Identification of Riccati Dynamics Under Perspective and Orthographic Observations

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***Abstract*—In this paper, the problem of identifying motion and shape parameters of a planar object undergoing a Riccati motion, from the associated optical flow generated on the image plane of a single CCD camera, has been studied. The optical flow is generated by projecting feature points on the object onto the image plane via perspective and orthographic projections. Riccati dynamics is to be viewed as a natural extension of the well-known affine dynamics that has been the subject of parameter estimation research for many years. An important result we show is that, under perspective projection, the parameters of a specific Riccati dynamics that extend the well-known "rigid motion" can be identified up to choice of a sign. This fact is in sharp contrast to many other results in the literature, where under perspective projection, parameters are recovered up to a possible depth ambiguity. The paper also discusses other Riccati equations obtained from quadratic extension of a rigid motion and affine motion. For each of the various motion models considered and for each of the two projection models, we show that the extent to which motion and shape parameters can be recovered from optical flow can in fact be recovered from the linear approximation of the optical flow. The quadratic part of the optical flow carries no additional information for the class of parameter identification problems considered. We also extend our analysis to a pair of cameras.**

***Index Terms*—Affine flow, orthographic projection, parameter identification, perspective projection, planar object, Riccati dynamics, rigid flow.**

I. Introduction and Background

An important problem that has been of interest in machine vision is to identify parameters of motion dynamics from observing projection of feature points on the image plane observed over time. Typically one assumes that the feature points are located on the surface of the moving object and that their projections describe an *optical flow* over time. The underlying problem of interest is to locate the surface and estimate its motion from the corresponding optical flow.

To illustrate the class of problems considered in this paper we assume that the motion dynamics is described by the following equation:

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} f_1 & f_2 & f_3 & 0 & 0 & 0 \\ 0 & f_1 & 0 & f_2 & f_3 & 0 \\ 0 & 0 & f_1 & 0 & f_2 & f_3 \end{pmatrix}\begin{pmatrix} X^2 \\ XY \\ XZ \\ Y^2 \\ YZ \\ Z^2 \end{pmatrix}. \tag{1.1}$$

Let us denote

$$\begin{aligned} b &= (\, b_1 \quad b_2 \quad b_3\,)^T \\ f^T &= (\, -f_1 \quad -f_2 \quad -f_3\,) \\ A &= (a_{ij}) \end{aligned} \tag{1.2}$$

where $i,\, j = 1,\, 2,\, 3$. The dynamical system (1.1) is a Riccati dynamics in $\mathbb{R}^3$, which has been illustrated in Figs. 1 and 2. It is a class of *quadratic motion models* more general than a *rigid flow* which preserves *shape*. Thus in Figs. 1 and 2 the shape of the planar surfaces remain planar, although the distance between two points on the plane may not remain constant. The motivation behind considering a Riccati dynamics as a model for motion will be described later in this paper. An important question, that is of interest in machine vision, is to ask the following.

Question 1.1: Consider the dynamical system (1.1) and assume that the state vector $(X,\, Y,\, Z)$ is observed by the function

$$x = X/Z, \qquad y = Y/Z, \qquad Z \neq 0. \tag{1.3}$$

To what extent are the motion parameters and the initial conditions $X(0), Y(0), Z(0)$ identifiable from the observation vector $(x(t),\, y(t))$ in a given time interval $[0,\, T)$, $T > 0$?

In general, the point $(X,\, Y,\, Z)$ may be assumed to be a feature point on a surface and we assume in this paper that this surface is a planar surface described by

$$\overline{p}X + \overline{q}Y + \overline{s}Z + 1 = 0. \tag{1.4}$$

It is fairly easy to show that the vector $(x,\, y)$ defined in (1.3) satisfies the following differential equation on the image plane $\mathbb{R}^2$ described by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \begin{pmatrix} d_3 & d_4 \\ d_5 & d_6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d_7 x^2 + d_8 xy \\ d_8 y^2 + d_7 xy \end{pmatrix} \tag{1.5}$$

where $d_1, \cdots, d_8$ are eight time-varying parameters that are functions of the shape parameters $\overline{p}$, $\overline{q}$, $\overline{s}$. The above equation

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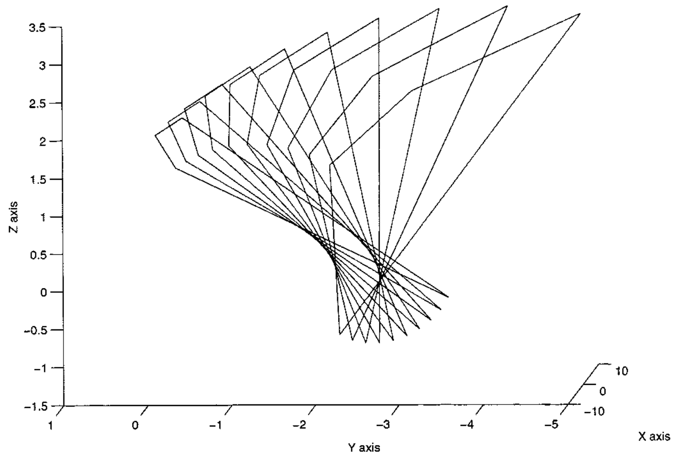

Fig. 1. A plane undergoing Riccati dynamics in $\mathbb{R}^3$. The plane has been represented by four points.

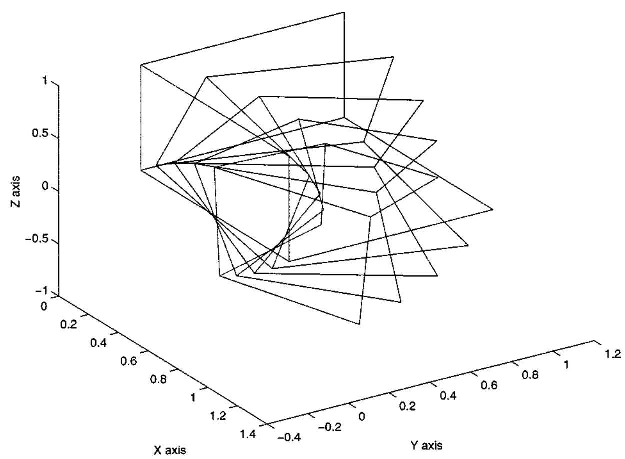

Fig. 2. A cube moving under Riccati dynamics in $\mathbb{R}^3$. One corner of the cube with three intersecting surfaces has been shown.

(1.5) is well known in the literature and is called the "optical flow." It is of great interest, starting probably from Horn [4], (see also [5]), as to how one can estimate the "optical flow" parameters $d_1, \cdots, d_8$ from the image data. We now ask the following modification to Question 1.1.

Question 1.2: Assume that the parameters $d_1, \cdots, d_8$ of the optical flow dynamics (1.5) have been obtained in some time interval $[0, T)$. To what extent are the motion parameters identifiable together with all the shape parameters $\overline{p}, \overline{q}, \overline{s}$?

Note in particular that in Question 1.2, one is interested not just in recovering motion parameters but also the location of the plane (1.4) as it moves in time.

We now review the background of Questions 1.1 and 1.2. If we assume that the vector $f = 0$ and that the matrix A is skew symmetric, i.e. $A = \Omega$ where

$$\Omega = \begin{pmatrix} 0 & \omega_1 & \omega_2 \\ -\omega_1 & 0 & \omega_3 \\ -\omega_2 & -\omega_3 & 0 \end{pmatrix} \tag{1.6}$$

then the dynamical system (1.1) reduces to an affine dynamics in $\mathbb{R}^3$ which models a rigid motion of rigid bodies and has been considered by several researchers in machine vision for several years (see Kanatani [1] for a background discussion). Note in particular that for this specific case, $\omega_1, \omega_2, \omega_3, b_1, b_2, b_3$ are six motion parameters and $\overline{p}, \overline{q}, \overline{s}$ are three shape parameters and Question 1.2 refers to the extent these nine motion and shape parameters can be recovered from the "brightness pattern," which is assumed to be directly connected to the optical flow parameters in (1.5). A partial answer to Question 1.2 has been provided by Kanatani [1] as follows. If we define

$$p = -\overline{p}/\overline{s},\ q = -\overline{q}/\overline{s},\ c_1 = -b_1\overline{s},\ c_2 = -b_2\overline{s},\ c_3 = -b_3\overline{s}$$

Kanatani showed that parameters can be recovered up to two alternatives.

Proposition 1.3 (Kanatani [1]): Assume that $c_3 \neq 0$, then parameters can be recovered up to the following alternatives:

$$\begin{aligned} &(c_1, c_2, c_3, \omega_1, \omega_2, \omega_3, p, q), \\ &(-c_3 p, -c_3 q, c_3, \omega_1 - c_1 q + c_2 p, \omega_2 + c_1 + c_3 p, \\ &\quad \omega_3 + c_2 + c_3 q, -c_1/c_3, -c_2/c_3). \end{aligned}$$

Similar results have also been reported earlier by Waxman and Ullman [3]. The results reported in [1] and [3] in general ignored the fact that $\overline{p}, \overline{q}, \overline{s}$ satisfy an equation of its own, called the shape dynamics and is given by

$$\frac{d}{dt}\begin{pmatrix} \overline{p} \\ \overline{q} \\ \overline{s} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} - \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}\begin{pmatrix} \overline{p} \\ \overline{q} \\ \overline{s} \end{pmatrix} + \begin{pmatrix} b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & b_1 & 0 & b_2 & b_3 & 0 \\ 0 & 0 & b_1 & 0 & b_2 & b_3 \end{pmatrix}\begin{pmatrix} \overline{p}^2 \\ \overline{pq} \\ \overline{ps} \\ \overline{q}^2 \\ \overline{qs} \\ \overline{s}^2 \end{pmatrix}. \tag{1.7}$$

Note in particular that (1.7) has the same structure as that of (1.1). Once again, if we assume that $f = 0$ and $A = \Omega$ described by (1.6), Ghosh and Loucks [2] considered this special case of the dynamical system (1.7) with the vector $(d_1, \cdots, d_8)$ as the observation vector and showed the following.

Proposition 1.4 (Ghosh and Loucks [2]): Consider the dynamical system (1.7) with $f = 0$ and $A = \Omega$. For a generic choice of parameters in the space $\mathbb{R}^9$ of parameters $\omega_1, \omega_2, \omega_3, b_1, b_2, b_3, \overline{p}, \overline{q}, \overline{s}$, it is possible to identify $\omega_1, \omega_2, \omega_3$ uniquely and the vector $(b_1, b_2, b_3, \overline{p}, \overline{q}, \overline{s})$ up to a nonzero scale factor α giving rise to the following scaling:

$$\begin{aligned} &\mathbb{R}^1 - \{0\} \times \mathbb{R}^6 \longrightarrow \mathbb{R}^6 \\ &(\alpha, b_1, b_2, b_3, \overline{p}, \overline{q}, \overline{s}) \mapsto (b_1\alpha, b_2\alpha, b_3\alpha, \overline{p}/\alpha, \overline{q}/\alpha, \overline{s}/\alpha). \end{aligned}$$

The parameter α is the depth parameter which scales the plane (1.4) to

$$\overline{p}X + \overline{q}Y + \overline{s}Z + \alpha = 0.$$

Remark 1.5: Throughout this paper, we shall use the word "generic" to mean "for all but possibly a proper algebraic set."

The conclusion of Proposition 1.4 is that using perspective projection and optical flow based methods, it is possible, at best, to identify the shape parameters $(\overline{p}, \overline{q}, \overline{s})$ and translational motion parameters (b_1, b_2, b_3), up to choice of a depth parameter ambiguity α, while the angular velocities can be precisely identified.

We now remark as to why it is important to consider a Riccati equation (1.1) as a suitable motion model for our problem. This is because if feature points on a plane satisfy (1.1) the parameters $(\overline{p}, \overline{q}, \overline{s})$ satisfy (1.7). If instead, an affine motion model for (1.1) is used, one obtains (1.7) with $f = 0$ for the dynamics of the shape parameters. Likewise if an affine model for (1.7) is used, one obtains (1.1) with $b = 0$ for the motion dynamics. In either of the two cases, we are still faced with a Riccati equation to analyze, assuming an affine motion model does not simplify the shape dynamics. On the other hand, assuming a Riccati motion dynamics does not render our problem to become more complicated, the shape dynamics still remains a Riccati equation. The other advantage of choosing Riccati dynamics for our motion model is that it is preserved under orthographic and perspective projections. Finally, choice of a Riccati dynamics includes all other dynamics considered in the literature as a special case. Moreover, in many problems, *viz*. medical image registration problems [7], [8], it is important to extend the rigid motion with a higher order perturbation, leading to a possible nonrigid mechanics for the associated motion model.

To discuss the results of this paper, let us consider the dynamical system (1.1) where $A = \Omega$ and $b_i = f_i$, $i = 1, 2, 3$. Such a dynamical system can be viewed as a quadratic extension of the periodic rotational dynamics

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & \omega_1 & \omega_2 \\ -\omega_1 & 0 & \omega_3 \\ -\omega_2 & -\omega_3 & 0 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

We ask Questions 1.1 and 1.2 related to this dynamics and obtain the following interesting result.

Theorem 1.6: Consider the dynamic system (1.1) with $A = \Omega$ and $b = -f$ together with an observation function (1.3). For a generic choice of parameters, it is possible to identify $(\omega_1, \omega_2, \omega_3)$ uniquely and $(b_1, b_2, b_3, X(0), Y(0), Z(0))$ up to a sign ambiguity.

The following theorem is also quite surprising.

Theorem 1.7: For the dynamic system (1.7) with $A = \Omega$ and $b = -f$, and for a generic choice of parameters in the space $\mathbb{R}^9$ of parameters $\omega_1, \omega_2, \omega_3, b_1, b_2, b_3, \overline{p}, \overline{q}, \overline{s}$, it is possible to identify $\omega_1, \omega_2, \omega_3$ uniquely and the vector $(b_1, b_2, b_3, \overline{p}, \overline{q}, \overline{s})$ up to a sign ambiguity, by observing optical flow generated by feature points on the moving plane (1.4).

Remark 1.8: The reason why Theorems 1.6 and 1.7 are surprising is that unlike Propositions 1.3 and 1.4, there is no depth ambiguity. In fact, the depth ambiguity has been reduced to a pair of sign ambiguity.

We shall see subsequently in this paper that the depth resolvability has been possible because of the special structure of (1.1) and (1.7) with $A = \Omega$ and $b = -f$. This would not be the case in general. Nevertheless, we find it quite interesting to note the existence of a nonlinear dynamics for which depth ambiguity can be resolved under perspective projection.

We shall also consider "orthographic projection," where the state vector $(X, Y, Z)^T$ is observed as follows:

$$x = X, \quad y = Y. \tag{1.8}$$

For the motion dynamics (1.1) with $A = \Omega$, the optical flow equation under orthographic projection (1.8) continues to be of the form (1.5) with a different functional description of $d_1, \cdots, d_8$.

In either of the two cases for (1.3), (1.8) "perspective" and "orthographic" projections, respectively, we may assume that the quadratic component of (1.5) is small and is therefore not observed. The parameter identification problem is discussed under the assumption that the parameters of the optical flow (1.5) are observed up to affine terms. We show, quite surprisingly, that in this case no additional information regarding motion and shape parameters is lost.

Finally the paper concludes with a discussion of the two-camera projection instead of one where we assume that the relative position and orientation of the two cameras are known. We assume that each of the two cameras, working in parallel, compute the affine component of the optical flow and show that all the 18 motion and shape parameters of a "plane" (1.4) undergoing a "Riccati dynamics" (1.7), can be identified uniquely. This emphasizes the role of a two-camera "nonstereo" approach to machine vision when the calibration parameters between the two cameras are assumed known.

II. Associated Homogeneous Dynamics

In this section, we consider a general formulation of the problem considered in Section I. We do this by rewriting the state vector in (1.1) with respect to a set of homogeneous coordinates as follows:

$$X = X_1/W_1, \qquad Y = Y_1/W_1, \qquad Z = Z_1/W_1. \tag{2.1}$$

The Riccati dynamics (1.1) redues to

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{bmatrix} = \begin{pmatrix} A & b \\ f^T & 0 \end{pmatrix}\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{bmatrix} \tag{2.2}$$

where the state vector $[X_1\ Y_1\ Z_1\ W_1]^T$ is an element of $\mathbb{RP}^3$, and the real projective space of homogeneous lines in $\mathbb{R}^4$. Since the 4×4 matrix in (2.2) is defined only up to addition of a term λI where λ is a scalar, we assume without any loss of generality that trace $A = 0$ throughout this paper. Assume as in Section I, that the state vector (X, Y, Z) lies on the plane given by (1.4). We now define the following set of homogeneous coordinates for $\overline{p}$, $\overline{q}$ and $\overline{s}$ given as follows:

$$\overline{p} = p/w, \qquad \overline{q} = q/w, \qquad \overline{s} = s/w. \tag{2.3}$$

In the homogeneous coordinates $[p, q, s, w]^T$, the shape dynamics is described as follows:

$$\frac{d}{dt}\begin{bmatrix} p \\ q \\ s \\ w \end{bmatrix} = \begin{pmatrix} -A^T & -f \\ -b^T & 0 \end{pmatrix}\begin{bmatrix} p \\ q \\ s \\ w \end{bmatrix} \tag{2.4}$$

where the state vector $[p \ \ q \ \ s \ \ w]^T$ is once again an element of $\mathbb{RP}^3$. In the homogeneous coordinates (2.1), (2.3), the plane (1.4) is described as follows:

$$pX_1 + qY_1 + sZ_1 + wW_1 = 0. \tag{2.5}$$

Finally we assume that we observe the vector (x, y) given by (1.3), which is described as perspective observation. The dynamics of the vector (x, y) can be easily shown to satisfy an equation of the form (1.5) where the parameters $d_1, \cdots, d_8$ can be written as

$$\begin{array}{ll} d_1 = a_{13} - b_1\bar{s}, & d_2 = a_{23} - b_2\bar{s} \\ d_3 = a_{11} - a_{33} - b_1\bar{p} + b_3\bar{s}, & d_4 = a_{12} - b_1\bar{q} \\ d_5 = a_{21} - b_2\bar{p}, & d_6 = a_{22} - a_{33} - b_2\bar{q} + b_3\bar{s} \\ d_7 = -a_{31} + b_3\bar{p}, & d_8 = -a_{32} + b_3\bar{q}. \end{array} \tag{2.6}$$

If we assume that we observe the vector (x, y) given by (1.8), which is described by orthographic observation, the dynamics of the vector (x, y) can be shown to satisfy an equation of the form (1.5) where $d_1, \cdots, d_8$ can be written as

$$\begin{array}{ll} d_1 = b_1 - a_{13}w/s, & d_2 = b_2 - a_{23}w/s \\ d_3 = a_{11} - a_{13}p/s - f_3w/s, & d_4 = a_{12} - a_{13}q/s \\ d_5 = a_{21} - a_{23}p/s, & d_6 = a_{22} - a_{23}q/s - f_3w/s \\ d_7 = f_1 - f_3p/s, & d_8 = f_2 - f_3q/s. \end{array} \tag{2.7}$$

In order to homogenize the eight parameters $d_1, \cdots, d_8$, we define

$$d_1 = y_1/y_9, \quad d_2 = y_2/y_9, \quad \cdots, \quad d_8 = y_8/y_9. \tag{2.8}$$

We can write (2.6) in homogeneous coordinates as follows:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \end{bmatrix} = \begin{pmatrix} 0 & 0 & -b_1 & a_{13} \\ 0 & 0 & -b_2 & a_{23} \\ -b_1 & 0 & b_3 & a_{11} - a_{33} \\ 0 & -b_1 & 0 & a_{12} \\ -b_2 & 0 & 0 & a_{21} \\ 0 & -b_2 & b_3 & a_{22} - a_{33} \\ b_3 & 0 & 0 & -a_{31} \\ 0 & b_3 & 0 & -a_{32} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} p \\ q \\ s \\ w \end{bmatrix}. \tag{2.9}$$

Likewise, we can rewrite (2.7) in homogeneous coordinates as follows:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \end{bmatrix} = \begin{pmatrix} 0 & 0 & b_1 & -a_{13} \\ 0 & 0 & b_2 & -a_{23} \\ -a_{13} & 0 & a_{11} & -f_3 \\ 0 & -a_{13} & a_{12} & 0 \\ -a_{23} & 0 & a_{21} & 0 \\ 0 & -a_{23} & a_{22} & -f_3 \\ -f_3 & 0 & f_1 & 0 \\ 0 & -f_3 & f_2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} p \\ q \\ s \\ w \end{bmatrix}. \tag{2.10}$$

Note that in each of the two equations, (2.9) and (2.10), the observation vector $[y_1, \cdots, y_9]^T$ is a vector in $\mathbb{RP}^8$. The parameter identification problem under perspective and orthographic projections is described as follows.

Problem 2.1 (Identification Under Perspective Observation): Consider the homogeneous dynamical system (2.4), (2.9). The problem is to identify A, b, f, $\bar{p}$, $\bar{q}$, $\bar{s}$ to the extent possible.

Problem 2.2 (Identification under orthographic observation): Consider the homogeneous dynamical system (2.4), (2.10). The problem is to identify A, b, f, $\bar{p}$, $\bar{q}$, $\bar{s}$ to the extent possible.

Additionally, we shall analyze Problems 2.1 and 2.2 under the special case when A is a skew symmetric matrix. We also consider the case when $A = \Omega$ and $b = -f$.

If we assume that the optical flow (1.5) is visible only up to affine terms at a given known point (x^*, y^*) on the image plane, we linearize the right-hand side of (1.5) and obtain

$$\begin{aligned} \dot{x} &= d_1 + (d_3 + 2d_7x^* + d_8y^*)x + (d_4 + d_8x^*)y + \text{h.o.t.} \\ \dot{y} &= d_2 + (d_5 + d_7y^*)x + (d_6 + 2d_8y^* + d_7x^*)y + \text{h.o.t.}. \end{aligned} \tag{2.11}$$

Thus the observed parameters are

$$\begin{array}{ll} \xi_1 = d_1, & \xi_2 = d_2 \\ \xi_3 = d_3 + 2d_7x^* + d_8y^*, & \xi_4 = d_4 + d_8x^* \\ \xi_5 = d_5 + d_7y^*, & \xi_6 = d_6 + 2d_8y^* + d_7x^*. \end{array} \tag{2.12}$$

Analogous to (2.8), if we define

$$\xi_1 = y_1/y_7, \quad \xi_2 = y_2/y_7, \quad \cdots, \quad \xi_6 = y_6/y_7$$

we can write (2.12) in homogeneous coordinates as follows:

$$[y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5 \quad y_6 \quad y_7]^T = \Psi[p \quad q \quad s \quad w]^T \tag{2.13}$$

where $\Psi =$

$$\begin{pmatrix} 0 & 0 & -b_1 & a_{13} \\ 0 & 0 & -b_2 & a_{23} \\ -b_1 + 2x^*b_3 & y^*b_3 & b_3 & a_{11} - a_{33} \\ & & & -2x^*a_{31} - y^*a_{32} \\ 0 & -b_1 + x^*b_3 & 0 & a_{12} - x^*a_{32} \\ -b_2 + y^*b_3 & 0 & 0 & a_{21} - y^*a_{31} \\ x^*b_3 & -b_2 + 2y^*b_3 & b_3 & a_{22} - a_{33} \\ & & & -x^*a_{31} - 2y^*a_{32} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for the perspective observation (2.9) observed up to linear terms. We can also write (2.12) in homogeneous coordinates as follows:

$$[y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5 \quad y_6 \quad y_7]^T = \Phi[p \quad q \quad s \quad w]^T \tag{2.14}$$

where $\Phi =$

$$\begin{pmatrix} 0 & 0 & b_1 & -a_{13} \\ 0 & 0 & b_2 & -a_{23} \\ -a_{13} - 2x^*f_3 & -y^*f_3 & a_{11} + 2x^*f_1 & -f_3 \\ & & +y^*f_2 & \\ 0 & -a_{13} - x^*f_3 & a_{12} + x^*f_2 & 0 \\ -a_{23} - y^*f_3 & 0 & a_{21} + y^*f_1 & 0 \\ -x^*f_3 & -a_{23} - 2y^*f_3 & a_{22} + x^*f_1 & -f_3 \\ & & +2y^*f_2 & \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

for the orthographic observation (2.10) observed up to affine terms.

We end this section describing the following two problems of parameter identification under perspective and orthographic projection, where the optical flow is observed up to affine terms.

Problem 2.3 (Identification Under Linearized Perspective Observation) : Consider the homogeneous dynamical system (2.4), (2.13). The problem is to identify A, b, f, $\overline{p}$, $\overline{q}$, $\overline{s}$ to the extent possible.

Problem 2.4 (Identification Under Linearized Orthographic Observation): Consider the homogeneous dynamical system (2.4), (2.14). The problem is to identify A, b, f, $\overline{p}$, $\overline{q}$, $\overline{s}$ to the extent possible.

As in Problems 2.1 and 2.2, we also analyze Problems 2.3 and 2.4 under the special case when $A = \Omega$ and when $A = \Omega$, $b = -f$.

III. Identification Under Perspective Projection

The objective of this section is to analyze Problem 2.1 in details and gain insight to Problems 2.2–2.4 that are topics of discussion in the subsequent sections of this paper. Before analyzing Problem 2.1, we refer to Question 1.1 and consider the homogeneous dynamical system (2.2) with a homogeneous observation function

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{bmatrix} \tag{3.1}$$

where λ is an arbitrary nonzero parameter. The pair (2.2), (3.1) is referred to as a perspective dynamical system with state space $\mathbb{RP}^3$ and observation space $\mathbb{RP}^2$. Let us abbreviate (2.2), (3.1) as

$$\frac{d}{dt}[\mathcal{X}(t)] = \mathcal{A}[\mathcal{X}(t)], \qquad [\mathcal{Y}(t)] = \mathcal{C}[\mathcal{X}(t)] \tag{3.2}$$

where the definitions of $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{A}$, and $\mathcal{C}$ are clear. The dynamical system (3.2) is referred to via the triplet $(\mathcal{C}, \mathcal{A}, [\mathcal{X}(\prime)])$. We now have the following.

Theorem 3.1: Consider the homogeneous dynamical system (3.2). For a generic choice of the matrix $\mathcal{A}$ and state vector $[X_1(0), Y_1(0), Z_1(0), W_1(0)]^T$, the set of all triplets that produce the same output as given by (3.1) is described as

$$(\lambda_1 \mathcal{C} P^{-1}, P\mathcal{A}P^{-1}, [P\mathcal{X}(0)]) \tag{3.3}$$

where λ_1 is a nonzero real number and P is a nonsingular 4×4 matrix of the form

$$P^{-1} = \begin{pmatrix} p_{11} & 0 & 0 & 0 \\ 0 & p_{11} & 0 & 0 \\ 0 & 0 & p_{11} & 0 \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix}. \tag{3.4}$$

Proof of Theorem 3.1: Recall from Ghosh and Loucks [2, Proposition 6.3] that if the triplet $(\mathcal{C}, \mathcal{A}, [\mathcal{X}(0)])$ is minimal then the set of all triplets that produce the same output as that of (3.1) is given precisely by an action of a perspective group. Restricted to the triplet $(\mathcal{C}, \mathcal{A}, [\mathcal{X}(0)])$, where $\mathcal{C}$ is the 3×4 matrix in (3.1) and $\mathcal{A}$ has trace zero, the action of the perspective group is given precisely by (3.3).

(Q.E.D.)

Let us denote

$$\gamma = (p_{41}\; p_{42}\; p_{43}).$$

The scaling on the matrix $\mathcal{A}$ described by (3.3) is given by

$$\begin{aligned} A &\longmapsto A + \frac{b\gamma + \gamma b I}{p_{11}} \\ b &\longmapsto b\frac{p_{44}}{p_{11}} \\ f^T &\longmapsto \frac{p_{11}}{p_{44}} f^T - \frac{\gamma A}{p_{44}} - \frac{\gamma b \gamma}{p_{11}p_{44}} \\ \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} &\longmapsto \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \frac{p_{44}/p_{11}}{1 - \frac{p_{41}}{p_{11}}X - \frac{p_{42}}{p_{11}}Y - \frac{p_{43}}{p_{11}}Z}. \end{aligned} \tag{3.5}$$

Note that (3.5) is a 4-parameter family of orbit of the perspective group, parameterized by $p_{41}/p_{11}, p_{42}/p_{11}, p_{43}/p_{11}, p_{44}/p_{11}$.

Remark 3.2: It follows from Theorem 3.1, that in the 15-dimensional motion parameter space (A, b, f) of the homogeneous dynamical system (2.2), the set of parameters that can be identified generically by the homogeneous observation function (3.1) is given precisely by the 4-parameter orbit (3.5) (see Fig. 3).

Let us now consider a special case when A is a skew symmetric matrix, i.e. $A = \Omega$ where Ω is defined in (1.6). The homogeneous dynamical system is given by

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{bmatrix} = \begin{pmatrix} 0 & \omega_1 & \omega_2 & b_1 \\ -\omega_1 & 0 & \omega_3 & b_2 \\ -\omega_2 & -\omega_3 & 0 & b_3 \\ -f_1 & -f_2 & -f_3 & 0 \end{pmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{bmatrix}. \tag{3.6}$$

Assume as before that the state vector is observed by the homogeneous observation function (3.1). We have the following theorem.

Theorem 3.3: Consider the homogeneous dynamical system (3.6). For a generic choice of parameters $(\omega_1, \omega_2, \omega_3, b_1, b_2, b_3, f_1, f_2, f_3)$ and state vector $[X_1(0), Y_1(0), Z_1(0), W_1(0)]^T$, the set of all parameters that can be identified by the homogeneous observation function (3.1) is given by the following one-parameter orbit:

$$\begin{aligned} A &\longmapsto A \\ b &\longmapsto b\frac{p_{44}}{p_{11}} \\ f^T &\longmapsto \frac{p_{11}}{p_{44}} f^T \\ \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} &\longmapsto \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \frac{p_{44}}{p_{11}}. \end{aligned} \tag{3.7}$$

Proof of Theorem 3.3: The proof follows from the description of the orbit in (3.5). In particular if $b \neq 0$, it follows that for $A = -A^T$ to be satisfied, we must restrict $\gamma = 0$. Under this restriction, the orbit of the perspective group action is given precisely by (3.7).

(Q.E.D.)

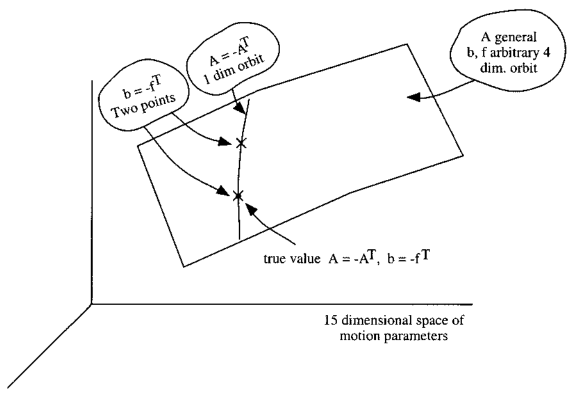

Fig. 3. Description of identifiable parameters by perspective projection (a) A is a general 3×3 matrix, b, f arbitrary: 4-dimensional orbit (b) $A = -A^T$, b, f arbitrary: 1 dimensional orbit (c) $A = -A^T$, $b = -f^T$: orbit is a pair of points.

Remark 3.4: Note in particular that the one parameter in (3.7) is precisely the depth parameter. Hence in this case, parameters are recoverable up to a one-parameter depth ambiguity (see Fig. 3).

The last special case considered in this section is to restrict the homogeneous dynamical system (3.6) even further by assuming $b_1 = f_1, b_2 = f_2, b_3 = f_3$. This way we obtain a special case of (3.5) and we have the following theorem.

Theorem 3.5: Consider the homogeneous dynamical system (3.6) with $b = -f$. For a generic choice of parameters $(\omega_1, \omega_2, \omega_3, b_1, b_2, b_3)$ and state vector $[X_1, Y_1, Z_1, W_1]^T$, the parameters $\omega_1, \omega_2, \omega_3$ can be identified exactly and the parameter vector (b_1, b_2, b_3, X, Y, Z) can be identified up to choice of a sign, by the homogeneous observation function (3.1) where $X = X_1/W_1, Y = Y_1/W_1$ and $Z = Z_1/W_1$.

Proof of Theorem 3.5: The proof follows from the description of the orbit in (3.7). In particular if $b = -f$, it follows that $-p_{11}^2 + p_{44}^2 = 0$, provided that $b \neq 0$. Thus we have $p_{11} = p_{44}$ or $p_{11} = -p_{44}$. Thus the orbit space (3.7) reduces to the following sign ambiguity:

$$\begin{aligned}(\omega_1, \omega_2, \omega_3) &\longmapsto (\omega_1, \omega_2, \omega_3)\\ (b_1, b_2, b_3) &\longmapsto \pm(b_1, b_2, b_3)\\ (X, Y, Z) &\longmapsto \pm(X, Y, Z).\end{aligned}$$

(Q.E.D.)

Remark 3.6: Note in particular that it is possible to resolve the depth ambiguity to a sign ambiguity because of the special structure of the matrix in (3.6) with $b = -f$ (see Fig. 3).

IV. Identification Under Orthographic Projection

In order to analyze the problem of parameter identification under orthographic projection, we consider the homogeneous dynamical system (2.2) with the following homogeneous observation function:

$$\begin{bmatrix} y_1\\ y_2\\ y_3\end{bmatrix} = \begin{pmatrix}\lambda & 0 & 0 & 0\\ 0 & \lambda & 0 & 0\\ 0 & 0 & 0 & \lambda\end{pmatrix}\begin{bmatrix}X_1\\ Y_1\\ Z_1\\ W_1\end{bmatrix} \tag{4.1}$$

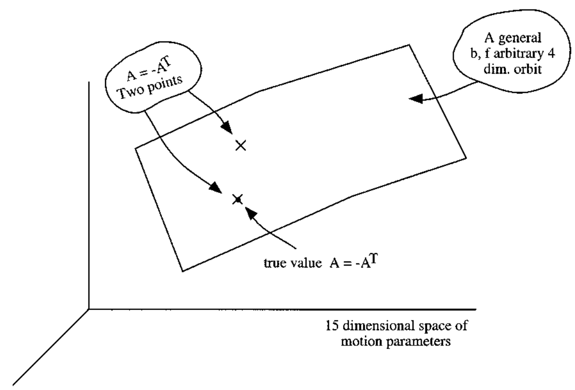

Fig. 4. Description of identifiable parameters by orthographic projection (a) A is a general 3×3 matrix, b, f arbitrary: 4-dimensional orbit (b) $A = -A^T$, b, f arbitrary: orbit is a pair of points.

which describes the associated orthographic projection (1.8). Note in particular that, as in (3.1), λ is an arbitrary nonzero parameter. As in Theorem 3.1, we shall abbreviate the dynamical system (2.2), (4.1) via (3.2) and consider the following theorem.

Theorem 4.1: Consider the homogeneous dynamical system (2.2), (4.1) abbreviated as (3.2). For a generic choice of the matrix $\mathcal{A}$ and state vector $[X_1(0)\ Y_1(0)\ Z_1(0)\ W_1(0)]^T$, the set of all triplets that produce the same output as given by (4.1) is described by (3.3) where λ_1 is a nonzero real number and P is a nonsingular matrix of the form

$$P^{-1} = \begin{pmatrix} p_{11} & 0 & 0 & 0\\ 0 & p_{11} & 0 & 0\\ p_{31} & p_{32} & p_{33} & p_{34}\\ 0 & 0 & 0 & p_{11}\end{pmatrix}. \tag{4.2}$$

Proof of Theorem 4.1: The proof is analogous to the proof of Theorem 3.1 and is omitted. (Q.E.D.)

As in Section III, if we now restrict to the special case when $A = \Omega$, b, f are arbitrary we obtain the following result.

Theorem 4.2: Consider the homogeneous dynamical system (3.6), (4.1). For a generic choice of parameters $(\omega_1, \omega_2, \omega_3, b_1, b_2, b_3, f_1, f_2, f_3)$ and state vector $[X_1(0), Y_1(0), Z_1(0), W_1(0)]^T$, the set of all parameters that can be identified by the homogeneous observation function (4.1) is given by the following:

$$\begin{matrix}\omega_1 \longmapsto \omega_1, & \omega_2 \longmapsto \pm\omega_2, & \omega_3 \longmapsto \pm\omega_3\\ b_1 \longmapsto b_1, & b_2 \longmapsto b_2, & b_3 \longmapsto \pm b_3\\ f_1 \longmapsto f_1, & f_2 \longmapsto f_2, & f_3 \longmapsto \pm f_3\end{matrix} \tag{4.3}$$

$$\begin{pmatrix}X\\ Y\\ Z\end{pmatrix} \longmapsto \begin{pmatrix}X\\ Y\\ \pm Z\end{pmatrix}.$$

Remark 4.3: Thus when the matrix A is skew symmetric, the 4-parameter orbit space reduces to a pair of points with only a sign ambiguity. In particular, $\omega_1, b_1, b_2, f_1, f_2, X$ and Y can be identified precisely, whereas there is a sign ambiguity on the vector $(\omega_2, \omega_3, b_3, f_3, Z)$ (see Fig. 4).

Proof of Theorem 4.2: Let us restrict the orbit described by (3.3) where P is defined by (4.2) to the case when $A = -A^T$. It would follow that $a_{11} = a_{22} = a_{33} = 0$. This restriction would imply that $p_{31} = p_{32} = p_{34} = 0$. From the restriction $a_{32} = -a_{23}$ it would follow that $p_{11}/p_{33} = \pm 1$ unless $a_{32} = 0$. Likewise from the restriction $a_{31} = -a_{13}$, it would follow that $p_{11}/p_{33} = \pm 1$ unless $a_{31} = 0$. Thus if $(a_{32}, a_{31}) \neq 0$, then $p_{11}/p_{33} = \pm 1$. Hence (4.2) would take the form

$$P^{-1} = \begin{pmatrix} p_{11} & 0 & 0 & 0 \\ 0 & p_{11} & 0 & 0 \\ 0 & 0 & \pm p_{11} & 0 \\ 0 & 0 & 0 & p_{11} \end{pmatrix}. \tag{4.4}$$

The orbit of (4.4) is given precisely by (4.3). (Q.E.D.)

V. Identification Using Optical Flow

In Sections III and IV, we have shown that if we have a point $(X, Y, Z)^T$ in $\mathbb{R}^3$ undergoing a Riccati dynamics of the form (1.1), then the parameters of the dynamics can be identified generically up to a 4-parameter orbit in a 15-dimensional motion parameter space, provided that the point $(X, Y, Z)^T$ is observed via perspective or orthographic observation function. Furthermore, if the matrix A is assumed to be skew symmetric, then the orbit space reduces to a 1-parameter orbit under perspective observation and a pair of two points under orthographic observation. Finally if $A = -A^T$ and $b_1 = f_1$, $b_2 = f_2$, $b_3 = f_3$, then the orbit space is also a pair of two points under perspective observation. The purpose of this section is to show that the structure of the orbit space does not change even when the projection $(x, y)^T$ of the state variables are not observed directly but only the coefficients of the optical flow (1.5) is assumed to be observed. The main conclusion of this section is that if one assumes the observation functions to be (2.9) and (2.10), for respectively perspective and orthographic projections, instead of (3.1) and (4.1), the structure of the orbit space remains unchanged.

Let us consider the homogeneous dynamics given by (2.4) which describes the motion of a plane (2.5) parameterized by the homogeneous vector $[p\ q\ s\ w]^T$. As described in Section II, we assume that the vector $[p\ q\ s\ w]^T$ is observed via the homogeneous transformation (2.9). This would be the case, if points on the plane (2.5) is projected perspectively using (1.3). Thus we have a homogeneous dynamical system (2.4), (2.9) which we are going to abbreviate as

$$\frac{d}{dt}[\eta(t)] = \mathcal{A}_1[\eta(t)], \qquad [\mathcal{Y}_1(t)] = \mathcal{C}_1[\eta(t)]. \tag{5.1}$$

We have the following theorem.

Theorem 5.1: Consider the homogeneous dynamical system (2.4), (2.9) abbreviated as (5.1). For a generic choice of the matrix $\mathcal{A}_1, \mathcal{C}_1$ and the state vector $[\eta(0)] = [p(0), q(0), s(0), w(0)]^T$, the set of all triplets that produce the same output as given by (2.9) is described as

$$(\lambda_1 \mathcal{C}_1 Q^{-1}, Q\mathcal{A}_1 Q^{-1}, [Q\eta(0)]) \tag{5.2}$$

where λ_1 is a nonzero real number and Q is a nonsingular 4×4 matrix of the form

$$Q^{-1} = \begin{pmatrix} q_{11} & 0 & 0 & q_{14} \\ 0 & q_{11} & 0 & q_{24} \\ 0 & 0 & q_{11} & q_{34} \\ 0 & 0 & 0 & q_{44} \end{pmatrix}. \tag{5.3}$$

Remark 5.2: By comparing (3.3) and (5.2), it follows that $Q^T = P^{-1}$ and is therefore given by (3.4). The scaling on the vector $[p\ q\ s\ \omega]$ is therefore given as follows:

$$[p\ q\ s\ w] \longmapsto [p\ q\ s\ w] \begin{pmatrix} p_{11} & 0 & 0 & 0 \\ 0 & p_{11} & 0 & 0 \\ 0 & 0 & p_{11} & 0 \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix}. \tag{5.4}$$

The scaling (5.4) can be equivalently written as follows:

$$\bar{p} \longmapsto \frac{p_{11}}{p_{44}}\bar{p} + \frac{p_{41}}{p_{44}}$$
$$\bar{q} \longmapsto \frac{p_{11}}{p_{44}}\bar{q} + \frac{p_{42}}{p_{44}}$$
$$\bar{s} \longmapsto \frac{p_{11}}{p_{44}}\bar{s} + \frac{p_{43}}{p_{44}}.$$

Thus the plane (1.4) is scaled as follows:

$$\left(\bar{p} + \frac{p_{41}}{p_{11}}\right) X + \left(\bar{q} + \frac{p_{42}}{p_{11}}\right) Y + \left(\bar{s} + \frac{p_{43}}{p_{11}}\right) Z + \frac{p_{44}}{p_{11}} = 0. \tag{5.5}$$

The 4-parameter ambiguity is essentially a depth and orientation ambiguity on the plane. Thus for a Riccati motion (1.1) and under perspective projection, both "depth" and "orientation" of the plane are ambiguous.

Proof of Theorem 5.1: It follows from Ghosh and Loucks ([2, Proposition 6.3]), that if the triplet $(\mathcal{C}_1, \mathcal{A}_1, [\eta(0)])$ is minimal, then the set of all triplets that produce the same output as that of (2.9) is given precisely by the following:

$$(\mathcal{C}_1, \mathcal{A}_1, [\eta(0)]) \longmapsto (\lambda_1 \mathcal{C}_1 Q^{-1}, \lambda I + Q\mathcal{A}_1 Q^{-1}, [Q\eta(0)])$$

where *a priori*, Q is any general element of $GL(4)$. Since trace of $\mathcal{A}_1$ is assumed to be zero, it follows that $\lambda = 0$. Denoting Q^{-1} to be of the form

$$Q^{-1} = \begin{pmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \\ q_{31} & q_{32} & q_{33} & q_{34} \\ q_{41} & q_{42} & q_{43} & q_{44} \end{pmatrix}$$

it follows in order for $\mathcal{C}_1 Q^{-1}$ to have the same structure as that of $\mathcal{C}_1$ we have

$$\begin{pmatrix} -b_1 & a_{13} \\ -b_2 & a_{23} \end{pmatrix} \begin{pmatrix} q_{31} & q_{32} \\ q_{41} & q_{42} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} -b_1 & a_{12} \\ b_3 & -a_{32} \end{pmatrix} \begin{pmatrix} q_{21} & q_{23} \\ q_{41} & q_{43} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} -b_2 & a_{21} \\ b_3 & -a_{31} \end{pmatrix} \begin{pmatrix} q_{12} & q_{13} \\ q_{42} & q_{43} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus under the generic condition

$$b_1 a_{23} \neq b_2 a_{13}, \qquad b_1 a_{32} \neq b_3 a_{12}, \qquad b_2 a_{31} \neq b_3 a_{21}$$

Q^{-1} is of the form

$$\begin{pmatrix} q_{11} & 0 & 0 & q_{14} \\ 0 & q_{22} & 0 & q_{24} \\ 0 & 0 & q_{33} & q_{34} \\ 0 & 0 & 0 & q_{44} \end{pmatrix}.$$

Finally if $(b_1\ b_2\ b_3) \neq 0$ we have $q_{11} = q_{22} = q_{33}$. Thus Q^{-1} is of the form as defined in (5.3). (Q.E.D.)

Let us consider the case when the homogeneous dynamics (2.4) is observed via the homogeneous transformation (2.10). This would be the case, if points on the plane (2.5) are projected orthographically using (1.8). We have a theorem analogous to Theorem 5.1 where Q is of the form

$$Q^{-1} = \begin{pmatrix} q_{11} & 0 & q_{13} & 0 \\ 0 & q_{11} & q_{23} & 0 \\ 0 & 0 & q_{33} & 0 \\ 0 & 0 & q_{43} & q_{11} \end{pmatrix} \tag{5.6}$$

under the generic condition

$$b_1 a_{23} \neq b_2 a_{13}, \qquad f_3 a_{12} \neq f_2 a_{13}, \qquad f_1 a_{23} \neq f_3 a_{21}.$$

The scaling on the vector $[p\ q\ s\ w]$ would be given analogously as

$$[p\ q\ s\ w] \longmapsto [p\ q\ s\ w] P^{-1}$$

where P^{-1} is defined in (4.2). The plane (1.4) is scaled in this case as follows:

$$\left(\overline{p} + \frac{p_{31}}{p_{11}}\,\overline{s}\right) X + \left(\overline{q} + \frac{p_{32}}{p_{11}}\,\overline{s}\right) Y + \left(\frac{p_{33}}{p_{11}}\,\overline{s}\right) Z + \left(1 + \frac{p_{34}}{p_{11}}\,\overline{s}\right) = 0. \tag{5.7}$$

Remark 5.3: The main idea behind this section is that if instead of observing projections of individual feature points on the moving plane, one observes parameters of the associated optical flow on the image plane, the extent to which motion and shape parameters that are recovered is the same. This picture is true for both orthographic and perspective projections.

Remark 5.4: For the special case when $A^T = -A$ and when $A^T = -A$, $b = -f$ one obtains theorems analogous to Theorems 3.3 and 3.5 under perspective projection and Theorem 4.2 under orthographic projection when the parameters of the associated optical flow are observed instead of projections of individual feature points.

VI. Identification Using Linearized Optical Flow

For a general class of "Riccati motion model" given by (1.1), we have seen that the associated optical flow is described by (1.5) for either perspective or orthographic observation functions. In homogeneous coordinates, the parameter vector $(d_1, \cdots, d_8)$ is described by (2.9) and (2.10) for perspective and orthographic observation functions, respectively. In Section V we have shown that as far as estimating the motion and shape parameters is concerned, it is immaterial whether or not one observes the projection of the state vector (X, Y, Z) via (1.3) or (1.8) as opposed to observing the parameter vector of the optical flow via (2.9) or (2.10), respectively.

In this section, we show that it is not necessary to observe all the eight parameters of the optical flow. As described in (2.11), if the optical flow equation is linearized at a known point (x^*, y^*) on the image plane, and if we observe all six parameters of the affine approximation of the optical flow equation, one obtains, in homogeneous coordinates, the observation equations given by (2.13) and (2.14), respectively, for perspective and orthographic projections, respectively.

Thus under perspective projection, we have the homogeneous dynamical system given by (2.4), (2.13) and for orthographic projection, we have the homogeneous dynamical system given by (2.4), (2.14). We now state and prove the following theorem.

Theorem 6.1: Consider the homogeneous dynamical system (2.4), (2.13) or (2.4), (2.14) for, respectively, perspective and orthographic projection models, abbreviated as (5.1). For a generic choice of matrix $\mathcal{A}_1, \mathcal{C}_1$ and state vector $[\eta(0)] = [p(0), q(0), s(0), w(0)]^T$, the set of all triplets that produce the same output as given by (2.13) or (2.14) respectively is described as (5.2) where Q^{-1} is defined by (5.3) or (5.6) respectively under perspective and orthographic projection models.

Proof of Theorem 6.1: This theorem is analogous to the proof of Theorem 5.1. The main idea is that in order for $\mathcal{C}_1 Q^{-1}$ to have the same structure as that of $\mathcal{C}_1$, Q^{-1} must be of the form (5.3) or (5.6). It is easy to see that generically this is indeed the case. In fact if

$$b_1 a_{23} \neq b_2 a_{13}, \qquad b_1 \neq x^* b_3, \qquad b_2 \neq y^* b_3$$

it will follow that the structure of Q^{-1} is of the form (5.3). Likewise, if

$$b_1 a_{23} \neq b_2 a_{13}, \qquad a_{13} \neq -x^* f_3, \qquad a_{23} \neq y^* f_3$$

then it follows that the structure of Q^{-1} is of the form (5.6). (Q.E.D.)

The main claim of this section is that the affine part of the optical flow is sufficient for parameter identification, *at least for planes in* $\mathbb{R}^3$ *undergoing Riccati motion*. No new information is present in the quadratic component.

VII. Identification Using a Pair of Cameras

So far in this paper, we have considered only one camera. Our main conclusion has been to show that under *cyclopean vision*, motion and shape parameters, for a large class of problems, can be identified generically up to a choice of four parameters. Of course, under additional restrictions, parameter ambiguity can be reduced at best to a choice of two alternatives. In this section, we would like to study to what extent parameters can be identified using a pair of cameras operating in parallel and possibly asynchronously. Our basic strategy is to compute the ambiguity surfaces for each camera and intersect the two surfaces and show that generically the intersection is always a unique point. This approach of ours should be compared with the ones existing in the literature (see for example [21]–[25]). The two-camera problem is described as follows.

Problem 7.1: Assume that the plane (1.4) in $\mathbb{R}^3$ undergoes a Riccati motion given by (1.7). Assume furthermore that we have a pair of CCD cameras each separately observing points on the plane and each separately reconstructing coefficients of the "linearized optical flow" equation on its corresponding image plane. Finally, assume that the relative position and orientation of the two cameras are known. The problem is to estimate the associated motion and shape parameters of the moving plane given by (1.7).

The main result of this section is to show that under the assumptions of Problem 7.1, all 18 motion and shape parameters are uniquely identifiable. What is probably surprising about this result is that each of the two cameras do not have to observe the same set of points on the plane. Moreover, the cameras do not even have to observe simultaneously and may have different sampling instances. Thus the two cameras may operate in parallel and possibly asynchronously. The two cameras may in fact be a single camera that has moved its position over time.

In order to describe the main result of this section, we shall need the following notation. Let $(X\ Y\ Z)^T$ be the coordinates of a point on the plane (1.4), with respect to a coordinate axis attached to camera 1. Homogenizing the vector $(X,\, Y,\, Z)$ we get the homogeneous coordinates $[X_1,\, Y_1,\, Z_1,\, W_1]^T$. Likewise let $[X_2\ Y_2\ Z_2\ W_2]^T$ be the homogeneous coordinates of the same point with respect to coordinates on camera 2. We assume the following transformation between the two coordinates:

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \\ W_2 \end{bmatrix} = \begin{pmatrix} R & \theta \\ 0 & 1 \end{pmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{bmatrix} \tag{7.1}$$

where $R \in SO(3)$ and θ is an arbitrary 3×1 vector. It is easy to see that under the transformation (7.1), the parameters $\overline{p},\, \overline{q},\, \overline{s}$ of the plane (1.4) are transformed as follows:

$$\begin{bmatrix} p_2 \\ q_2 \\ s_2 \\ w_2 \end{bmatrix} = \begin{pmatrix} R & 0 \\ -\theta^T R & 1 \end{pmatrix} \begin{bmatrix} p_1 \\ q_1 \\ s_1 \\ w_1 \end{bmatrix} \tag{7.2}$$

where $[p_i,\, q_i,\, s_i,\, w_i]^T$ $i = 1,\, 2$ are homogeneous coordinates of the plane defined via (2.3). We assume that R and θ are known and fixed and we denote the 4×4 matrix in (7.2) by $\mathcal{R}$. We assume that points on the plane (1.4) are observed via perspective projection either via coordinates (1.3) or via coefficients of the optical flow (2.9) or via coefficients of the linearized optical flow (2.13). From (5.4) we know the scalings on the parameters of the plane that cannot be identified by each of the two cameras observing separately. This is given by

$$\begin{bmatrix} p_i \\ q_i \\ s_i \\ w_i \end{bmatrix} \longmapsto \begin{pmatrix} p_{11}^{(i)} & 0 & 0 & p_{14}^{(i)} \\ 0 & p_{11}^{(i)} & 0 & p_{24}^{(i)} \\ 0 & 0 & p_{11}^{(i)} & p_{34}^{(i)} \\ 0 & 0 & 0 & p_{44}^{(i)} \end{pmatrix} \begin{bmatrix} p_i \\ q_i \\ s_i \\ w_i \end{bmatrix} \tag{7.3}$$

where the index $i = 1,\, 2$ refers to each of the two cameras. For a generic choice of A, b and f in (2.4) we have

$$\mathcal{R}\pi_1 = \pi_2 \mathcal{R} \tag{7.4}$$

where π_i is the 4×4 nonsingular matrix in (7.3). For a generic value of R and θ, i.e. $\mathcal{R}$, in order for (7.4) to be satisfied it follows that $\pi_1 = \pi_2 = I$.

Thus we have the following main result of this section and an interesting fact about machine vision with two cameras described as follows.

Theorem 7.2: Consider a homogeneous dynamical system (2.4) with an observation function (2.13) given by camera 1 and an analogous observation function given by camera 2. Assume that the relative calibration parameters between cameras 1 and 2 are given by R and θ defining a transformation (7.1) where we assume that R and θ are known. It follows that generically the motion and shape parameters are unique.

Note in particular that Theorem 7.2 is not a theorem about stereo vision because the two cameras compute entirely in parallel and possibly even asynchronously.

Under orthographic projection, the transformation (7.3) is of the form

$$\begin{bmatrix} p_i \\ q_i \\ s_i \\ w_i \end{bmatrix} \longmapsto \begin{pmatrix} p_{11}^{(i)} & 0 & p_{13}^{(i)} & 0 \\ 0 & p_{11}^{(i)} & p_{23}^{(i)} & 0 \\ 0 & 0 & p_{33}^{(i)} & 0 \\ 0 & 0 & p_{43}^{(i)} & p_{11}^{(i)} \end{pmatrix} \begin{bmatrix} p_i \\ q_i \\ s_i \\ w_i \end{bmatrix} \tag{7.5}$$

where the index $i = 1,\, 2$ refers to each of the two cameras. As in (7.4) we have

$$\mathcal{R}\pi_1 = \pi_2 \mathcal{R}$$

where π_i's are the 4×4 nonsingular matrices in (7.5). Since π_1 and π_2 have the same eigenvalues, it follows that

$$p_{11}^{(1)} = p_{11}^{(2)}, \qquad p_{33}^{(1)} = p_{33}^{(2)}.$$

Additionally we have

$$R \begin{pmatrix} p_{11}^{(1)} & 0 & p_{13}^{(1)} \\ 0 & p_{11}^{(1)} & p_{23}^{(1)} \\ 0 & 0 & p_{33}^{(1)} \end{pmatrix} = \begin{pmatrix} p_{11}^{(1)} & 0 & p_{13}^{(2)} \\ 0 & p_{11}^{(1)} & p_{23}^{(2)} \\ 0 & 0 & p_{33}^{(1)} \end{pmatrix} R. \tag{7.6}$$

Since $p_{11}^{(1)}$ is a double eigenvalue for each of the two matrices in (7.6) with eigenvector given by $(\alpha,\, \beta,\, 0)^T$, it follows that

$$R \begin{pmatrix} \alpha_1 \\ \beta_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ 0 \end{pmatrix} \tag{7.7}$$

for suitable choice of $\alpha_1,\, \beta_1,\, \alpha_2,\, \beta_2$ provided $p_{33}^{(1)} \neq p_{11}^{(1)}$. For a generic R, (7.7) is not satisfied. Hence we must have $p_{33}^{(1)} = p_{11}^{(1)}$. We therefore have a constraint of the form

$$\begin{pmatrix} R & 0 \\ \theta & 1 \end{pmatrix} \begin{pmatrix} p_{11} & 0 & p_{13} & 0 \\ 0 & p_{11} & p_{23} & 0 \\ 0 & 0 & p_{11} & 0 \\ 0 & 0 & p_{43} & p_{11} \end{pmatrix} = \begin{pmatrix} p_{11} & 0 & p_{13}^{(1)} & 0 \\ 0 & p_{11} & p_{23}^{(1)} & 0 \\ 0 & 0 & p_{11} & 0 \\ 0 & 0 & p_{43}^{(1)} & p_{11} \end{pmatrix} \begin{pmatrix} R & 0 \\ \theta & 1 \end{pmatrix}.$$

It is easy to see that for a generic R and θ, we have

$$p_{13} = p_{23} = p_{43} = 0.$$

Thus we have a theorem equivalent to Theorem 7.2 for the orthographic projection as well, described as follows.

Theorem 7.3: Consider a homogeneous dynamical system (2.4) with an observation function (2.14) given by camera 1 and an analogous observation function given by camera 2. Assume that the relative calibration parameters between cameras 1 and 2 are given by R and θ defining a transformation (7.1) where we assume that R and θ are known. It follows that generically the motion and shape parameters are unique.

VIII. Example

In this section we consider a plane initially at

$$2X + 3Y + 5Z + 1 = 0 \tag{8.1}$$

and undergoing a dynamics of the form (1.7) where $b_1 = 3, b_2 = 1, b_3 = 5, f_1 = b_1, f_2 = b_2, f_3 = b_3, \omega_1 = 5, \omega_2 = 7$, and $\omega_3 = 9$ and the initial conditions are given by $\overline{p}(0) = 2$, $\overline{q}(0) = 3$ and $\overline{s}(0) = 5$.

Example 8.1 (Perspective Projection): Assume that points on the moving plane described by (8.1), (1.7) are observed perspectively using observation function (1.3). We define

$$\pi_i = p_{4i}/p_{11}, \quad i = 1, 2, 3, 4.$$

Using Theorem 3.1, it follows that the motion and shape parameters that can be identified are given by

$$A = \begin{pmatrix} 6\pi_1+\pi_2+5\pi_3 & 5+3\pi_2 & 7+3\pi_3 \\ -5+\pi_1 & 3\pi_1+2\pi_2+5\pi_3 & 9+\pi_3 \\ -7+5\pi_1 & -9+5\pi_2 & 3\pi_1+\pi_2+10\pi_3 \end{pmatrix} \tag{8.2}$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 3\pi_4 \\ \pi_4 \\ 5\pi_4 \end{pmatrix} \tag{8.3}$$

$$\begin{aligned} f_1 &= (3 - 5\pi_2 - 7\pi_3 + 3\pi_1^2 + \pi_1\pi_2 + 5\pi_1\pi_3)/\pi_4 \\ f_2 &= (1 + 5\pi_1 - 9\pi_3 + 3\pi_1\pi_2 + \pi_2^2 + 5\pi_2\pi_3)/\pi_4 \\ f_3 &= (5 + 7\pi_1 + 9\pi_2 + 3\pi_1\pi_3 + \pi_2\pi_3 + 5\pi_3^2)/\pi_4. \end{aligned} \tag{8.4}$$

Equations (8.2)–(8.4) have been derived from (3.5). Likewise from (5.5) it follows that the scaling on (8.1) is given by

$$\left(\frac{2+\pi_1}{\pi_4}\right) X + \left(\frac{3+\pi_2}{\pi_4}\right) Y + \left(\frac{5+\pi_3}{\pi_4}\right) Z + 1 = 0.$$

Thus, every dynamical system of the form (1.7) with motion parameters given by (8.2)–(8.4) and with initial condition

$$\overline{p}(0) = \frac{2+\pi_1}{\pi_4}, \qquad \overline{q}(0) = \frac{3+\pi_2}{\pi_4}, \qquad \overline{s}(0) = \frac{5+\pi_3}{\pi_4}$$

cannot be distinguished from (8.1), (1.7) via perspective projection (1.3). Qualitatively, we have a 4-parameter ambiguity which includes ambiguity on depth and orientation of the plane.

As an illustration of this ambiguity, we consider the following two distinct planes with equations given by (8.1) and

$$10X + 15Y + 5Z + 1 = 0. \tag{8.5}$$

Each of the two planes has a corresponding motion dynamics such that the associated optical flow for the two planes are identical. This serves to illustrate the point that there is not only depth but also an orientation ambiguity in these problems. The dynamics of plane (8.1) is chosen as

$$\frac{d}{dt}\begin{bmatrix} p \\ q \\ s \\ w \end{bmatrix} = \begin{pmatrix} 0 & 5 & 7 & 3 \\ -5 & 0 & 9 & 1 \\ -7 & -9 & 0 & 5 \\ -3 & -1 & -5 & 0 \end{pmatrix} \begin{bmatrix} p \\ q \\ s \\ w \end{bmatrix}$$

and the plane (8.5) is chosen as

$$\frac{d}{dt}\begin{bmatrix} p \\ q \\ s \\ w \end{bmatrix} = \begin{pmatrix} -24 & -3 & -33 & 231 \\ -41 & -12 & -51 & 473 \\ -7 & -9 & 0 & 169 \\ -3 & -1 & -5 & 36 \end{pmatrix} \begin{bmatrix} p \\ q \\ s \\ w \end{bmatrix}.$$

Remark 8.2: Interestingly, if we know *a priori* that the initial position of the plane is at (8.1), then the ambiguity on motion can be constrained to be

$$\pi_1 = 2\pi_4 - 2, \qquad \pi_2 = 3\pi_4 - 3, \qquad \pi_3 = 5\pi_4 - 5. \tag{8.6}$$

This will lead to a *1-parameter ambiguity on motion assuming known shape*. To illustrate our point, we consider two planes with identical initial position given by (8.1), with different dynamics given by

$$\frac{d}{dt}\begin{bmatrix} p \\ q \\ s \\ w \end{bmatrix} = \begin{pmatrix} 0 & 5 & 7 & 3 \\ -5 & 0 & 9 & 1 \\ -7 & -9 & 0 & 5 \\ -3 & -1 & -5 & 0 \end{pmatrix} \begin{bmatrix} p \\ q \\ s \\ w \end{bmatrix}$$

and

$$\frac{d}{dt}\begin{bmatrix} p \\ q \\ s \\ w \end{bmatrix} = \begin{pmatrix} -6 & 3 & -3 & 21/2 \\ -14 & -3 & -6 & 34 \\ -22 & -14 & -25 & 108 \\ -6 & -2 & -10 & 34 \end{pmatrix} \begin{bmatrix} p \\ q \\ s \\ w \end{bmatrix}$$

so that their subsequent positions are not identical. The optical flow generated by each of the planes is nevertheless the same. This example illustrates the fact that there is a motion ambiguity even though the initial position and orientation of the plane is identical. It is not too difficult to justify using (3.5) and (8.6) that the motion ambiguity is in general restricted to a single parameter. In fact if we choose

$$\overline{p} = 2, \qquad \overline{q} = 3, \qquad \overline{s} = 5$$

and assume that

$$\begin{aligned} p_{41} &= 2(\lambda - 1)p_{11}, \quad p_{42} = 3(\lambda - 1)p_{11} \\ p_{43} &= 5(\lambda - 1)p_{11}, \quad p_{44} = \lambda p_{11} \end{aligned}$$

we have the required one-parameter motion ambiguity parameterized by λ, whereas the initial depth and orientation on the plane is held fixed.

If we now constrain the matrix A in (8.2) to be skew symmetric and choose

$$\pi_1 = \pi_2 = \pi_3 = 0$$

we obtain

$$A = \begin{pmatrix} 0 & 5 & 7 \\ -5 & 0 & 9 \\ -7 & -9 & 0 \end{pmatrix}$$

a fixed matrix and

$$\begin{aligned} b_1 &= 3\pi_4, \quad b_2 = \pi_4, \quad b_3 = 5\pi_4 \\ f_1 &= 3/\pi_4, \quad f_2 = 1/\pi_4, \quad f_3 = 5/\pi_4. \end{aligned}$$

The scaling on (8.1) is given by

$$2X + 3Y + 5Z + \pi_4 = 0.$$

The scale π_4 contributes to the depth ambiguity in this case. Note that the motion parameters in A is completely determined.

Finally if we constrain $b_i = f_i$, $i = 1, 2, 3$, we have $\pi_4 = \pm 1$. Thus in this case the depth ambiguity is resolved and we have a sign ambiguity.

Example 8.3 (Orthographic Projection): Assume that points on the moving plane described by (8.1), (1.7) are observed orthographically using observation function (1.8). We define

$$\Delta_i = p_{3i}/p_{11}, \qquad i = 1, 2, 3, 4. \tag{8.7}$$

The motion and shape parameters that can be identified is described by four parameters $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ using (8.7). From (5.7), it follows that the scaling on (8.1) is given by

$$\frac{2+5\Delta_1}{1+5\Delta_4}X + \frac{3+5\Delta_2}{1+5\Delta_4}Y + \frac{5\Delta_3}{1+5\Delta_4}Z + 1 = 0. \tag{8.8}$$

Thus, as in the Example 8.1, we have a 4-parameter ambiguity which includes ambiguity on both depth and orientation of the plane.

If we constrain the matrix A in (8.2) to be skew symmetric, we obtain $\Delta_1 = \Delta_2 = \Delta_4 = 0$, $\Delta_3 = \pm 1$ and (8.8) reduces to

$$2X + 3Y \pm 5Z + 1 = 0.$$

Using Theorem 4.2, we conclude that the dynamical system (1.7) can be identified up to sign given by

$$\frac{d}{dt}\begin{pmatrix}\overline{p}\\ \overline{q}\\ \overline{s}\end{pmatrix} = \begin{pmatrix}3\\ 1\\ \pm 5\end{pmatrix} + \begin{pmatrix}0 & 5 & \pm 7\\ -5 & 0 & \pm 9\\ \mp 7 & \mp 9 & 0\end{pmatrix}\begin{pmatrix}\overline{p}\\ \overline{q}\\ \overline{s}\end{pmatrix} + \begin{pmatrix}3 & 1 & \pm 5 & 0 & 0 & 0\\ 0 & 3 & 0 & 1 & \pm 5 & 0\\ 0 & 0 & 3 & 0 & 1 & \pm 5\end{pmatrix}\begin{pmatrix}\overline{p}^2\\ \overline{pq}\\ \overline{ps}\\ \overline{q}^2\\ \overline{qs}\\ \overline{s}^2\end{pmatrix}.$$

Thus under the assumption of skew symmetry on the motion parameter matrix A, the 4-parameter ambiguity reduces to a sign ambiguity.

IX. Conclusion

Riccati dynamics forms an interesting class of motion dynamics in $\mathbb{R}^3$ which preserves the shape (planar shape) of surfaces in motion. However, it does not preserve distance between two points. Hence a polyhedral object would be deformed nonrigidly under a Riccati flow while maintaining the polyhedral shape.

This paper unifies motion and shape estimation under Riccati dynamics from many different points of view. First of all, using one camera we show that for a general class of Riccati dynamics, perspective and orthographic projections both give rise to a quadratic optical flow. Specifically, orthographic projection is not any simpler than perspective projection and there is a 4-parameter ambiguity in either of the two problems. The four parameters, in particular, include arbitrary initial position and orientation of the plane. When $A = -A^T$, orthographic projection has an advantage over perspective projection because in the former there is only a sign ambiguity, whereas in the latter there is a depth ambiguity. A suitable class of nonlinear dynamics has been presented in this paper where there is no depth ambiguity even under perspective projection.

An important result of this paper is to show that the extent to which motion and shape parameters can be identified is the same if projection of individual feature points are observed as opposed to observing parameters of the associated optical flow.

Two of the most important results of this paper is that 1) it suffices to observe only the linearization of the optical flow dynamics and 2) two eyes (cameras) provide unique parameter estimates of motion and shape even when they are not used in stereo mode.

Thus parameters from linearized optical flow from two cameras observing a Riccati dynamics of a plane in $\mathbb{R}^3$ is sufficient to uniquely identify the motion and shape parameters under both perspective and orthographic projections, even though the cameras may be observing asynchronously and may not be observing the same set of points on the moving plane.

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