the Coordinated Science Laboratory, University of Illinois. Since August 1982 he has been with M/A-COM LINKABIT, Inc., San Diego, CA. His interests include modeling and control of large scale systems with applications to power systems, economic models, and communication networks.

Petar V. Kokotovic (SM'74–F'80) did his graduate work with Prof. Mitrovic in his native Belgrade, Yugoslavia, in 1961–1963, and with Prof. Feldbaum and Prof. Tsypkin in Moscow, USSR, in 1966.

During this period he was with the Pupin Research Institute and the University of Belgrade. Since 1966 he has been on the Electrical Engineering Faculty, Coordinated Science Laboratory, University of Illinois, Urbana-Champaign. He has taught control theory and conducted research in modeling, sensitivity analysis, optimal and adaptive control, singular perturbations, and large scale systems. He has held visiting appointments with ETH Zürich, Switzerland, INRIA, France, Stanford University, and lectured at other American and European institutions. His consulting activities include General Electric Company, Ford Motor Company, and Systems Control, Inc.

Dr. Kokotovic has served on committees of the International Federation of Automatic Control (IFAC) and is an Associate Editor of the IFAC journal *Automatica*. He was also on the IDC and AdCom of the IEEE Control Systems Society.

Simultaneous Stabilization and Simultaneous Pole-Placement by Nonswitching Dynamic Compensation

BIJOY K. GHOSH AND CHRISTOPHER I. BYRNES, MEMBER, IEEE

Abstract—**In this paper, motivated by questions in fault tolerance, we investigate the existence of a compensator which simultaneously renders a given r-tuple of plants internally stable. Sufficient conditions are derived for simultaneous pole-assignability of the generic r-tuple by dynamic output feedback, which are also shown to be necessary (and equivalent to generic stabilizability) in the case where the number of either input or output channels is one. We also derive an upper bound on the order of a simultaneous pole-assigning compensator. If $r = 1$, this reduces to the condition derived by Brasch and Pearson, while if $r = 2$, this contains the recent theorem by Vidyasagar and Viswanadham. The cases $r \geqslant 3$ are new.**

I. Introduction and Statement of the Main Results

The "simultaneous stabilization problem"—in either discrete or continuous time—consists in answering the following question:

Given an r-tuple $G_1(s),\cdots,G_r(s)$ of $p \times m$ proper transfer functions, does there exist a compensator $K(s)$ such that the closed-loop systems $G_1(s)(I+K(s)G_1(s))^{-1},\cdots,$ $G_r(s)(I+K(s)G_r(s))^{-1}$ are (internally) stable?

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B. K. Ghosh is with the Division of Applied Sciences, Harvard University, Cambridge, MA.

C. I. Byrnes is with the Department of Mathematics and the Division of Applied Sciences, Harvard University, Cambridge, MA.

As pointed out in [15], this question arises in reliability theory, where $G_2(s),\cdots,G_r(s)$ represent a plant $G_1(s)$ operating in various modes of failure, and $K(s)$ is a nonswitching stabilizing compensator. Of course, for the same reason, it is important in the stability analysis and design of a plant which can be switched into various operating modes. The simultaneous stabilization problem can also apply to the stabilization of a nonlinear system which has been linearized at several equilibria. Finally, it has been shown [14], [20] that to solve the case $r = 2$ is to solve the well-known problem considered by Youla *et al.* in [21]: When can a single plant be stabilized by a stable compensator? This correspondence also serves to give some measure of the relative depth of this problem.

In order to describe the results obtained via this correspondence, we need some notation. First, set n_i = McMillan degree of $G_i(s)$. In the scalar input–output setting ($m = p = 1$), we regard each $G_i(s)$ as a point in $\mathbb{R}^{2n_i+1}$, viz. if

$$G_i(s) = p_i(s)/q_i(s)$$

where

$$p_i(s) = a_{oi} + \cdots + a_{n_i i}s^{n_i}$$

and

$$q_i(s) = b_{1i} + \cdots + b_{n_i i}s^{n_i-1} + s^{n_i}$$

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then $G_i(s)$ corresponds to the vector $(a_{oi},\cdots,a_{n_i i},b_{1i},\cdots,b_{n_i i})\in\mathbb{R}^{2n_i+1}$.

Moreover, since p_i and q_i are relatively prime, this vector lies in the open dense set $\mathrm{Rat}(n_i)\subset\mathbb{R}^{2n_i+1}$ (see [3] for the strictly proper case). In [14] Saeks and Murray used the techniques of fractional representations [8] and the correspondence mentioned above to give explicit inequalities defining the open set

$$U\subset \mathrm{Rat}(n_1)\times\mathrm{Rat}(n_2)$$

of pairs $(G_1(s),G_2(s))$ which are simultaneously stabilizable. In [20] Vidyasagar and Viswanadham showed, using similar techniques, that, provided $\max(m,p)>1$, the open set U of pairs $(G_1(s),G_2(s))$, which can be stabilized, is in fact dense.

This can be made precise by topologizing a point $G_i(s)$ in the set

$$\Sigma^{n_i}_{m,p}=\{\,p\times m\ G_i(s);\ \text{degree}\ G_i(s)=n_i\}$$

as a vector in $\mathbb{R}^{(2n_i+1)(mp)}$ via its Hankel parameters: If

$$G_i(s)=\sum_{j=0}^{\infty}H_{ij}s^{-j}$$

then $G_i(s)$ corresponds to the $n+1$ $p\times m$ block matrices $\{H_{i0},\cdots,H_{i,2n_i}\}$ which determines $G_i(s)$. It is known that $\Sigma^{n}_{m,p}$ is an $(n(m+p)+mp)$-manifold (see [7], [12], [5]), although this is not important here. What is important is that $\Sigma^n_{m,p}$ is a topological space.

One of our main results concerns the generic stabilizability problem, that is:

Question 1.1. Fix m, p, r, and n_i. Is the set U of r-tuples $G_1(s),\cdots,G_r(s)$, which can be simultaneously stabilized, open and dense in $\Sigma^{n_1}_{m,p}\times\cdots\times\Sigma^{n_r}_{m,p}$?

It is also important to ask, for reasons of global robustness of algorithms finding such a compensator, for compensators with a fixed degree of complexity.

Question 1.2. Fix m, p, r, and n_i. What is the minimal value of q (if one exists) for which the set W_q of r-tuples, which can be simultaneously stabilized by a compensator of degree $\leqslant q$, is open and dense in $\Sigma^{n_1}_{m,p}\times\cdots\times\Sigma^{n_r}_{m,p}$?

It should be noted that, in the case $r=1$, Question 1.2 is an outstanding, unsolved, classical problem. In this paper, we prove:

Theorem 1.1. In either discrete or continuous time, a sufficient condition for generic simultaneous stabilizability is

$$\max(m,p)\geqslant r. \tag{1.1}$$

Indeed, if (1.1) holds, then the generic r-tuple can be stabilized by a compensator of degree less than or equal to q, where q satisfies

$$q[\max(m,p)+1-r]\geqslant\sum_{i=1}^{r}n_i-\max(m,p). \tag{1.2}$$

In the case $r=1$, it is unknown whether generic stabilizability implies generic pole assignability; that is, whether or not these properties of m, n, and p are really different (see [4]). Perhaps not surprisingly then, Theorem 1.1 follows in the strictly proper case from

Theorem 1.2. A sufficient condition for generic simultaneous pole assignability of an r-tuple of strictly proper plants is (1.1), where the compensator $K(s)$ can be taken to be of degree q satisfying (1.2).

Here, simultaneous pole assignability means the assignability of r sets of self-conjugate sets of numbers $\{s_{1i},\cdots,s_{n_i+q,i}\}\subset\mathbb{C}$. In fact, sharper bounds on q can be obtained (see [18], [11]). Our proof relies on the recent pole-placement techniques derived for $r=1$ by Stevens in his thesis [18], which contains an improvement on existing results in the literature (see also [9], [17]). We have stated Theorem 1.2 only in the strictly proper case; the proper case involves more technical arguments from algebraic geometry which can be found in [11]. Indeed there we show that a sufficient condition for generic simultaneous pole assignability is (1.1), where the compensator $K(s)$ is taken to be of degree q satisfying

$$q[\max(m,p)+1-r]+\max(m,p)-r\geqslant\sum_{i=1}^{r}n_i.$$

Furthermore, if the closed-loop poles are topologized by the coefficients of the r closed-loop characteristic polynomials of degree (n_i+q) $i=1,2,\cdots,r$, a sufficient condition for generic simultaneous pole assignability of all but possibly a proper algebraic subset of poles is (1.1) where q satisfies (1.2). We shall, however, give an independent proof of Theorem 1.1 in the nonstrictly proper case, based on the equivalence of generic stabilizability and existence of a solution to a generic "deadbeat control" problem, which we can solve if (1.1) is satisfied. This argument extends the argument given in [4] for the case $r=1$ and $q=0$.

Note that if $r=1$, then (1.1) is always satisfied, in which case (1.2) implies:

Corollary 1.3. (Brasch–Pearson [2]). The generic $p\times m$ plant $G(s)$ of degree n can be stabilized by a compensator of order q, where q satisfies

$$(q+1)\max(m,p)\geqslant n. \tag{1.3}$$

If $r=2$ and $\max(m,p)>1$, then (1.1) is again satisfied, so we obtain rather easily:

Corollary 1.4. (Vidyasagar–Viswanadham [20]). If $r=2$ and $\max(m,p)>1$, then the generic pair $(G_1(s),G_2(s))$ is simultaneously stabilizable.

Moreover, in this case, we know an upper bound on the order of the required compensator. For example, if $m=p=2$, $r=2$, then q can be taken to satisfy

$$q\geqslant n_1+n_2-2.$$

On the other hand, in [20] the explicit conditions defining the closed set

$$\Sigma^{n_1}_{m,p}\times\Sigma^{n_2}_{m,p}-U$$

of pairs not simultaneously stabilizable were derived. Such conditions can be derived from our proof, but instead we refer to [10], where Theorem 1.1 (excepting (1.2)) is proved by interpolation methods also yielding a set of explicit conditions in the range $r \leqslant \max(m, p)$.

Finally, we prove that the condition (1.1) is sharp in the following sense:

Theorem 1.5. If $\min(m, p) = 1$, then for fixed m, p, r, and n_i the following statements are equivalent for proper plants.

(i) $q \in \mathbb{N}$ satisfies $q(\max(m, p) + 1 - r) + \max(m, p) \geqslant \sum_{i=1}^r n_i$.

(ii) The generic r-tuple $G_1(s), \cdots, G_r(s)$ is simultaneously stabilizable in discrete or continuous time by a compensator of degree $\leqslant q$.

(iii) The generic r-tuple $G_1(s), \cdots, G_r(s)$ is simultaneously stabilizable in discrete or continuous time.

In the strictly proper case, it follows that (i)–(iii) is also equivalent to generic simultaneous pole assignability. This holds in the proper case as well, in case we ask for generic simultaneous pole assignability of all but a proper algebraic subset of poles, but requires a separate argument [11].

Corollary 1.6. If $\min(m, p) = 1$ and $r \leqslant \max(m, p)$, then the generic r-tuple is simultaneously stabilizable by a compensator of order precisely given by the least integer q satisfying (1.2).

As a further corollary, we obtain one of the results obtained by Saeks and Murray in [14] (see also [15]):

Corollary 1.7. (Saeks–Murray [14]). Suppose $m = p = 1$ and $r = 2$. Simultaneous stabilizability is not a generic property.

We remark that these results hold also over the field $\mathbb{C}$ of complex numbers—in particular, the complex analogue of Corollary 1.7 dispels a folklore conjecture concerning simultaneous stabilization using compensators with complex coefficients.

Finally, over any field, the method of proof of Theorem 1.2 gives linear equations for a compensator simultaneously placing $\sum n_i + rq$ poles when the generic hypothesis is satisfied.

II. Pole Placement and the Generalized Sylvestor Matrix: A Proof of Theorem 1.2

In this section we proceed to prove Theorem 1.2. Note that Theorem 1.1 and Corollaries 1.3 and 1.4 follow immediately in the strictly proper case from this theorem. Without any loss of generality we can assume that $m \geqslant p$, for if $K(s)$ stabilizes $G_i^t(s)$, then $K^t(s)$ stabilizes $G_i(s)$.

Suppose, first of all, that $p = 1$, so that we are given a set of r, m input 1 output plants of Mcmillan degree $\leqslant n_i$ represented as

$$\left[\sum_{j=0}^{n_i} p^i_{m+p,j} s^j\right]^{-1} \left[\sum_{j=0}^{n_i} p^i_{1j} s^j, \cdots, \sum_{j=0}^{n_i} p^i_{m+p-1,j} s^j\right] \quad (2.1)$$

for $i = 1, 2, \cdots, r$. A 1-input–m-output compensator of Mcmillan degree $\leqslant q$ is represented as

$$\left[\sum_{j=0}^{q} a_{m+p,j} s^j\right]^{-1} \left[\sum_{j=0}^{q} a_{1j} s^j, \cdots, \sum_{j=0}^{q} a_{m+p-1,j} s^j\right] \quad (2.2)$$

with the restriction $p^i_{m+p,n_i}, a_{m+p,q} \neq 0 \ \forall i = 1, 2, \cdots, r$.

Note that in (2.1) and (2.2) the coefficients p^i_{kj} $\forall i$ and a_{kj} has been defined up to a nonzero scale factor. Moreover, for a strictly proper plant or compensator, $p^i_{kn_i} = 0$, $a_{kq} = 0 \ \forall k = 1, \cdots, m + p - 1$; $i = 1, \cdots, r$.

The associated return difference equation, $\det[I + K(s)G_k(s)] = 0$, is given by

$$\Pi_i(s) = \sum_{k=1}^{m+p} \left[\sum_{j=0}^{n_i} p^i_{kj} s^j\right]\left[\sum_{j=0}^{q} a_{kj} s^j\right] \triangleq \sum_{j=0}^{n_i+q} c_{ij} s^j \qquad \forall i = 1, 2, \cdots, r. \quad (2.3)$$

A generic r-tuple of plants defines a mapping χ, via (2.3), between the compensator parameters and the coefficient of the return difference polynomials given by

$$\chi : \mathbb{R}^{(q+1)(m+1)} \to \mathbb{R}^{\sum n_i + r(q+1)} \quad (2.4)$$

where

$$\chi(A_1, \cdots, A_{m+p}) = (A_1, \cdots, A_{m+p})\bar{S} = \left(c_{10}, \cdots, c_{1n_1+q}, \cdots, c_{r0}, \cdots, c_{r,n_r+q}\right) \quad (2.5)$$

where

$$\bar{S} = \operatorname{col}(Q_1, \cdots, Q_{m+p})$$
$$Q_k = (P_{k1}, P_{k2}, \cdots, P_{kr}) \quad (2.6)$$

$$P_{ki} = \underbrace{\begin{bmatrix} p^i_{k0} & p^i_{k1} & \cdots & \cdots & p^i_{kn_i} & & \bigcirc \\ & \ddots & & & & \ddots & \\ \bigcirc & & p^i_{k0} & p^i_{k1} & & & p^i_{kn_i} \end{bmatrix}}_{n+q+1} \Big\updownarrow q+1 \quad (2.7)$$

$$A_k = (a_{k0}, a_{k1}, \cdots, a_{kq}). \quad (2.8)$$

The matrix $\bar{S}$ in (2.5) is of order $(q+1)(m+p)$ by $\sum n_i + r(q+1)$. By row and column transposition, $\bar{S}$ can be reduced to the form

$$\bar{S}' = \overbrace{\begin{bmatrix} P'_0 & \cdots & \cdots & \cdots & P'_n & & \bigcirc \\ & \ddots & & & & \ddots & \\ \bigcirc & & P'_0 & \cdots & \cdots & \cdots & P'_n \end{bmatrix}}^{(n+q+1)} \Big\updownarrow (q+1) \quad (2.9)$$

where $n = \max n_i$ and P_j', $j = 0, 1, \cdots, n$ have been appropriately defined. $\bar{S}'$ is classically known as the generalized Sylvestor matrix. For $n_i = n \ \forall i$ and $m + p \geqslant r$, its rank, as computed by Bitmead *et al.* [1], is given by

$$(q+1)(m+p) - \sum_{i:\, \nu_i \leqslant q+1} (q+1-\nu_i) \tag{2.10}$$

where ν_i is the observability index of the $(m+p-r)\times r$ transfer function $H = DC^{-1}$ and where

$$D = \sum_{i=0}^{n} \bar{P}_i' s^{n-i} \qquad C = \sum_{i=0}^{n} \bar{P}_i'' s^{n-i} \tag{2.11}$$

where

$$P_i' = \mathrm{col}\left(\bar{P}_i', \bar{\bar{P}}_i'\right) \quad \text{and} \quad i = 0, 1, \cdots, n.$$

We now state the following:

Lemma 2.1. The generalized Sylvestor matrix is of full rank for a generic r-tuple of proper m-input–1-output plants.

Note: For simplicity, we prove this lemma for the restricted case $n_i = n \ \forall i$. The proof of the more general case has been sketched in [11] wherein we have explicitly constructed a principal minor of $\bar{S}$, the generalized Sylvestor matrix which has nonzero determinant for a generic r-tuple of plants.

Proof: Assume $n_i = n \ \forall i$. Without any loss of generality let $m + p \geqslant r$ for otherwise the rows of $\bar{S}'$ are clearly independent. Notice that (2.11) defines a bijection between an r-tuple of plants and a $(m+p-r)\times r$ transfer function H of Mcmillan degree rn. Thus a generic r-tuple of plants correspond to an H with observability indices given by ν_0 or $\nu_0 + 1$ where ν_0 is the largest integer less than or equal to $rn/(m+p-r)$. Thus noting that

$$\sum_{i=1}^{m+p-r} \nu_i = rn$$

the rank of the generalized Sylvestor matrix is given by (2.10) as

$$\min\left[(q+1)(m+p), r(n+q+1)\right]. \qquad \text{Q.E.D}$$

Lemma 2.2. Assume $\min(m, p) = 1$. A sufficient condition for generic pole assignment, for an r-tuple of strictly proper plants by a proper compensator is given by

$$(q+1)(m+p-r) \geqslant \sum_{i=1}^{r} n_i - r + 1. \tag{2.12}$$

Proof: Assume $a_{m+p,q} = 1$ and $p^i_{m+p,n_i} = 1$, $c_{i,n_i+q} = 1$ $\forall i = 1, 2, \cdots, r$. Since the last column of P_{ki} $\forall i = 1, 2, \cdots, r$ is identically equal to $[0, 0, \cdots, 1]^T$ a sufficient condition for generic pole assignment is that χ' is onto. Here the mapping

$$\chi' : \mathbb{R}^{(q+1)(m+p)-1} \to \mathbb{R}^{\Sigma n_i + rq} \tag{2.13}$$

is given by

$$\chi'(A_0, \cdots, A_{m+p-1}, A'_{m+p}) = (A_0, \cdots, A_{m+p-1}, A'_{m+p})\bar{S}_1 \tag{2.14}$$

where

$$A'_{m+p} = (a_{m+p0}, \cdots, a_{m+p,q-1})$$

and $\bar{S}_1$ is obtained from $\bar{S}$ of (2.6) by deleting the last column of P_{ki} $\forall i = 1, \cdots, r$ and the last row of Q_{m+p}. It is easy to see that when (2.12) is satisfied, the generic rank of $\bar{S}$ is unaltered by deleting its last row. By applying Lemma 2.1 and specializing $p^i_{kn_i} = 0 \ \forall i, k$ $\bar{S}_1$ may be shown to be a full rank matrix of order $[(q+1)(m+p)-1]\times r(n+q)$. Therefore, a sufficient condition for generic pole placement is given by (2.8). Q.E.D.

The proof of Theorem 1.2 now proceeds by a reduction to the case $\min(m, p) = 1$, which has been treated in Lemmas 2.1–2.2. This procedure, which is called "vectoring down", is adopted from the case $r = 1$, studied in Stevens' thesis [18], and from Brasch–Pearson [2].

Lemma 2.3. Given an r-tuple of $p \times m$ plants $G_i(s)$ of degrees n_i, each with n_i simple poles, there is an open dense set of $1 \times p$ vectors $v \in \mathbb{R}^p$ such that $vG_i(s)$ has degree n_i, for all i.

Proof: If $r = 1$, then we may expand $G(s)$

$$G(s) = \sum_{i=1}^{n} \frac{R_i}{s - \lambda_i}$$

in a partial fraction expansion, where $\lambda_i \in \mathbb{C}$ and each R_i has rank 1. Now, the set U_1 of real vectors v such that $vR_1 \neq 0$ is clearly open and dense in $\mathbb{R}^p$. Defining $U_2, \cdots, U_n$ similarly, set

$$V = \bigcap_{i=1}^{n} U_i.$$

Thus V is an open dense set of vectors with the required property.

If $r > 1$, one obtains, as above, sets $V_1, \cdots, V_r$ in $\mathbb{R}^p$ having an open dense intersection $\cap_{i=1}^{n} V_i$. Q.E.D.

Lemma 2.4. Given an r-tuple of $p \times m$ plants $G_i(s)$, there exists a constant gain output feedback K such that the closed-loop systems $G_i(s)(I + KG_i(s))^{-1}$ have distinct simple poles.

Proof: For $r = 1$, the set W_1 of K such that the closed-loop system has simple poles is the complement in $\mathbb{R}^{mp}$ of an algebraic set. It is well known [2] that this set is nonempty; therefore, W_1 is open and dense. Taking any K in the open dense set $\cap_{i=1}^{r} W_i$ gives the desired conclusion. Q.E.D.

Thus choosing any $(v, K) \in \mathbb{R}^p \times \mathbb{R}^{mp}$ we have a mapping from an open dense set

$$\Phi_{(v,K)} : \Sigma^{n_1}_{m,p} \times \cdots \times \Sigma^{n_r}_{m,p} \to \Sigma^{n_1}_{m,1} \times \cdots \times \Sigma^{n_r}_{m,1}$$

$$\Phi_{(v,K)}\left(G_i(s)\right)_{i=1}^{r} = \left(vG_i(s)\left(I + KG_i(s)\right)^{-1}\right)_{i=1}^{r}$$

which is rational in the Hankel parameters (H_{ij}) of (G_i). Applying Lemmas 2.1–2.2 to the case $\min(m, p) = 1$, i.e.,

$$\Sigma^{n_1}_{m,1} \times \cdots \times \Sigma^{n_r}_{m,1}$$

gives—via composition with Φ—an open dense set of

$$\Sigma^{n_1}_{m,p} \times \cdots \times \Sigma^{n_r}_{m,p}$$

which can be simultaneously pole assigned. Q.E.D.

III. Generic Stabilizability Condition of an r-Tuple of Proper Plants

In this section we proceed to prove Theorem 1.1 independent of Theorem 1.2. We first show that the following three statements are equivalent.

I) A generic r-tuple of proper plants is stabilizable with respect to the open left-half plane by a proper compensator of degree $\leqslant q$.

II) A generic r-tuple of proper plants is stabilizable with respect to the interior of the unit disc, by a proper compensator of degree $\leqslant q$.

III) A generic r-tuple of proper plants is pole assignable at the origin by a proper compensator of degree $\leqslant q$.

Lemma 3.1. I $\Leftrightarrow$ II.

Proof: Consider the conformal transformation

$$\phi(s)=(s+1)/(s-1) \tag{3.1}$$

which maps the r-tuple of proper plants $g_1, g_2,\cdots,g_r$ onto the r-tuple of proper plants $g_1',\cdots,g_r'$ where $g_i'(s)=g_i(\phi(s))$ except for the algebraic set of plants satisfying "$g_i(s)$ has a pole at $s=1$ for some $i=1,\cdots,r$". The proof now follows from the two facts.

1) $\phi(s)$ maps the open left-half plane onto the interior of the unit disc.

2) The mapping

$$(g_1,\cdots,g_r)\mapsto(g_1',\cdots,g_r')$$

and its inverse, map the generic r-tuple of proper plants to the generic r-tuple of proper plants. Q.E.D.

Lemma 3.2. II $\Leftrightarrow$ III.

Proof: Sufficiency is clear and follows by an analogous argument of Lemma 3.1 with $\phi(s)=s+a$, $a>0$, $a\in\mathbb{R}$.

To prove necessity, we have the following: For each $r=1,2,\cdots$,(shown easily by assuming statement II and considering $\phi(s)=as$, $a>0$, $a\in\mathbb{R}$). $\exists$ an open dense set of U_r of r-tuple of plants for which there exist a compensator of degree $\leqslant q$ which places the poles in the interior of the disc D_r of radius $1/r$ centered at the origin. Consider the set

$$U=\bigcap_{r=1}^{\infty} U_r.$$

Clearly, U is a dense set by the Baire Category Theorem [13]. Since the mapping χ given by (2.4) is linear, it has a closed image. Moreover, every r-tuple of plants in U admits a sequence of compensators which places the poles arbitrary close to the origin. Thus for generic r-tuples, U is contained in the set V of all r-tuple of plants for which there exists a compensator which places the poles at the origin. By the Tarski [19]–Seidenberg [16] theory of elimination over $\mathbb{R}$, V is indeed defined by union and/or intersection of sets given by polynomial equations or inequations $f_\alpha>0$, $f_\beta=0$. Finally, since U is dense in V, $f_\beta(U)=0\Rightarrow f_\beta\equiv 0$ so that V is defined by strict polynomial inequalities. Hence V is open. Moreover, since U is dense, V is also dense. Q.E.D.

Lemma 3.3. For a generic r-tuple $(r\leqslant m+p)$ of $\min(m,p)=1$ plants

$$\text{III}\Leftrightarrow(q+1)(m+p)\geqslant\sum_{i=1}^{r}n_i+rq+1.$$

Proof: We want to obtain a necessary and sufficient condition that there exists $(A_1,\cdots,A_{m+p})\in\mathbb{R}^{(q+1)(m+p)}$ such that

$$a_{m+p,q}\neq 0 \tag{3.2}$$

$$(A_1,\cdots,A_{m+p})\bar{S}=(\underbrace{0,\cdots,0,f_1}_{\leftarrow n_1+q+1\rightarrow}|\underbrace{0,\cdots,f_2}_{\leftarrow n_2+q+1\rightarrow}|,\cdots,|\underbrace{0,\cdots,f_r}_{\leftarrow n_r+q+1\rightarrow}) \tag{3.3}$$

for some $f_i\neq 0$, $i=1,2,\cdots,r$.

Equation (3.3) may be written as

$$(A_1,\cdots,A_{m+p})S_1=\underbrace{(0,\cdots,0)}_{\leftarrow rq+\Sigma n_i\rightarrow} \tag{3.4}$$

$$(A_1,\cdots,A_{m+p})S_2=(f_1,\cdots,f_r) \tag{3.5}$$

where S_1, S_2 may be appropriately defined with S_1 of order $(q+1)(m+p)$ by Σn_i+q.

(Necessity): If $(q+1)(m+p)\leqslant\Sigma n_i+rq$, the unique solution of (3.3) is given by

$$(A_1,\cdots,A_{m+p})=\mathbf{0} \tag{3.6}$$

since, for a generic r-tuple of plants, S_1 is of full rank. This can be easily seen by our arguments in Lemma 2.2. Finally, note that (3.6) does not satisfy (3.2), (3.3).

(Sufficiency): Under the condition $(q+1)(m+p)\geqslant\Sigma n_i+rq+1$, there exists a vector $(A_1,\cdots,A_{m+p})$ satisfying (3.4) and (3.2). This again follows from the fact that generically S_1 is of full rank and its rank is unaltered by deleting its last row. Finally, to see that $(A_1,\cdots,A_{m+p})$ can also satisfy (3.5) let b_j, $i=1,2,\cdots,r$, be the columns of S_2. Then the proof follows by noting that generically

$$\dim\operatorname{Ker}S_1>\dim\operatorname{Ker}[S_1|b_j],\qquad j=1,\cdots,r.$$

Q.E.D.

Theorem 1.1 then follows from Lemma 3.1, 3.2, 3.3, and the vectoring down technique used in the proof of Theorem 1.2 in Section II.

IV. Proof of Theorem 1.5

To say there exists $q\in\mathbb{N}$ satisfying (1.2) is to $\max(m,p)\geqslant r$. Thus (ii) follows from (i) by Theorem 1.1.

(ii) $\Rightarrow$ (iii) since (iii) is weaker than (ii).

By Lemma 3.1, in order to prove (iii) $\Rightarrow$ (i), it suffices to assume that $G_1(s),\cdots,G_r(s)$ are simultaneously stabilizable in continuous time.

Proposition 4.1. The generic $(m+1)$-tuple of $1\times m$ proper continuous-time plants of degree n is not simultaneously stabilizable by a proper compensator of finite (but not *a priori* bounded) degree.

Proof: Consider the domain of (simultaneous) stability

$$\mathfrak{D} = \left\{(c_{ij}): \sum_{j=0}^{n_i+q} c_{ij}s^j \text{ has all roots in the open left-half complex plane, } i=1,2,\cdots,r\right\}$$

and its convex hull $\Omega(\mathfrak{D}) \subset \mathbb{R}^{n_1+q} \times \cdots \times \mathbb{R}^{n_r+q}$. Clearly, a necessary condition for generic simultaneous stabilizability is

$$\text{image}\,(\chi_\eta) \cap \Omega(\mathfrak{D}) \neq \varnothing$$

for an open dense set of η. Since

$$\Omega(\mathfrak{D}) \subset \{(c_{ij}): c_{ij} > 0\}$$

it will suffice to prove:

Lemma 4.2. If $r = m + p$, then there exists an open set of r-tuples η such that image (χ_η), as defined in (2.5), contains no vector with only positive entries.

We fix the value of q and construct the associated Sylvestor matrix $\bar{S}'$ as given by (2.9). We claim that the open set E of plants defined by

$$E \triangleq \{(P_0', P_1', \cdots, P_n') | P_0'^{-1\eta} \text{ has all the entries negative, where } \eta \text{ is a nonzero column of } P_j', j=1,2,\cdots,\eta\}$$

or in other words $\exists \boldsymbol{\alpha} | \alpha_i > 0 \ \forall i = 1,2,\cdots,\Sigma n_i + r(q+1)$

$$\boldsymbol{a}\bar{S}' = \boldsymbol{\alpha} \tag{4.1}$$

has a solution. Writing $\bar{S}'$ as

$$\bar{S}' \triangleq [S' \quad S'']$$

where

$$S' = \begin{bmatrix} P_0' & P_1' & \cdots & P_q' \\ 0 & P_0' & \cdots & P_{q-1}' \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & P_0' \end{bmatrix} \tag{4.2}$$

and $P_j' = 0$ for all $j > n$.

Equation (4.1) can be written as

$$\boldsymbol{a}'[I | S'^{-1}S''] = \boldsymbol{\alpha} \tag{4.3}$$

where S'^{-1} is given as follows

$$S'^{-1} = \begin{bmatrix} X_0 & X_1 & \cdots & X_q \\ & X_0 & \cdots & X_{q-1} \\ \cdots & \cdots & \cdots & \cdots \\ & & & X_0 \end{bmatrix}$$

where $X_0 = P_0'^{-1}$

$$-P_0'^{-1}(P_1', P_2', \cdots, P_{r+1}') \begin{bmatrix} X_r \\ X_{r-1} \\ \vdots \\ X_0 \end{bmatrix} = X_{r+1}$$

$$\forall r = 0, \cdots, q-1.$$

The identity matrix of order $(q+1)(m+p)$ in (4.3) forces a' to have all the entries positive. Moreover, since $\eta \in E$, $S'^{-1}S''$ has all its entries negative so that $\boldsymbol{a}'(S'^{-1}S'')$ has all the entries negative which is a contradiction since $\boldsymbol{\alpha}$ is a positive vector.

Finally it is shown that E is not an empty set. For a fixed $P_0' = P_0^*$ choose the vector δ to be so that $P_0^{*-1}\delta$ has all its entries negative. Define the nonzero columns of P_j' to be δ for $j = 1, \cdots, n$ and call it P_j^* so that

$$(P_0^*, P_1^*, \cdots, P_n^*) \in E. \qquad \text{Q.E.D.}$$

Remark: If image (χ_η) is affine hyperplane, then the necessary condition

$$\text{image}\,(\chi_\eta) \cap \Omega(\mathfrak{D}) \neq \varnothing$$

of course is sufficient, i.e., implies

$$\text{image}\,(\chi_\eta) \cap \mathfrak{D} \neq \varnothing.$$

This fact was used by Chen, together with

Lemma 4.3. (Chen [6]). If $r = 1$, $\Omega(\mathfrak{D}) = \{(c_1, \cdots, c_n): c_i > 0\}$

to give precise conditions for stabilizability in the case $r = 1$, $q = 0$, $\min(m, p) = 1$, and $\max(m, p) = n - 1$. Chen's technique can be adapted in the cases $r \geq 1$ to give explicit conditions—in certain cases—defining the open set of simultaneously stabilizable plants when $r > \max(m, p)$ (see [11]).

Note that Corollary 1.6 now follows from our previous results on the generic rank of the generalized Sylvestor matrix, while Corollary 1.7 follows either from Theorem 1.5 or Proposition 4.1.

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Bijoy K. Ghosh was born in Calcutta, India, on March 5, 1956. He received the B.Tech. degree from the Birla Institute of Technology and Science, Pilani, India, and the M.Tech. degree in electrical engineering from the Indian Institute of Technology, Kanpur, India. Currently he is a graduate student in the Division of Applied Sciences at Harvard University, Cambridge, MA.

His primary area of research has been in large scale systems with particular emphasis on reliability, especially the simultaneous pole placement and stabilization problems. He is also interested in the two-dimensional spectral estimation problems.

Christopher I. Byrnes (M'81) received the Ph.D. degree from the University of Massachusetts, Amherst, in 1975.

From 1975 to 1978 he served as Instructor of Mathematics at the University of Utah, and held visiting positions at the University of Kansas, Harvard University, and Stanford University. Since 1978 he has been on the faculty at Harvard University in the Department of Mathematics and in the Division of Applied Sciences where he currently holds an appointment as Associate Professor. His primary areas of research have been in algebraic system theory and in the applications of algebraic geometry and topology to linear system theory, especially to global analysis and the frequency domain analysis of linear, multivariable systems. He is an Associate Editor of the journal *Systems and Control Letters*, as well as the Editor of the volumes *Partial Differential Equations and Differential Geometry, Proc. 1977 Park City Conference* (New York: Marcel Dekker, June 1977), *Geometrical Methods for the Theory of Linear Systems* (with C. F. Martin) (Dordrecht: D. Reidel, 1980), and *Algebraic and Geometric Methods in Linear System Theory* (with C. F. Martin), vol. 18 in *Lectures in Applied Mathematics* (Providence, RI: American Mathematical Society).

Dr. Byrnes is a member of the American Mathematical Society and the Society for Industrial and Applied Mathematics.

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