), (1.5) and the pair of vectors ξ_1^* and ξ_2^* satisfying (4.1). The pair of points $[\xi_1^*]$ and $[\xi_2^*]$ in $\mathbb{CP}^{n-1}$ are perspective unobservable iff

$$\dim Ce^{A\sigma}(H_1 + H_2) \leq 1 \tag{4.2}$$

for every σ in $\mathbb{R}$.

Proof of Lemma 4.2: Assume that the pair of points $[\xi_1^*]$ and $[\xi_2^*]$ are perspective unobservable. It follows that for every $\sigma \in \mathbb{R}$, every $\overline{\xi}_1, \overline{\xi}_2 \in \mathbb{C}^n : [\xi_1^*] = [\overline{\xi}_1], [\xi_2^*] = [\overline{\xi}_2]$, and for every control signal $u(t)$ we have

$$\begin{aligned} Ce^{A\sigma}\overline{\xi}_1 &+ \int_0^\sigma Ce^{A(\sigma-\tau)}Bu(\tau)\,d\tau \\ &= \alpha\left(Ce^{A\sigma}\overline{\xi}_2 + \int_0^\sigma Ce^{A(\sigma-\tau)}Bu(\tau)\,d\tau\right) \end{aligned} \tag{4.3}$$

for some nonzero scalar α, which may be dependent on σ. It follows, by choosing $u(t) = 0$ in particular, that for every fixed σ, there must exist a homogeneous line ℓ, possibly dependent on σ, in $\mathbb{C}^n$ such that

$$Ce^{A\sigma}\overline{\xi}_1 \in \ell, \qquad Ce^{A\sigma}\overline{\xi}_2 \in \ell. \tag{4.4}$$

Additionally, it follows that:

$$Ce^{A\sigma}\int_0^\sigma e^{-A\tau}Bu(\tau)\,d\tau \tag{4.5}$$

is a subset of ℓ for otherwise we can always choose $\overline{\xi}_1 = \xi_1^*$, $\overline{\xi}_2 = \xi_2^*$ and can conclude that under the restriction (4.1), there would always exist a control $u(t)$ such that (4.2) is not satisfied for σ in some interval $[T_1, T_2]$. Thus, we infer that

$$Ce^{A\sigma}H \subset \ell. \tag{4.6}$$

Combining (4.4) and (4.6), we have

$$Ce^{A\sigma}(H_1 + H_2) \subset \ell \tag{4.7}$$

which implies (4.2).

Conversely, assume that (4.2) is satisfied for every σ in $\mathbb{R}$. It follows that there exists a homogeneous line ℓ, possibly dependent on σ such that (4.7) is satisfied and that for every $\overline{\xi}_1$, $\overline{\xi}_2$ such that $[\overline{\xi}_1] = [\xi_1^*]$ and $[\overline{\xi}_2] = [\xi_2^*]$ we have (4.4) and (4.6) satisfied for every σ. We conclude that the vectors

$$Ce^{A\sigma}\overline{\xi}_1 + \int_0^\sigma Ce^{A(\sigma-\tau)}Bu(\tau)\,d\tau$$

and

$$Ce^{A\sigma}\overline{\xi}_2 + \int_0^\sigma Ce^{A(\sigma-\tau)}Bu(\tau)\,d\tau$$

in $\mathbb{C}^p$ are linearly dependent for any choice of control input $u(t)$ and for every σ. Thus $[\xi_1^*]$ and $[\xi_2^*]$ are perspective unobservable. (Q.E.D.)

Before we state the main result on perspective observability, we consider the following definition.

Definition 4.3: We shall say that the dynamical system (1.4) is perspective observable if problem 1.2 has an affirmative answer for every pair of distinct points $[\xi_1]$, $[\xi_2] \in \mathbb{CP}^{n-1} \times \mathbb{CP}^{n-1}$.

We are now in a position to state and prove one of the main theorems of this section which generalizes an earlier result reported in [29] and [30], wherein the control input was not present. The next theorem is about a PBH rank condition to test perspective observability of a dynamical system in the presence of a control.

Theorem 4.4: Assume that the matrix pair (C, A) is an observable pair, i.e.,

$$rank(C^T, (CA)^T, (CA^2)^T, \ldots, (CA^{n-1})^T)^T = n. \tag{4.8}$$

The dynamical system (1.4), (1.5) is perspective unobservable over the base field $\mathbb{C}$ iff there exists a pair of eigenvalues λ_0, λ_1 of A, such that for all pairs of complex numbers μ_0, μ_1 (may be the same) in the set $\{\lambda_0, \lambda_1, \delta_1, \ldots, \delta_s\}$ one has

$$\text{rank}\begin{pmatrix} (A - \mu_0 I)(A - \mu_1 I) \\ C \end{pmatrix} < n \tag{4.9}$$

where $\delta_1, \ldots, \delta_s$ is the set of eigenvalues of A such that the subspace spanned by the corresponding eigenvectors or generalized eigenvectors of A is H.

Remark 4.5: Note that the existence of $\delta_1, \ldots, \delta_s$ follows from the fact that H is A-invariant and the property in question is true for any such subspace. Note also that if the above observability rank condition (4.8) is not satisfied, then the system (1.4) is perspective unobservable.

Remark 4.6: In view of the results in Section III on the perspective observability in the absence of any control input, it is possible to state the results of Theorem 4.4 over the base field $\mathbb{R}$. Such a generalization has not been attempted in this paper.

Theorem 4.7: In linear systems theory, adding control does not change observability. In the perspective setting, this is not the case. Sometimes, an unobservable pair of initial conditions can be rendered observable by a proper choice of control.

Proof of Theorem 4.4: Before we sketch the formal proof, we note the following. If we assume that (4.9) is satisfied it follows that there exist a nonzero vector $v \in \mathbb{C}^n$ such that:

$$\begin{aligned} (A - \mu_0 I)(A - \mu_1 I)v &= 0 \\ Cv &= 0. \end{aligned} \tag{4.10}$$

It may be remarked that the pair of matrices $(A - \mu_0 I)$ and $(A - \mu_1 I)$ commute, a fact that would be used subsequently in the proof. Let S be the eigenspace spanned by the eigenvectors or generalized eigenvectors u_0, u_1 of A corresponding to eigenvalues μ_0, μ_1 it follows that v is an element of S and can be written as:

$$v = \alpha_0 u_0 + \alpha_1 u_1 \tag{4.11}$$

for some scalars α_0 and α_1. Finally, we have $Cv = 0 \Rightarrow \alpha_0 C u_0 + \alpha_1 C u_1 = 0$. It follows that:

$$Ce^{A\sigma} v = \alpha_0 e^{\mu_0 \sigma} C u_0 + \alpha_1 e^{\mu_1 \sigma} C u_1 = (e^{\mu_0 \sigma} - e^{\mu_1 \sigma}) \alpha_0 C u_0.$$

Hence, we conclude that

$$\dim Ce^{A\sigma}(S) \leq 1. \tag{4.12}$$

Sufficiency: Assume that (4.9) is satisfied for a pair of eigenvalues λ_0, λ_1 of A. It follows from (4.12) that if S is the subspace spanned by the eigenvectors or generalized eigenvectors of A corresponding to eigenvalues $\lambda_0, \lambda_1, \delta_1, \ldots, \delta_s$, we have

$$\dim Ce^{A\sigma}(S) \leq 1. \tag{4.13}$$

Let ξ_1^* and ξ_2^* be two linearly independent vectors in S, it follows from (4.13) that (4.2) would be satisfied. Thus, the pair $[\xi_1^*]$, $[\xi_2^*]$ cannot be observed.

Necessity: Assume that (1.4), (1.5) is perspective unobservable. It follows that there exist two independent vectors ξ_1^*, ξ_2^* such that $[\xi_1^*]$ and $[\xi_2^*]$ cannot be observed by (1.4), (1.5). Moreover, (4.1) is automatically satisfied by ξ_1^*, ξ_2^* because of the rank assumption (4.8). Using Lemma 4.2, we conclude that (4.2) is satisfied, for every σ in $\mathbb{R}$. It follows that there exists a homogeneous line ℓ, possibly depending on σ such that

$$Ce^{A\sigma}[\xi_i^*] \subset \ell, \qquad i = 1, 2 \tag{4.14}$$

and

$$Ce^{A\sigma} H \subset \ell. \tag{4.15}$$

In what follows we show that from (4.14) and (4.15) we can infer the following. *There exists a pair of eigenvectors (generalized eigenvectors) v_r and v_s of A with associated eigenvalues λ_r and λ_s such that for all pairs of complex numbers μ_0, μ_1 in the set*

$$\{\lambda_r, \lambda_s, \delta_1, \ldots, \delta_s\} \tag{4.16}$$

the rank condition (4.9) *is satisfied,* which would complete the proof.

Let $v_1, v_2, \ldots, v_s$ be a set of eigenvectors or generalized eigenvectors of A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$ such that

$$\begin{aligned} \xi_1^* &= \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_s v_s \\ \xi_2^* &= \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_r v_r \end{aligned} \tag{4.17}$$

where $\alpha_s \neq 0$ and $\beta_r \neq 0$ and where we assume that $r < s$ without any loss of generality (if not, replace ξ_2^* by $\xi_2^* + a\xi_1^*$ for some choice of a). One can order the eigenvalues in such a way that

$$\lambda_i + \lambda_j = \lambda_r + \lambda_s, \quad \Longleftrightarrow \quad \lambda_i = \lambda_r, \, \lambda_j = \lambda_s$$

and when $\lambda_i + \lambda_j = \lambda_r + \lambda_s$, we have

$$p_i + p_j = p_r + p_s, \quad \Longleftrightarrow \quad p_i = p_r, \, p_j = p_s$$

where p_i is the multiplicity of the eigenvalue λ_i. For details on the existence of this ordering see [29]. It follows from (4.14) that $Ce^{A\sigma} v_s$ and $Ce^{A\sigma} v_r$ are linearly dependent for all values of σ. In particular, since Cv_s and Cv_r are linearly dependent, there would exist a scalar θ such that $v_s + \theta v_r$ would be in the null space of the matrix

$$\operatorname{rank} \begin{pmatrix} (A - \lambda_s I)(A - \lambda_r I) \\ C \end{pmatrix}. \tag{4.18}$$

The above argument can actually be repeated for each and every pair of eigenvectors in $H_1 + H_2$ indicating that for every pair of eigenvalues μ_0, μ_1 in the set (4.16), the rank condition (4.9) would be satisfied. (Q.E.D.)

Remark 4.8: To clarify an important point made in the proof of Theorem 4.4, if μ_0 and μ_1 are two eigenvalues of a matrix A, and if v is a vector in the kernel of $(A - \mu_0 I)(A - \mu_1 I)$, such a v must necessarily be spanned by a pair of eigenvectors, generalized eigenvectors of A. If S is the associated eigenspace, spanned by the pair, it follows that $Ce^{At}S$ is at most one-dimensional. The choice of u_0 and u_1 in (4.11) is not unique but what is important in the proof is that the image of the subspace S under the map $Ce^{At}*$ is at most one-dimensional provided that the rank condition (4.9) is satisfied. In the case when S is higher dimensional, it can be defined as the span of the eigenvectors, generalized eigenvectors, of the eigenvalues of A in the set $\{\lambda_0, \lambda_1, \delta_1, \ldots, \delta_s\}$. Assuming that (4.9) is satisfied for every pair of eigenvalues μ_0, μ_1 in the above set, it would follow once again that the image of the subspace S under the map $Ce^{At}*$ is at most one dimensional.

To end this section, we write down the condition for perspective observability explicitly.

Remark 4.9: Assuming the observability rank condition (4.8) for the matrix pair (C, A), the dynamical system (1.4), (1.5) is perspective observable, over the base field $\mathbb{C}$ iff for every pair of eigenvalues λ_1, λ_2 of A, there exist some pair μ_0, μ_1 in the set $\{\lambda_1, \lambda_2, \delta_1, \ldots, \delta_s\}$ such that

$$\operatorname{rank} \begin{pmatrix} (A - \mu_0 I)(A - \mu_1 I) \\ C \end{pmatrix} = n. \tag{4.19}$$

Over the base field $\mathbb{R}$, the rank condition (4.19) is only sufficient.

The following corollary of the Theorem 4.4 is perhaps surprising.

Corollary 4.10: Assume that the dynamical system (1.4) and (1.5) is such that

$$\operatorname{rank}(B, AB, A^2B, \ldots, A^{n-1}B) \geq 2 \tag{4.20}$$

then the same dynamical system is perspective observable over the base field $\mathbb{C}$ iff the *observability rank condition* (4.8) is satisfied.

Proof of Corollary 4.10: If the observability rank condition is not satisfied, the dynamical system is clearly perspective unobservable. Conversely, it follows from (4.8) and (4.20) that:

$$\dim Ce^{A\sigma} H \geq 2.$$

Thus, from Lemma 4.2, it would follow that every pair of vectors would be perspective observable. (Q.E.D.)

V. Realizability via the Rational Exponential Interpolation

So far in the previous sections we have considered the problem of perspective control and observation assuming that the parameters of the homogeneous dynamical system (1.4) are known. If on the other hand, the parameters of the system are unknown, it is important to be able to identify the parameters from a record of the observation over a certain interval of time. In particular, it is important to realize homogeneous dynamical systems with minimum state dimension, if possible, and ascertain if the choice of the parameters is unique. If not, it is important to be able to classify the extent of the nonuniqueness. The interpolation problem considered in this section is a step in that direction.

Roughly speaking, the problem we consider is described as follows. Assume $B = 0$ and that the output of a dynamical system (1.4) has been observed over a finite interval of time. The problem is to identify the parameters of the system from the observed output. We shall see that the parameter identification problem is connected with a certain class of interpolation problem. In particular, for a linear dynamical system, one considers an *exponential interpolation problem.* On the other hand, for a homogeneous dynamical system, of the kind described in (1.4) one obtains a *rational exponential interpolation problem.*

Before we describe the "rational exponential interpolation" problem, let us consider the exponential interpolation problem already described in [32]. We consider a linear autonomous system

$$\dot{x} = Ax \quad y = Cx \quad x(0) = x_0 \tag{5.1}$$

solution of which can be expressed as follows, assuming distinct eigenvalues of A:

$$y(t) = Ce^{At}x_0 = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t}. \tag{5.2}$$

We assume $y \in \mathbb{R}^p$, $x \in \mathbb{R}^n$ and that $y(t)$ is given at discrete data points $t = t_0 + hk$, $k = 0, 1, 2, \ldots$; where t_0, h are given fixed constants. We obtain

$$y(t_0 + hk) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t_0 + \lambda_i hk}. \tag{5.3}$$

Let us define

$$\gamma_i = \alpha_i e^{\lambda_i t_0}, \qquad \xi_i = e^{\lambda_i h}, \qquad y_k = y(t_0 + hk) \tag{5.4}$$

and rewrite (5.3) as

$$y_k = \sum_{i=1}^{n} \gamma_i \xi_i^k, \qquad k = 0, 1, 2, \ldots. \tag{5.5}$$

Finally, we write (5.5) in vector form and define two $p \times n$ matrices

$$\hat{y}_k = (y_k, y_{k+1}, \ldots, y_{k+n-1}) \tag{5.6}$$

and

$$\hat{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n). \tag{5.7}$$

We now obtain

$$\hat{y}_k = \hat{\gamma} V_k \tag{5.8}$$

where V_k is a $n \times n$ square matrix defined as

$$V_k = \begin{pmatrix} \xi_1^k & \xi_1^{k+1} & \cdots & \xi_1^{k+n-2} & \xi_1^{k+n-1} \\ \xi_2^k & \xi_2^{k+1} & \cdots & \xi_2^{k+n-2} & \xi_2^{k+n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ \xi_{n-1}^k & \xi_{n-1}^{k+1} & \cdots & \xi_{n-1}^{k+n-2} & \xi_{n-1}^{k+n-1} \\ \xi_n^k & \xi_n^{k+1} & \cdots & \xi_n^{k+n-2} & \xi_n^{k+n-1} \end{pmatrix}. \tag{5.9}$$

It is easy to see that

$$\hat{y}_{k+1} = \hat{y}_k V_k^{-1} V_{k+1} \tag{5.10}$$

where

$$V_k^{-1} V_{k+1} = V_0^{-1} V_1 \tag{5.11}$$

and is of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -f_1 \\ 1 & 0 & & 0 & -f_2 \\ 0 & 1 & \cdots & 0 & -f_3 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & -f_n \end{pmatrix} \tag{5.12}$$

where

$$\prod_{i=1}^{n} (\lambda - \xi_i) = \lambda^n + f_n \lambda^{n-1} + f_{n-1} \lambda^{n-2} + \cdots + f_1. \tag{5.13}$$

To see (5.13), note that $V_1 = DV_0$ where

$$D = diag(\xi_1, \xi_2, \ldots, \xi_n).$$

It now follows from (5.10) and (5.12) that:

$$y_{k+n} = f_1 y_k + f_2 y_{k+1} + \cdots + f_n y_{k+n-1}.$$

From the $y_j - s$, one computes the coefficients $f_1, \ldots, f_n$ and the variables $\xi_1, \ldots, \xi_n$. In principle, one can compute λ_1, λ_2, $\ldots$, λ_n. Using (5.8), one can also compute $\gamma_1, \gamma_2, \ldots, \gamma_n$ and hence $\alpha_1, \alpha_2, \ldots, \alpha_n$.

This is a very brief explanation of Prony's method. In the form described here it is obvious that the technique is numerically unstable. However, it can be implemented in a more stable fashion [32]. A standard reference for Prony's method is the numerical analysis text by Hildebrand [35].

We now proceed to consider the exponential rational interpolation problem and consider the homogeneous dynamical system (1.4), (1.5). Writing the output $y(t)$ in homogeneous coordinates of $\mathbb{RP}^{p-1}$ as

$$y(t) = [Ce^{At}x_0] = \left[\sum_{i=1}^{n} \alpha_i e^{\lambda_i t}\right]. \tag{5.14}$$

Abbreviating $y(t) = [y_1(t), y_2(t), \ldots, y_p(t)]$, a specific choice of coordinates of $\mathbb{RP}^{p-1}$ is given by

$$y_c(t) = \left(\frac{y_1(t)}{y_p(t)}, \frac{y_2(t)}{y_p(t)} \ldots, \frac{y_{p-1}(t)}{y_p(t)}\right).$$

In the notation of (5.14), the vector $y_c(t) \in \mathbb{R}^{p-1}$ can be defined as a rational exponential function

$$y_c(t) = \left(\sum_{i=1}^{n} \beta_i e^{\lambda_i t}\right) \Bigg/ \left(\sum_{i=1}^{n} \delta_i e^{\lambda_i t}\right)$$

where $\beta_i \in \mathbb{R}^{p-1}$ and $\delta_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$. Discretizing $y_c(t)$ as before, we obtain

$$y_c(t_0 + hk) = \frac{\sum_{i=1}^{n} (\beta_i e^{\lambda_i t_0})(e^{\lambda_i h})^k}{\sum_{i=1}^{n} (\delta_i e^{\lambda_i t_0})(e^{\lambda_i h})^k}.$$

Choosing the notations as follows:

$$y_k = y_c(t_0 + hk) \quad \xi_i = e^{\lambda_i h} \quad \gamma_i = \beta_i e^{\lambda_i t_0} \quad \rho_i = \delta_i e^{\lambda_i t_0}$$

we obtain

$$y_k \left(\sum_{i+1}^{n} \rho_i \xi_i^k\right) = \sum_{i=1}^{n} \gamma_i \xi_i^k \tag{5.15}$$

where $\gamma_i \in \mathbb{R}^{p-1}$, $\rho_i \in \mathbb{R}$, $y_k \in \mathbb{R}^{p-1}$. For notational simplicity, we assume at this point that $p = 2$. The general case can be easily constructed and will be remarked later on. We would now like to write (5.15) in matrix notation by defining

$$\hat{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)$$
$$\hat{\rho} = (\rho_1, \rho_2, \ldots, \rho_n)$$

and V_k as in (5.9) as follows:

$$\hat{\gamma} V_k = \hat{\rho} V_k D_k \tag{5.16}$$

where $D_k = diag(y_k, y_{k+1}, \ldots, y_{k+n-1})$. It is easy to eliminate $\hat{\gamma}$ from (5.16) and obtain

$$\hat{\rho} V_k D_k V_k^{-1} V_{k+1} D_{k+1}^{-1} = \hat{\rho} V_{k+1}. \tag{5.17}$$

Recalling the structure of $V_k^{-1} V_{k+1}$ from (5.12), we obtain

$$D_k V_k^{-1} V_{k+1} D_{k+1}^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -f_1 \dfrac{y_k}{y_{k+n}} \\ 1 & 0 & & 0 & -f_2 \dfrac{y_{k+1}}{y_{k+n}} \\ 0 & 1 & \cdots & 0 & -f_3 \dfrac{y_{k+2}}{y_{k+n}} \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & -f_n \dfrac{y_{k+n-1}}{y_{k+n}} \end{pmatrix}. \tag{5.18}$$

We can rewrite (5.17) as

$$\hat{\rho} V_k \Sigma V_{k+1}^{-1} = 0 \tag{5.19}$$

where

$$\Sigma = \begin{pmatrix} 0 & 0 & \cdots & 0 & f_1 \left(1 - \dfrac{y_k}{y_{k+n}}\right) \\ 0 & 0 & & 0 & f_2 \left(1 - \dfrac{y_{k+1}}{y_{k+n}}\right) \\ 0 & 0 & \cdots & 0 & f_3 \left(1 - \dfrac{y_{k+2}}{y_{k+n}}\right) \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & f_n \left(1 - \dfrac{y_{k+n-1}}{y_{k+n}}\right) \end{pmatrix}.$$

We need to find conditions on ξ_i such that (5.19) can be solved for a nonzero $\hat{\rho}$. In principle, this can be done by rewriting (5.19) as

$$\Pi \Delta = 0 \tag{5.20}$$

where

$$\Delta = (f_1\rho_1, \ldots, f_1\rho_n, f_2\rho_1, \ldots, f_2\rho_n, \ldots, f_n\rho_1, \ldots, f_n\rho_n)^T$$

and where Π can be appropriately defined. Note that Δ is an n^2 dimensional vector independent of k. The matrix Π depends on k and is a function of y_k and ξ_i. In order to solve (5.20) for a nonzero Δ, the matrix Π has to have a corank at least one, giving rise to a set of determinant conditions on ξ_i. By choosing n such determinants, one would solve for the variables ξ_i. The rest of the algorithm is same as the classical Prony's algorithm.

Remark 5.1: When $p > 2$, one can easily write a set of $p-1$ matrix equations of the form (5.20). The details can be easily constructed.

Remark 5.2: The exponential rational interpolation algorithm described above depends on our ability to solve a set of at least n polynomial equations in the variables ξ_i. Unfortunately, this process is not numerically robust. The modified Prony's algorithm described in this section needs to be compared with the rescaling algorithm described in [8], wherein the exponential rational interpolation problem has been rescaled to a regular exponential interpolation problem. This can then be solved by the classical Prony's algorithm.

VI. Summary and Conclusion

To summarize, the important contributions of this paper are three fold. First of all, *it introduces a new perspective control problem which is important in steering a vector in* $\mathbb{R}^n$ *up to its direction.* It is not totally surprising that these problems have a connection with the Riccati flow. What would be important in this context of gaze control is to introduce dynamical systems that are somewhat more general than (1.4) (see, for example, [37]) and possibly makes contact with the dynamics of mechanical systems. One would like to be able to visually actuate these systems up to direction. We would like to refer to [31] wherein (1.4) has been replaced by a more general Riccati dynamics. Second of all, *we revisit the perspective observability problem* [29], [30] *and obtain a necessary and sufficient condition over the real base field, in the absence of any control.* We also obtain a rank condition (4.9) as an observability criterion in the presence of a control. The rank condition is necessary and sufficient over $\mathbb{C}$ and is claimed to be sufficient over $\mathbb{R}$. The third contribution that we present in this paper is described as follows: *if the output vector function is known only up to its direction, how to synthesize a perspective system, that would match this data.* We show that this problem is equivalent to a rational exponential interpolation problem and emphasize that it is an important problem. Obtaining a robust solution to this problem is currently an open research topic.

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