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A necessary and sufficient condition for the perspective observability problem

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Abstract

In this paper we derive a necessary and sufficient condition for observing the initial condition of a new class of system known as the perspective system. Such a system has already been applied in the field of computer vision especially in the area of motion estimation problems. Our result generalizes an earlier result by Popov–Belevitch–Hautus on the problem of observing a linear dynamical system.

Keywords: Nonlinear systems; Perspective observability

1. Introduction

In this paper we obtain a complete solution to a problem introduced earlier by Ghosh et al. [1]. The main problem, which we shall call the perspective observability problem can be described as follows.

Perspective Observability Problem. Assume that $\mathscr{F}$ is the field of either real or complex numbers. Let A be a $n \times n$ matrix and C be a $m \times n$ matrix where we assume that $2 \leqslant m \leqslant n$. Consider the dynamical system

$$\dot{x} = Ax, \qquad x(0) = x_0 \in \mathscr{F}^n \tag{1.1}$$

together with the observation function

$$Z : \mathscr{F}^n - B \to \mathscr{F}\mathscr{P}^{m-1} \tag{1.2}$$

given by

$$x \mapsto [Cx],$$

where $B = \{x \in \mathscr{F}^n : Cx = 0\}$ and where $[Cx]$ is the vector of the homogeneous coordinates of Cx as an element of $\mathscr{F}\mathscr{P}^{m-1}$, the $m-1$ dimensional projective space of all homogeneous lines in $\mathscr{F}^m$. Assume that $[Cx(t)]$ is given for all $t \geqslant 0$. The perspective observation problem is to derive condition on A, C such that the initial condition $[x_0]$ is observable from this data.

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Remark 1.1. For the purpose of application, we are interested only in the case when $\mathscr{F}$ is $\mathscr{R}$, the field of real numbers. However over the field $\mathscr{C}$ of complex numbers, we obtain a necessary and sufficient condition to the observability problem. Over $\mathscr{R}$ this condition is shown to be sufficient.

The above problem has been originally posed in [1] wherein the basic motivation for considering observability problems was derived from problems in Computer Vision. This is now described briefly as follows. One considers a rigid body which is rotating and translating with known constant velocities. The moving object in turn produces a time varying image on the screen through a system of lenses, possibly a camera. An important problem in computer vision is to observe the initial condition of the object from an observed sequence of images.

Typically the observability question is resolved not from the projection of the entire object but possibly from the projections of various features on the object. In practice however the projection of these features have to be identified on the image and their motions have to be traced. Typically one considers features given by either a point, a straight line or a certain polynomial curve of a given fixed degree. These features are parameterized as points on a suitably defined feature space that has the structure of a manifold. The motion of the rigid body prescribes a dynamical system in the feature space. From the projections of these features on the screen, one constructs an observation function defined on the manifold. In this way we obtain an observed dynamical system. A problem of interest is to suitably observe the initial condition of the dynamical system provided that the output function is given for all $t \geqslant 0$. We shall now describe some specific instances as to how a perspective system arises in many problems involving computer vision.

We begin by considering a rigid body undergoing rotation with a constant angular velocity with respect to a fixed axis of rotation. It follows that if (x_t, y_t, z_t) is a point on the rigid body at the time instant t, then

$$(\dot{x}_t \;\; \dot{y}_t \;\; \dot{z}_t)^{\mathrm{T}} = \Omega(x_t \;\; y_t \;\; z_t)^{\mathrm{T}}, \tag{1.3}$$

where Ω is a skew symmetric matrix given by

$$\begin{pmatrix} 0 & \omega_1 & \omega_2 \\ -\omega_1 & 0 & \omega_3 \\ -\omega_2 & -\omega_3 & 0 \end{pmatrix}. \tag{1.4}$$

Assume that the points on the rigid body are observed perspectively, i.e. every nonzero point is observed upto a line which the point makes with the origin. This is described as

$$Z : \mathscr{R}^3 - \{(0,0,0)\} \rightarrow \mathscr{RP}^2, \tag{1.5}$$

defined as

$$(x_t, y_t, z_t) \rightarrow [x_t, y_t, z_t].$$

In fact if we assume that the screen is located at $z = 1$, then every point (x, y, z) where $z \neq 0$ is projected on the screen as the point $(x/z, y/z)$. This process of projection is known as the perspective projection and it has been studied in the computer vision literature [6, 5] for many years. Clearly a projected point on the screen identifies a homogeneous line through the point (x, y, z). The linear system (1.3) together with the observation function (1.5) constitutes a trivial example of a perspective system.

As a slightly more nontrivial example we now consider a situation where the body is undergoing both rotation and translation. Let us consider two coordinate frames, one attached to the camera (call it (x_c, y_c, z_c)) and the other coordinate frame (call it the body coordinate frame (x_b, y_b, z_b)) with respect to which the body is translating at a constant velocity. We therefore have

$$(\dot{x}_b \;\; \dot{y}_b \;\; \dot{z}_b)^{\mathrm{T}} = (\zeta_x \;\; \zeta_y \;\; \zeta_z)^{\mathrm{T}}. \tag{1.6}$$

Furthermore assume that the camera coordinate frame is rotating with respect to the body frame at a constant angular velocity. From this assumption we conclude that

$$[x_c \; y_c \; z_c]^{\mathrm{T}} = \mathrm{e}^{\Omega t}[x_b \; y_b \; z_b]^{\mathrm{T}}, \tag{1.7}$$

where Ω is given by (1.4). With respect to the camera coordinate frame, the dynamics of the body is described as

$$\begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} = e^{\Omega t} \left[\begin{pmatrix} x_c(0) \\ y_c(0) \\ z_c(0) \end{pmatrix} + \begin{pmatrix} \zeta_x t \\ \zeta_y t \\ \zeta_z t \end{pmatrix} \right]. \tag{1.8}$$

It is not too hard to check that (1.8) can be obtained as a solution to the dynamical system

$$\begin{pmatrix} \dot{X}_c \\ \dot{Z}_c \end{pmatrix} = \begin{pmatrix} \Omega & I \\ 0 & \Omega \end{pmatrix} \begin{pmatrix} X_c \\ Z_c \end{pmatrix}, \tag{1.9}$$

where

$$X_c(t) = (x_c(t)\ y_c(t)\ z_c(t))^T,$$

$$Z_c(0) = (\zeta_x\ \zeta_y\ \zeta_z)^T.$$

Moreover if points on the body are observed up to homogeneous lines, we construct the observation function

$$Z : \mathscr{R}^6 - B \to \mathscr{RP}^2 \tag{1.10}$$

given by

$$(X_c\ Z_c) \mapsto [x_c\ y_c\ z_c],$$

where

$$B = \{(X_c\ Z_c) : X_c = 0\}.$$

The pair of Eqs. (1.9) and (1.10) is an example of a perspective system. Many other perspective systems have been introduced and the corresponding observability questions have been analysed in [1]. The main result of this paper is to obtain a necessary and sufficient condition for the perspective observability problem. This is described as follows.

Main Theorem. *The dynamical system* (1.1) *and* (1.2) *over the field* $\mathscr{C}$ *is perspectively observable if and only if*

$$\operatorname{rank} \begin{bmatrix} (A - \lambda_i I)(A - \lambda_j I) \\ C \end{bmatrix} = n \tag{1.11}$$

for every pair of eigenvalues λ_i *and* λ_j *(may be the same) of* A. *Over* $\mathscr{R}$ *the above condition* (1.11) *is sufficient for perspective observability.*

It may be noted that the above result is an immediate generalization of the Popov–Belevitch–Hautus test [4] on the observability of a linear dynamical system. This is described as follows.

PBH Eigenvector Test. A linear dynamical system

$$\dot{x} = Ax, \qquad x(0) = x_0 \in \mathscr{F}^n, \tag{1.12}$$

$$z = Cx, \qquad z \in \mathscr{F}^m \tag{1.13}$$

is observable (i.e. a nonzero $x(0)$ would not produce a zero $z(t)$ for all $t \geqslant 0$) if and only if

$$\operatorname{rank} \begin{bmatrix} A - \lambda_i I \\ C \end{bmatrix} = n \tag{1.14}$$

for every eigenvalue λ_i of A.

It may also be noted that the main theorem presented here improves the necessary and sufficient condition obtained in [1]. In particular the rank condition is considerably easier to check.

2. Proof of the main theorem

Before we construct the proof of the main theorem, we introduce some notations, definitions and preliminary results. Define the vector space $\mathscr{F}^n \wedge \mathscr{F}^n$ (for a definition of wedge product see [2]) and consider the linear map

$$\hat{A} : \mathscr{F}^n \wedge \mathscr{F}^n \to \mathscr{F}^n \wedge \mathscr{F}^n,$$

given by

$$x \wedge y \mapsto Ax \wedge y + x \wedge Ay.$$

Let us also consider the linear map

$$\hat{C} : \mathscr{F}^n \wedge \mathscr{F}^n \to \mathscr{F}^m \wedge \mathscr{F}^m,$$

given by

$$x \wedge y \mapsto Cx \wedge Cy.$$

Finally consider the linear system

$$\dot{\hat{x}} = \hat{A}\hat{x}, \qquad \hat{z} = \hat{C}\hat{x}, \tag{2.1}$$

where $\hat{x} \in \mathscr{F}^n \wedge \mathscr{F}^n$, $\hat{z} \in \mathscr{F}^m \wedge \mathscr{F}^m$. Let U be the unobservable subspace of (2.1). The following proposition follows quite easily.

Proposition 2.1. *The perspective system* (1.1) *and* (1.2) *is perspectively observable if and only if there is no decomposable vector $x \wedge y$ contained in U.*

Define $Grass_{\mathscr{F}}(2,n)$ (see [3] for details) to be the space of all 2-dimensional linear subspaces in $\mathscr{F}^n$. One could embed $Grass_{\mathscr{F}}(2,n)$ as a subset of $\mathscr{P}(\mathscr{F}^n \wedge \mathscr{F}^n)$ by Plücker embedding where $\mathscr{P}(\mathscr{F}^n \wedge \mathscr{F}^n)$ is the space of all lines through 0 in $\mathscr{F}^n \wedge \mathscr{F}^n$. We also consider the unobservable subspace U as a subspace of $\mathscr{P}(\mathscr{F}^n \wedge \mathscr{F}^n)$. Note that to say U contains a decomposable vector is to say that U intersects $Grass_{\mathscr{F}}(2,n)$. We therefore have the following proposition.

Proposition 2.2. *The following statements are equivalent.*
1. *The system* (1.1) *and* (1.2) *is perspectively observable.*
2. *The intersection of U and $Grass_{\mathscr{F}}(2,n)$ is empty.*
3. *Let $x_1,\ldots,x_n$ be a set of coordinate vectors of $\mathscr{F}^n$. If a_{ij} are such that*

$$\sum_{i<j} a_{ij}(x_i \wedge x_j) \in U, \tag{2.2}$$

then the following quadratic condition is not satisfied.

$$a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk} = 0 \quad \textit{for all } i < j < k < l. \tag{2.3}$$

Note that the coordinates a_{ij} are called the Plücker coordinates and (2.3) is the set of quadratic conditions these coordinates have to satisfy in order that the point in U is actually contained in $Grass_{\mathscr{F}}(2,n)$. It may be remarked that in order to check the observability of a perspective system, the necessary and sufficient conditions provided by the above proposition are difficult to check. Over $\mathscr{C}$, these conditions are equivalent to a relatively easy to check rank condition (1.11) which illustrates the significance of the main theorem.

Proof of the main theorem. We first prove necessity. Assume that there exists a vector $x \neq 0$ such that $Cx = 0$ and $(A - \lambda_i I)(A - \lambda_j I)x = 0$. Let us define $y = (A - \lambda_j I)x$. Then we have

$$Ay = \lambda_i y. \tag{2.4}$$

If $y = kx$ for some number k (may be 0), then x is an eigenvector of A and

$$Cx = 0. \tag{2.5}$$

It follows that the pair (A, C) is not observable and so is not perspectively observable.

If x and y are linearly independent, we claim that

$$e^{tA}x = \begin{cases} e^{t\lambda_j}x + \dfrac{e^{t\lambda_i} - e^{t\lambda_j}}{\lambda_i - \lambda_j}y & \text{if } \lambda_i \neq \lambda_j, \\ e^{t\lambda_j}x + te^{t\lambda_j}y & \text{if } \lambda_i = \lambda_j. \end{cases} \tag{2.6}$$

Notice that $Ax = \lambda_j x + y$. When $\lambda_i \neq \lambda_j$, $x - (1/(\lambda_i - \lambda_j))y$ is an eigenvector of A associated with λ_j. So

$$e^{tA}x - \frac{e^{t\lambda_i}}{\lambda_i - \lambda_j}y = e^{tA}\left(x - \frac{1}{\lambda_i - \lambda_j}y\right) = e^{t\lambda_j}\left(x - \frac{1}{\lambda_i - \lambda_j}y\right),$$

or

$$e^{tA}x = e^{t\lambda_j}x + \frac{e^{t\lambda_i} - e^{t\lambda_j}}{\lambda_i - \lambda_j}y.$$

When $\lambda_i = \lambda_j$,

$$A^n x = \lambda_i^n x + n\lambda_i^{n-1}y, \quad n > 0.$$

So

$$e^{tA}x = e^{t\lambda_i}x + te^{t\lambda_i}y.$$

Since x and y are independent, we have $x \wedge y \neq 0$. Moreover from (2.6) we have

$$Ce^{tA}x = \begin{cases} \dfrac{e^{t\lambda_i} - e^{t\lambda_j}}{\lambda_i - \lambda_j}Cy & \text{if } \lambda_i \neq \lambda_j, \\ te^{t\lambda_j}Cy & \text{if } \lambda_i = \lambda_j. \end{cases} \tag{2.7}$$

Finally from (2.4) we conclude that

$$Ce^{tA}y = e^{t\lambda_i}Cy. \tag{2.8}$$

Thus from (2.7) and (2.8) it follows that

$$(Ce^{tA}x) \wedge (Ce^{tA}y) = 0.$$

Hence the system is not perspectively observable.

To prove sufficiency, let us choose a set of generalized eigenvectors $\{x_1, \ldots, x_n\}$ of A with

$$x_i = (A - \lambda_j I)x_j \quad \text{if } x_i \wedge (A - \lambda_j I)x_j = 0$$

as a basis of $\mathscr{R}^n$ and arrange the order such that

$$\begin{aligned} &\operatorname{Re}\lambda_i < \operatorname{Re}\lambda_{i+1} \quad \text{or} \\ &\operatorname{Re}\lambda_i = \operatorname{Re}\lambda_{i+1} \quad \text{and} \quad \operatorname{Im}\lambda_i < \operatorname{Im}\lambda_{i+1} \quad \text{or} \\ &\lambda_i = \lambda_{i+1} \quad \text{and} \quad p_i \leqslant p_{i+1}, \end{aligned} \tag{2.9}$$

where λ_i is the eigenvalue associated with x_i and p_i is the order of x_i, i.e. $(A - \lambda_i I)^{p_i}x_i \neq 0$ and $(A - \lambda_i I)^{p_i+1}x_i = 0$. Let us define

$$\hat{A}_{ij} = \hat{A} - (\lambda_i + \lambda_j)I \quad \text{and} \quad A_i = A - \lambda_i I.$$

Then it follows that

$$\hat{A}^p_{ij}(x_i \wedge x_j) = \begin{cases} 0 & \text{if } p > p_i + p_j, \\ \binom{p_i + p_j}{p_i} A_i^{p_i} x_i \wedge A_j^{p_j} x_j & \text{if } p = p_i + p_j, \\ \left(\binom{2p_i}{p_i - 1} - \binom{2p_i}{p_i} \right) A_i^{p_i - 1} x_i \wedge A_j^{p_j} x_j & \text{if } x_i = A_j x_j, \\ & \quad p = p_i + p_j - 1. \end{cases} \tag{2.10}$$

Assume that there is a vector

$$x \wedge y = \sum_{i<j} c_{ij} x_i \wedge x_j \in U.$$

We shall show that the rank condition (1.11) is not satisfied. Let $r < s$ be the two integers such that $c_{rs} \neq 0$ and $c_{ij} = 0$ for all $i < j$ with either $i > r$ or $j > s$. Without loss of generality, one can assume that

$$x = \sum_{i=1}^{r} a_i x_i \quad \text{and} \quad y = \sum_{j=1}^{s} b_j x_j.$$

(If not, replace x by $x + ay$ for some a.) Then $a_r \neq 0$ and $b_s \neq 0$. For any $i < j$ with $i \leqslant r$ and $j \leqslant s$,

$$\lambda_i + \lambda_j = \lambda_r + \lambda_s \text{ if and only if } \lambda_i = \lambda_r \text{ and } \lambda_j = \lambda_s, \tag{2.11}$$

and when $\lambda_i + \lambda_j = \lambda_r + \lambda_s$,

$$p_i + p_j = p_r + p_s \text{ if and only if } p_i = p_r \text{ and } p_j = p_s. \tag{2.12}$$

Let

$$p = \max\{p_i \mid i = 1, \ldots, n\}$$

and define $f : \mathscr{C}^n \wedge \mathscr{C}^n \to \mathscr{C}^n \wedge \mathscr{C}^n$ by

$$f = \hat{A}^q_{rs} \prod_{\lambda_i + \lambda_j \neq \lambda_r + \lambda_s} \hat{A}^{2p+1}_{ij}, \tag{2.13}$$

where q is the largest integer such that $f(x \wedge y) \neq 0$. Then $f(U) \subset U$ because U is $\hat{A}$-invariant. Since $\hat{A}_{rs} f(x \wedge y) = 0$, $f(x \wedge y) \in U$ is an eigenvector of $\hat{A}$ associated with the eigenvalue $\lambda_r + \lambda_s$.

For any $k < l$ with $k \leqslant r$ and $l \leqslant s$, if $\lambda_k \neq \lambda_r$ or $\lambda_l \neq \lambda_s$, then $\lambda_k + \lambda_l \neq \lambda_r + \lambda_s$ by (2.11) and

$$f(x_k \wedge x_l) = \hat{A}^q_{rs} \prod_{\lambda_i + \lambda_j \neq \lambda_r + \lambda_s, \{i,j\} \neq \{k,l\}} \hat{A}^{2p+1}_{ij} (\hat{A}^{2p+1}_{kl} x_k \wedge x_l) = 0$$

by (2.11). Also for any $k < l$ with $k \leqslant r$ and $l \leqslant s$, if $\lambda_k = \lambda_r$ and $\lambda_l = \lambda_s$ but $p_k + p_l < q$, then

$$f(x_k \wedge x_l) = \prod_{\lambda_i + \lambda_j \neq \lambda_r + \lambda_s} \hat{A}^{2p+1}_{ij} (\hat{A}^q_{kl} x_k \wedge x_l) = 0.$$

The q in (2.13) either equals $p_r + p_s$ if there are $i < j$ such that $c_{ij} \neq 0$, $\lambda_i = \lambda_r$, $p_i = p_r$, $\lambda_j = \lambda_s$, $p_j = p_s$ and $x_i \neq A_j x_j$, or equals $p_r + p_s - 1$ if there are no such ij. In the latter case we would necessarily have $x_r = A_s x_s$ in particular. We claim that $f(x \wedge y) = u \wedge v$ and $u, v \in \ker(A_r A_s)$. We prove the claim by considering the following two possible cases.

Case 1. ($q = p_r + p_s$): For any $i \leqslant r$, $j \leqslant s$, we have

$$f(x_i \wedge x_j) = \begin{cases} k \begin{pmatrix} p_i + p_j \\ p_i \end{pmatrix} A_i^{p_i} x_i \wedge A_j^{p_j} x_j & \text{if } \lambda_i = \lambda_r, \lambda_j = \lambda_s, \\ & \quad p_i = p_r, p_j = p_s, \\ 0 & \text{otherwise,} \end{cases} \tag{2.14}$$

where

$$k = \prod_{\lambda_i + \lambda_j \neq \lambda_r + \lambda_s} ((\lambda_r + \lambda_s) - (\lambda_i + \lambda_j))^{2p+1}. \tag{2.15}$$

So

$$f(x \wedge y) = u \wedge v,$$

where

$$u = k \begin{pmatrix} p_i + p_j \\ p_i \end{pmatrix} \sum_{i \leqslant r, \lambda_i = \lambda_r, p_i = p_r} a_i A_i^{p_i} x_i \in \ker A_r$$

and

$$v = \sum_{j \leqslant s, \lambda_j = \lambda_s, p_j = p_s} b_j A_j^{p_j} x_j \in \ker A_s.$$

Case 2. ($q = p_r + p_s - 1$): In this case we would have $x_r = A_s x_s$, $\lambda_r = \lambda_s$ and $p_r = p_s - 1$. If there is an $i < r$ such that $a_i \neq 0$, $\lambda_i = \lambda_s$ and $p_i = p_r$, then $\hat{A}_{rs} f((a_i x_i) \wedge (b_s)(x_s)) \neq 0$ which contradicts the definition of q. Similarly, there is no $j < s$ with $b_j \neq 0$, $\lambda_j = \lambda_s$ and $p_j = p_s$.

$$f(c_{ij} x_i \wedge x_j) = \begin{cases} \dfrac{-kc_{ij}}{p_r} \begin{pmatrix} 2p_r \\ p_r - 1 \end{pmatrix} A_s^{p_r - 1} x_r \wedge A_s^{p_s} x_s & \text{if } i = r, j = s, \\ kc_{ij} \begin{pmatrix} 2p_r \\ p_r - 1 \end{pmatrix} A_s^{p_i} x_i \wedge A_s^{p_s} x_s & \text{if } j = s, \lambda_i = \lambda_s, \\ & \quad p_i = p_r - 1, \\ -kc_{ij} \begin{pmatrix} 2p_r \\ p_r \end{pmatrix} A_s^{p_j} x_j \wedge A_s^{p_s} x_s & \text{if } i = r, \lambda_j = \lambda_s, \\ & \quad p_j = p_r, \\ 0 & \text{otherwise,} \end{cases} \tag{2.16}$$

i.e. either

$$f(c_{ij} x_i \wedge x_j) = x' \wedge (A_s^{p_s}) x_s$$

for some vector x' in $\ker A_s^2$ or

$$f(c_{ij} x_i \wedge x_j) = 0.$$

Therefore,

$$f(x \wedge y) = u \wedge A_s^{p_s} x_s$$

and both u and $A_s^{p_s} x_s$ are in $\ker(A_s^2)$. This completes the proof of the above claim.

Since $\hat{C}(u \wedge v) = Cu \wedge Cv = 0$, there exist a and b such that $au + bv \neq 0$ and $C(au + bv) = 0$, i.e.

$$au + bv \in \ker \begin{bmatrix} (A - \lambda_r I)(A - \lambda_s I) \\ C \end{bmatrix}.$$

This completes the proof of the theorem over the field $\mathscr{C}$ of complex numbers.

Over the field $\mathscr{R}$ the condition (1.11) is not necessary. A nonperspectively observable system over $\mathscr{C}$ can still be perspectively observable over $\mathscr{R}$. As an example consider the following:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

The unobservable subspace of $\hat{A}$ and $\hat{C}$ is

$$U = \{ae_1 \wedge e_3 + be_1 \wedge e_4 + be_2 \wedge e_3 - ae_2 \wedge e_4\},$$

where

$$e_i = \begin{bmatrix} e_{i1} \\ e_{i2} \\ e_{i3} \\ e_{i4} \end{bmatrix}, \qquad e_{i,j} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

The intersection of U and $Grass(2,4)$ contains only two complex points

$$\begin{aligned} &\{(e_1 \wedge e_3 - e_2 \wedge e_4) \pm i(e_1 \wedge e_4 + e_2 \wedge e_3)\} \\ &\quad = \{(e_1 + ie_2) \wedge (e_3 + ie_4), (e_1 - ie_2) \wedge (e_3 - ie_4)\} \end{aligned}$$

and there is no real point in $U \cap Grass(2,4)$.

3. Conclusion

To conclude, we describe the main result of this paper. Motivated from motion estimation problems in computer vision, we consider the perspective observability problem and obtain a necessary and sufficient condition for this problem over the field of complex numbers. Our result generalizes the well known result due to Popov–Belevitch–Hautus on the problem of observing a linear dynamical system.

References

[1] B.K. Ghosh, M. Jankovic and Y.T. Wu, Perspective problems in system theory and its application to machine vision, *J. Math. Systems, Estimation and Control* **4**(1) (1994) 3–38.

[2] W.H. Greub, *Linear Algebra* (Springer, Berlin, 1975).

[3] P. Griffiths and J. Harris, *Principles of Algebraic Geometry* (Wiley, New York, 1978).

[4] M.L.J. Hautus, Controllability and observability conditions of linear autonomous systems, *Ned. Akad. Wetenschappen, Proc. Ser. A* **72** (1969) 443–448.

[5] B.K.P. Horn, *Robot Vision*, The MIT Electrical Engineering and Computer Science Series (1986).

[6] K. Kanatani, *Group-Theoretical Methods in Image Understanding*, Springer Series in Information Sciences, Vol. 20 (1990).