Syllabus and suggested problems for the mid term. are included here.

Please go through

- suggested problems.
- H.W. 1
- H.W 2 (problem 2).

while preparing for the midterm.

Material to revise for the test:

- Vector space, basis, linear independence of vectors.
- Linear Transformation, Range, Null space.
- Inner product of vector.
- Adjoint of a linear transformation.
- Trace and determinant of a square matrix.
- Eigenvalues, Eigenvectors, gen. eigenvectors, similarity trans. Jordan canonical forms.

- Sol^{n} to homogeneous eq^{n}.

  $$\dot{x} = A(t)x(t).$$

  Peano Baker series.

  Fundamental matrix. $\phi(t, t_0)$.

- Calculating exponential of matrices.

- Controllability / controllability Gramian.

- Rank test for controllability of Time Invariant systems.

  $$\dot{x} = Ax + bu.$$

Note: Floquet Theory will be excluded from the test.

1. Let V be a set of matrices defined as follows:

(a) $V_1 \triangleq \{ A \in \mathbb{R}^{2\times 2} : \text{trace } A = 0 \}$.

Is V a vector space under usual matrix addition and scalar multiplication?

(b) $V_2 \triangleq \{ A \in \mathbb{R}^{2\times 2} : \det A = 0 \}$.

Is V a vector space under usual matrix addition and scalar multiplication?

Find a basis for those V that are vector spaces?

2. Is it true that

$$e^{(A+A^{-1})t} = e^{At} e^{A^{-1}t}$$

$$e^{(A+A^{3})t} = e^{At} e^{A^{3}t}$$

why or why not?

3. Consider a mapping L from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$ as follows:

$$L : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$$

$$A \longmapsto \frac{1}{2}\left[A - A^T\right]$$

(a) Is L a linear transformation? Argue why or why not.

(b) Describe the range and nullspace of L

(c) Calculate rank and nullity of L.

(d) Can you describe L using any matrix.

- you need to choose a basis of $\mathbb{R}^{2\times 2}$
- you need to decide an order of the matrix
- Find it.

(e) Define inner product in $\mathbb{R}^{2\times 2}$ as follows:

$$\langle A, B \rangle = \mathrm{trace}(AB^T).$$

What would be the adjoint of L??

(f) Describe the null space of adj L and show that this space is perpendicular to every vector in the range space of L.

(4)
$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \\ 0 & t \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

Assume $z_1(0) = z_2(0) = z_3(0) = 0$. Describe the set of all pts in $\mathbb{R}^3$ that you can reach by a suitable choice of $u_1(t), u_2(t), 0 \le t \le 1$

- Take $t_0 = 0, t_1 = 1$
- Find range of $W(0,1)$.

(5) Is the state equation on Lec 3, page 29, controllable??

- calculate rank $[B, AB, A^2B]$.

6 Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

(i) Calculate A^{99}.

(ii) Calculate e^{At}.

7 Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(i) Calculate A^{103}

(ii) Calculate e^{At}.

8 Consider the state space eqn.

$$\dot{\underline{x}} = A\underline{x} + bu$$

where A is given by problem 7

$$b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{x}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$u(\sigma) = 1 \quad \sigma \geqslant 0$$

(i) Calculate $\underline{x}(t)$.

(ii) Write down the controllability gramian matrix $\overline{W}(0,1)$ from page 12, Lec 4.

(5)

(9) Let $A(t)$ be a square matrix :

$$A(t)\int_{t_0}^{t} A(\sigma)\,d\sigma =$$

$$\left[\int_{t_0}^{t} A(\sigma)\,d\sigma\right] A(t) \quad \forall t.$$

then.

$$\phi(t,t_0) = \exp\left[\int_{t_0}^{t} A(\sigma)\,d\sigma\right]$$

(show the above fact).

If $A(t) = \begin{pmatrix} t & t^2 \\ t^2 & t \end{pmatrix}$

(i) Calculate the 1st 3 terms of the $\phi(t,t_0)$.

(ii) Calculate $\det[\phi(t,t_0)]$ using Lec 5 page 28.

10 Cayley Hamilton Th$^{\underline{m}}$ says that if A is a $n \times n$ matrix, A^n can be written as a linear combination of lower powers of A. ie.

$$A^n = \alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1}.$$

It would follow that

$$e^A = \beta_0 I + \beta_1 A + \cdots + \beta_{n-1} A^{n-1}.$$

if A is $n \times n$.

Using the above recipe, calculate

$$\phi(t, 0)$$

in problem 9.