Binocular eye tracking control satisfying Hering's law ⋆

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Abstract

Human eye movement can be looked at, as a rotational dynamics on the space **SO(3)** with constraints that have to do with the axis of rotation. A typical eye movement can be decomposed into two components, that go by the name 'version' and 'vergence'. Hering's law proposes that the version component of the eye movement is identical in both the eyes, and versional eye movement is used to follow a target located far away. In order to focus on a closer target, the eyes rotate in opposite directions, using the vergence component. A typical eye movement would be regarded as a concatenation of version followed by vergence. In this paper, we shall represent such eye movements using unit quaternion, with constraints. Assuming that the eyes are perfect spheres with their mass distributed uniformly and rotating about their own centers, eye movement models are constructed using classical mechanics. For targets moving in near field, for which both version and vergence eye movements are required, optimal eye movement trajectories are simulated, where the goal is to minimize a quadratic cost function on the energy of the applied control torques.

Key words: Eye Movement, Binocular Vision, Listing's Plane, Mid-Sagittal Plane, Euler Lagrange's Equation, Optimal Control.

1 Introduction

Neurologists, physiologists and engineers have been interested in modeling and control of a single human eye (monocular control) since 1845 with notable studies conducted by Listing [12], Donders [6] and Helmholtz [20]. Specifically, it has been observed that the oculomotor system chooses just one angle of ocular torsion for any one gaze direction (see Donders [6]). Several studies have focussed on three dimensional eye movements [5], [8], including rigorous treatment of the topic from the point of view of modern control theory and geometric mechanics [13]. Assuming the human eye to be a rigid sphere, the oculomotor system can be viewed as a mechanical control system and one can apply geometric theory with Lagrangian and Hamiltonian viewpoints [4], [14]. This paper extends our earlier studies [16], [7], [22] from monocular control to controlling a pair of eyes.

Any eye orientation can be reached, starting from one specific orientation called the primary orientation, by rotation about a single axis. Listing's law states that, starting from a frontal gaze, any other gaze direction is obtained by a rotation matrix whose axis of rotation is constrained to lie on a plane, called the Listing's plane. Consequently, the set of all orientations the eye can assume is a submanifold [3] of **SO(3)** called **LIST**. Listing had shown and subsequently verified by others [18], [19], that while gazing targets located at optical infinity and keeping the head fixed, eye orientations are restricted to this submanifold **LIST** [16]. In Binocular vision, Listing's law is not valid for fixation of nearby targets, which is the main point of this paper.

It has been observed [17] (see also [15]), that when a pair of human eyes fixate on a nearby point target, the axes of rotations of the two eyes are not located on the Listing's plane. The eye rotations are not independently controlled (as proposed by Helmholtz [20] in (1866)), but can be viewed as a concatenation of *version* followed by *vergence* (in the spirit of what was originally proposed by Hering [10] in 1868). The versional component of the eye movement is identical for both the eyes and satisfies Listing's law. This is equivalent to saying that the versional part of the eye rotation belongs to **LIST**. On the other hand, the vergence component of the eye movement rotates the two eyes in opposite directions, in order to fixate nearby point targets. Following [17], we would assume that for the vergence part, the rotation vector is restricted to the mid-sagittal plane with respect to the fixed head coordinate system [1]. Starting from the primary orientation, the set of all orientations that are achievable using rotations with axes in the mid-sagittal plane is a submanifold **MS** of **SO(3)**. Identical to what was done earlier for **LIST** [16], we introduce a Riemannian metric for **MS** and write down the associated Euler Lagrange equa-

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[1] The point midway between the center of the two eyes is called the ego-center. The mid-sagittal plane is the plane perpendicular to the segment joining the center of the two eyes and passing through the ego-center.

tion that describes the *vergence eye dynamics* [2]. The paper combines version and vergence dynamics as a proposal for binocular control.

Specifically, the version and vergence control problems are separately looked at as an optimal control problem where the goal is to restrict the states of the two eyes to respectively **LIST** and **MS**. Via simulation, we display the combined effects of the two controllers to the left and the right eyes.

2 Quaternionic Representations

Representation of 'eye orientation' using quaternion has already been described in [16]. For the sake of clarity, we revisit some of the main ideas in this section. A quaternion is a four tuple of real numbers denoted by Q. We write each element $a \in Q$ as

$$a = a_0\, 1 + a_1\, i + a_2\, j + a_3\, k.$$

The space of unit quaternion is identified with the unit sphere in $\mathbb{R}^4$ and denoted by S^3. Each $q \in \mathrm{S}^3$ can be written as

$$q = \cos\frac{\phi}{2}\, 1 + \sin\frac{\phi}{2}\, n_1\, i + \sin\frac{\phi}{2}\, n_2\, j + \sin\frac{\phi}{2}\, n_3\, k, \quad (1)$$

where $\phi \in [0, 2\pi]$, and $n = (n_1, n_2, n_3)$ is a unit vector in $\mathbb{R}^3$. If q is an unit quaternion represented as in (1), one can show [2], [11] the following, using simple properties of quaternion multiplication (denoted by $\bullet$) –

"The vector component of $[q \bullet (v_1\, i + v_2\, j + v_3\, k) \bullet q^{-1}]$ is rotation of the vector (v_1, v_2, v_3) around the axis n by a counterclockwise angle ϕ."

If $\mathbf{S}^3$ is the space of unit quaternions, we define a map between $\mathbf{S}^3$ and **SO(3)** described as follows

$$rot\, :\, S^3 \rightarrow SO(3) \qquad (2)$$

where

$$q = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \longmapsto \begin{pmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_1q_2+q_0q_3) & q_0^2+q_2^2-q_1^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2+q_3^2-q_1^2-q_2^2 \end{pmatrix}. \qquad (3)$$

Recall that $SO(3)$ is the space of all 3×3 matrices W such that $WW^T = I$, the identity matrix and $det\, W = 1$. It can be verified that for any nonzero vector v in $\mathbb{R}^3$ we have

$$rot(q)\, v = vec[q \bullet v \bullet q^{-1}].$$

Note that the map 'rot' in (2) is surjective but not $1-1$. This is because both q and $-q$ in S^3 has the same image. We now write down a parametrization of the unit vector 'n' in (1) using polar coordinates as

$$n = \Big(\cos\theta\, \cos\alpha\;\; \sin\theta\, \cos\alpha\;\; \sin\alpha\Big)^T. \qquad (4)$$

Combining (1) and (4), we have the following parametrization of unit quaternions

$$q = \begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2}\cos\theta\, \cos\alpha \\ \sin\frac{\phi}{2}\sin\theta\, \cos\alpha \\ \sin\frac{\phi}{2}\sin\alpha \end{pmatrix}, \qquad (5)$$

called the axis-angle parametrization. Using the coordinates (θ, ϕ, α) we construct the following sequence of maps

$$[0,\pi] \times [0,2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \xrightarrow{\rho} S^3 \xrightarrow{rot} SO(3) \xrightarrow{proj} S^2, \quad (6)$$

where

$$\rho((\theta,\phi,\alpha)) = q \;\; (\text{in } (5)),$$
$$rot(q) = W$$

and

$$proj(W) = \begin{pmatrix} \sin\theta\, \sin\phi\, \cos\alpha + \cos\theta\, \sin^2\frac{\phi}{2}\, \sin 2\alpha \\ -\cos\theta\, \sin\phi\, \cos\alpha + \sin\theta\, \sin^2\frac{\phi}{2}\, \sin 2\alpha \\ \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}\, \cos 2\alpha \end{pmatrix}. \qquad (7)$$

Note that the matrix W in $SO(3)$ can be easily written from (3) and the details are omitted. The points in S^2 described by (7) provide a description of the gaze directions as a function of the coordinate angles θ, ϕ, α with respect to an initial gaze direction of $(0,0,1)^T$, i.e. obtained by rotating the vector $(0,0,1)^T$ using the rotation matrix W. The submanifolds **LIST** and **MS** can be easily parameterized by restricting $\alpha = 0$ and $\theta = \frac{\pi}{2}$ respectively in (5). This is done in the next section.

3 The Submanifold LIST

The law of rotation for the version eye movement, the Listing's law, asserts that the axis of rotation 'n' in (4) is restricted to the plane

$$\alpha = 0, \qquad (8)$$

and one obtains the axis of rotation as

$$n = (\cos\theta,\;\; \sin\theta,\;\; 0)^T.$$

The corresponding unit quaternion vector is given by

$$q_L = (\cos\frac{\phi}{2},\;\; \sin\frac{\phi}{2}\cos\theta,\;\; \sin\frac{\phi}{2}\sin\theta,\;\; 0)^T.$$

[2] The versional part of the eye dynamics was studied earlier in [16].

The rotation matrix W is given by

$$W = \begin{pmatrix} \cos^2\frac{\phi}{2} + \cos 2\theta\, \sin^2\frac{\phi}{2} & \sin 2\theta\, \sin^2\frac{\phi}{2} & \sin\theta\, \sin\phi \\ \sin 2\theta\, \sin^2\frac{\phi}{2} & \cos^2\frac{\phi}{2} - \cos 2\theta\, \sin^2\frac{\phi}{2} & -\cos\theta\, \sin\phi \\ -\sin\theta\, \sin\phi & \cos\theta\, \sin\phi & \cos\phi \end{pmatrix}. \tag{9}$$

Under the Listing's constraint, we define **LIST** to be the associated submanifold of $\mathbf{S}^3$ and $\mathbf{SO}_L(3)$ to be the associated submanifold of **SO(3)**. They are both two dimensional submanifolds parameterizing the versional eye rotation in $\mathbf{S}^3$ and $\mathbf{SO}(3)$ respectively. The gaze direction $(0,\ 0,\ 1)^T$ is transformed to the direction

$$(\sin\theta\, \sin\phi,\ -\cos\theta\, \sin\phi,\ \cos\phi)^T$$

by the rotation matrix (9). Thus we have the following sequence of maps

$$[0,\pi] \times [0,2\pi] \xrightarrow{\rho} \mathbf{LIST} \xrightarrow{rot} \mathbf{SO}_L(3) \xrightarrow{proj} \mathbf{S}^2, \tag{10}$$

obtained by restricting (6) under the constraint $\alpha = 0$.

4 The Submanifold MS

The law of rotation for the vergence eye movement (proposed in [17]), asserts that the axis of rotation 'n' in (4) is restricted to the plane [3]

$$\theta = \frac{\pi}{2}, \tag{11}$$

and obtain the axis of rotation as

$$n = (0,\ \cos\alpha,\ \sin\alpha)^T.$$

The corresponding unit quaternion vector is given by

$$q_M = (\cos\frac{\phi}{2},\ 0,\ \sin\frac{\phi}{2}\cos\alpha,\ \sin\frac{\phi}{2}\sin\alpha)^T. \tag{12}$$

The rotation matrix W is computed to be

$$W = \begin{pmatrix} \cos\phi & -\sin\phi\, \sin\alpha & \sin\phi\, \cos\alpha \\ \sin\phi\, \sin\alpha & \cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}\, \cos(2\alpha) & \sin^2\frac{\phi}{2}\, \sin(2\alpha) \\ -\sin\phi\, \cos\alpha & \sin^2\frac{\phi}{2}\, \sin(2\alpha) & \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}\, \cos(2\alpha) \end{pmatrix}. \tag{13}$$

Under this new constraint (11), we define **MS** to be the associated submanifold of $\mathbf{S}^3$ and $\mathbf{SO}_M(3)$ to be the associated submanifold of **SO(3)**. They are both two dimensional submanifolds parameterizing vergence eye rotation in $\mathbf{S}^3$ and $\mathbf{SO}(3)$ respectively. The gaze direction $(0,\ 0,\ 1)^T$ is transformed to the direction

$$(\sin\phi\cos\alpha,\ \sin^2\frac{\phi}{2}\sin(2\alpha),\ \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}\cos(2\alpha))^T$$

3 We shall call the constraint (11), the extended Listing's constraint. The plane (11) is precisely the mid-sagittal plane, justifying the acronym **MS** for the submanifold.

by the rotation matrix (13). Thus we have the following sequence of maps

$$[0,2\pi] \times [-\frac{\pi}{2},\frac{\pi}{2}] \xrightarrow{\rho} \mathbf{MS} \xrightarrow{rot} \mathbf{SO}_M(3) \xrightarrow{proj} \mathbf{S}^2, \tag{14}$$

obtained by restricting (6) under the constraint $\theta = \frac{\pi}{2}$.

The following two results are easy consequences of the Listing's Theorem already sketched in [7].

Theorem (Listing): Under the Listing's constraint (8), the map

$$\mathbf{SO}_L(3) - \left\{ \begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\} \xrightarrow{\mathbf{proj}} \mathbf{S}^2 - \left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

described by (10) is one to one and onto.

Theorem (Extended Listing): Under the Extended Listing's constraint (11), the map

$$\mathbf{SO}_M(3) - \left\{ \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \xrightarrow{\mathbf{proj}} \mathbf{S}^2 - \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{15}$$

described by (14) is one to one and onto.

Proof of Extended Listing's Theorem: There are two parts of the proof. The first part is to show that the mapping **'proj'** in (15) is $1-1$. In the second part we show that the mapping **'proj'** in (15) is onto.

Part I (Proof of $1-1$): Consider $\chi = proj \circ rot$ as a mapping from S^3 to S^2 in (6). It is known [9], that if p and q are points in S^3 and S^2 respectively such that $\chi(p) = q$, then the set of all $p_1 \in S^3$ that satisfies $\chi(p_1) = q$ has the property that

$$p_1 = p \bullet (a,0,0,b) \tag{16}$$

where $a^2 + b^2 = 1$ and the symbol $\bullet$, as before, denotes multiplication as a quaternion. Writing p as in (12), it follows that if $p \in MS$, then every p_1 satisfying (16) belongs to MS if $\sin\frac{\phi}{2}\cos\alpha = 0$ or $b = 0$. If $b = 0$, it would follow that $p_1 = p$ and χ is $1-1$. If $\phi = 0$, it would follow that $p = (1,0,0,0)^T$ and $q = (0,0,1)^T$ which is excluded from the range of **'proj'** in (15). Finally if $\alpha = \pm\frac{\pi}{2}$, it follows that $p = (\cos\frac{\phi}{2},0,0,\pm\sin\frac{\phi}{2})^T$ and $q = (0,0,1)^T$ which is excluded from the range of **'proj'** in (15).

Part II (Proof of onto): To show that the map **'proj'** is surjective, we consider an arbitrary vector

$$(\xi_1, \xi_2, \xi_3) \neq (0, 0, 1),$$

and solve the following set

$$\begin{aligned} \sin\phi\ \cos\alpha &= \xi_1, \\ \sin^2\frac{\phi}{2}\ \sin(2\alpha) &= \xi_2, \\ \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}\ \cos(2\alpha) &= \xi_3, \end{aligned} \tag{17}$$

of equations, for a suitable ϕ and α. We define the variable

$$\Delta = \frac{1-\xi_3}{\xi_2}.$$

Eliminating ϕ, from the last two equations of (17), we obtain

$$\xi_2\cos(2\alpha) + (\xi_3 - 1)\sin(2\alpha) + \xi_2 = 0. \tag{18}$$

Using the identity, $\sin^2(2\alpha) + \cos^2(2\alpha) = 1$, it would follow from (18) that when $\sin(2\alpha) \neq 0$, we have

$$\sin(2\alpha) = \frac{2\Delta}{1+\Delta^2}. \tag{19}$$

In fact, $\sin(2\alpha) = 0$ would imply $\cos(2\alpha) = -1$ from (18), and it would follow from (17) that $(\xi_1, \xi_2, \xi_3) = (0, 0, 1)$, which has been excluded from the range. One can also deduce that $\cos(2\alpha)$ satisfies

$$[1+\Delta^2]\cos^2(2\alpha) + 2\cos(2\alpha) + [1-\Delta^2] = 0,$$

which implies that either $\cos(2\alpha) = -1$ (which is impossible) or

$$\cos(2\alpha) = \frac{\Delta^2 - 1}{\Delta^2 + 1}. \tag{20}$$

The parameter α can be solved uniquely from (19) and (20). Going back to the (17), one can now solve for a suitable ϕ. □

It is easily inferred from the above two theorems that for both, versional and vergence eye movements, the gaze direction uniquely specifies the orientation of the eye, except perhaps when the gaze is backwards for the versional eye movement and axis of rotation is 'pure torsional' for the vergence eye movement [4].

To end this section, we make the following remark:

Remark: Under vergence eye movement, the left and the right eyes rotate in opposite directions. This can be implemented by considering the unit quaternion (12) for the left eye and defining

$$q_M = (\cos\frac{\phi}{2},\ \ 0,\ \ -\sin\frac{\phi}{2}\cos\alpha,\ \ -\sin\frac{\phi}{2}\sin\alpha)^T, \tag{21}$$

for the right eye. This would be equivalent to reversing the sign of the axis of rotation going from left to the right eye.

5 Version and Vergence as Dynamical Systems

Following [16], one can compute a Riemannian Metric [21] on **LIST** given by

$$g = \sin^2(\phi_L/2)\, d\theta_L^2 + \frac{1}{4}d\phi_L^2. \tag{22}$$

The associated geodesic equation on **LIST** reduces to the following pair of equations, already described in [16], given by

$$\begin{aligned} \ddot\theta_L + \dot\theta_L\dot\phi_L\cot(\phi_L/2) &= 0, \\ \ddot\phi_L - \left(\dot\theta_L\right)^2\sin(\phi_L) &= 0. \end{aligned} \tag{23}$$

The above calculation can be easily repeated, and one can compute a Riemannian Metric on **MS** given by

$$g = \sin^2(\phi_M/2)\, d\alpha_M^2 + \frac{1}{4}d\phi_M^2. \tag{24}$$

The associated geodesic equation on **MS** reduces to the pair of equations given by

$$\begin{aligned} \ddot\alpha_M + \dot\alpha_M\dot\phi_M\cot(\phi_M/2) &= 0, \\ \ddot\phi_M - (\dot\alpha_M)^2\sin(\phi_M) &= 0. \end{aligned} \tag{25}$$

Note that in the above equations (22) - (25) the subscripts L and M describe variables in **LIST** and **MS** respectively. Combining the version and the vergence systems, we define a control system given by

$$\begin{aligned} \ddot\theta_L &= -\dot\theta_L\dot\phi_L\cot(\phi_L/2) + \tau_{\theta_L}, \\ \ddot\phi_L &= \left(\dot\theta_L\right)^2\sin(\phi_L) + \tau_{\phi_L}, \\ \ddot\alpha_M &= -\dot\alpha_M\dot\phi_M\cot(\phi_M/2) + \tau_{\alpha_M}, \\ \ddot\phi_M &= (\dot\alpha_M)^2\sin(\phi_M) + \tau_{\phi_M}, \end{aligned} \tag{26}$$

where the τ-s in the right hand side are the generalized torques (see [16]). If we assume that (26) is the dynamics of the left eye, then the corresponding dynamics of the right eye is given by

$$\begin{aligned} \ddot\theta_L &= -\dot\theta_L\dot\phi_L\cot(\phi_L/2) + \tau_{\theta_L}, \\ \ddot\phi_L &= \left(\dot\theta_L\right)^2\sin(\phi_L) + \tau_{\phi_L}, \\ \ddot\alpha_M &= \dot\alpha_M\dot\phi_M\cot(\phi_M/2) + \tau_{\alpha_M}, \\ \ddot\phi_M &= -(\dot\alpha_M)^2\sin(\phi_M) + \tau_{\phi_M}, \end{aligned} \tag{27}$$

[4] Human eye does not rotate with pure torsion.

obtained by switching the sign of the angle ϕ_M.

6 Eye Movement under Binocular Control

We would now describe how equations (26) and (27) would be used in binocularly controlling the human left and the right eyes. The version and the vergence movements of the two eyes would be combined to obtain a product quaternion given by

$$q = \begin{pmatrix} \cos\frac{\phi_L}{2} \\ \sin\frac{\phi_L}{2}\cos\theta_L \\ \sin\frac{\phi_L}{2}\sin\theta_L \\ 0 \end{pmatrix} \bullet \begin{pmatrix} \cos\frac{\phi_M}{2} \\ 0 \\ \sin\frac{\phi_M}{2}\ \cos\alpha_M \\ \sin\frac{\phi_M}{2}\sin\alpha_M \end{pmatrix}. \tag{28}$$

The variables θ_L and ϕ_L that control the version part of the dynamics are identical. Under the influence of version, starting from the primary position, the two eyes move parallel (as illustrated in Fig. 1). The variables θ_M and ϕ_M that control the vergence part of the dynamics are not identical and follow equations (26) and (27) for the two eyes. For an arbitrary point target, the state variables θ_M and ϕ_M take values in such a way that the gaze directions of the left and right eye meet at a point, i.e. the target. The vergence movement of the eye has been displayed in Fig. 2.

Starting from an initial primary coordinate frame, the versional control system rotates the frame while keeping the gaze direction pointed in the general direction of the target. One can interpret the vergence control system to be entirely defined on the rotated frame providing gaze adjustment to focus onto a target. In this way, the two control systems can be computed independently, except that the state variables for vergence system have to be interpreted using local coordinates computed by rotating a primary frame controlled by version.

7 Optimally Controlling Binocular Vision

The set of generalized control vector functions that would steer the pair of eyes between two target points, while keeping the points in focus, is not unique. We would like to rotate the eye pair from one target point to another in a fixed time interval T, while minimizing a cost function in the form

$$\int_0^T \left[\frac{1}{2}w_1\ \tau^T\cdot\tau\ +\ w_2\ c_1^T c_1\ +\ w_3\ c_2^T c_2\ +\ p(t)^T\ (F-\ddot{x})\right] dt, \tag{29}$$

where we define

$$c_1\ =\ (\sin\theta_L\ \sin\phi_L - \xi_1(t),\ -\cos\theta_L\ \sin\phi_L - \xi_2(t),\ \cos\phi_L - \xi_3(t))^T$$

and

$$c_2\ =\ \begin{pmatrix} \sin\phi_M\ \cos\alpha_M - \zeta_1(t) \\ \sin^2\frac{\phi_M}{2}\ \sin(2\alpha_M) - \zeta_2(t) \\ \cos^2\frac{\phi_M}{2} - \sin^2\frac{\phi_M}{2}\cos(2\alpha_M) - \zeta_3(t) \end{pmatrix}$$

The vector $(\xi_1(t), \xi_2(t), \xi_3(t))$ are coordinates of the trajectory being tracked, by the version control, with respect to a global coordinate system. The vector $(\zeta_1(t), \zeta_2(t), \zeta_3(t))$ are coordinates of the trajectory being tracked, by the vergence control, with respect to a local coordinate system (obtained by rotating the primary frame using version). The parameters w_1, w_2 and w_3 are constant weights whereas $p(t)$ is a time varying weight that needs to be computed. τ is the vector of generalized torques. Vector x is the state vector $(\theta_L, \phi_L, \alpha_M, \phi_M)^T$ and F is the right hand side of (26) for the left eye and (27) for the right eye.

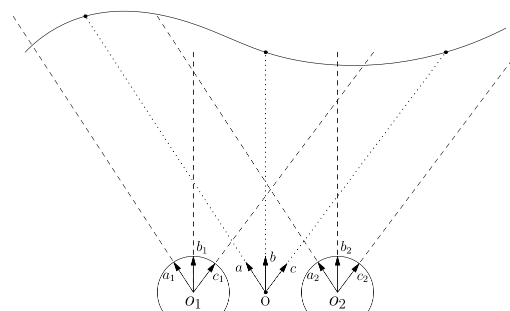


Fig. 1. Version eye movement in binocular vision while the eye tracks a trajectory defined in global coordinates. The vectors a_1, a_2 and a remain parallel. Likewise for the other triplets b_1, b_2, b and c_1, c_2, c.

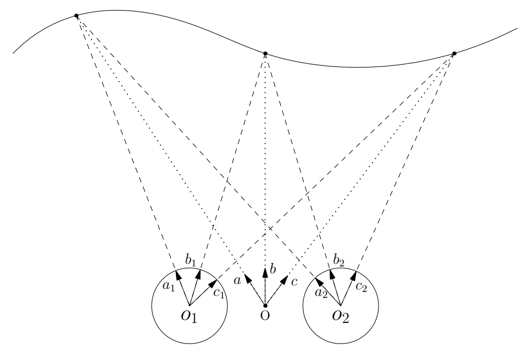


Fig. 2. Vergence eye movement in binocular vision while the eye tracks a trajectory. The vectors a_1, a_2 and a intersect at the target. Likewise for the other triplets b_1, b_2, b and c_1, c_2, c. The trajectory being tracked is the projection of the trajectory in Fig. 1 onto the local coordinates, viz. the head fixed coordinate rotated by the versional eye movement.

In order to obtain a necessary condition for optimality, we define the function

$$H(x, \dot{x}, p, \tau)\ =\ p^T\cdot F + \frac{1}{2}\tau^T\cdot\tau, \tag{30}$$

and write down the equation given by [5]

$$\ddot{x}\ =\ \frac{\partial H}{\partial p},\ \ \ddot{p}\ =\ \frac{\partial H}{\partial x} - \frac{d}{dt}\frac{\partial H}{\partial \dot{x}}, \tag{31}$$

5 The standard Hamilton's equation (see [1]), can be rewritten as the second order equation (31).

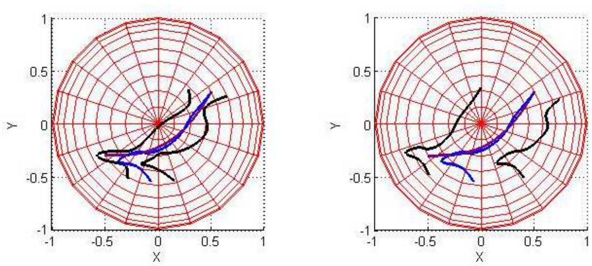
that are similar to the Hamilton's equations. The optimal values of the control are obtained by setting

$$\frac{\partial H}{\partial \tau} = 0.$$

In order to compute the optimal control, we would require to solve (31). Since only initial and final condition on $x(t)$ and $\dot{x}(t)$ is known, and we do not have any boundary condition on $p(t)$ and $\dot{p}(t)$, the optimal control problem we need to solve is a 'two point boundary value problem'.

8 Example

Simulation is carried out where we choose an a priori given target trajectory (shown in purple), that goes from left to right in Figs. 3a, 3b. The figures display only the gaze directions of the target. For the dynamical systems (26) and (27), optimal tracking controllers are computed that minimize the cost function (29). The optimal control is used to display the versional eye movement in Figs. 3a, 3b (shown in blue). Finally the combined effect of version and vergence eye movements are separately displayed in the figures for the left and the right eyes (shown in black).



(a) Target distance 4 units from the eye

(b) Target distance 2 units from the eye

Fig. 3. In the figures, the eye trajectories start below and go above. The purple curve shows the trajectory of the target being tracked and the blue curve shows the trajectory of the versional component. The two black curves show the trajectories of the two eyes (version + vergence) where the left trajectory corresponds to the left eye and the right trajectory corresponds to the right eye. The unit of distance is half the distance between the two eyes.

9 Conclusion

Binocular control problem is modeled using two separate control systems, consistent with the framework originally posed by Hering [10]. Defining appropriate submanifolds for each of the two systems, together with their corresponding Riemannian metric, we define 'version' and 'vergence' as a pair of dynamical systems. Subsequently we pose an optimal control problem in order to track a given trajectory in $\mathbb{R}^3$. The optimal control is computed by numerically solving the corresponding Two Point Boundary Value Problem.

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