A solution to Midterm 1

1 Ans:

char poly $p(\lambda)$ of A is given by

$$p(\lambda) = (\lambda+1)^2$$

Eigenvalues at $-1, -1, -1$.

To find eigenvector v_1 we solve.

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -u_1 \\ -u_2 \\ -u_3 \end{pmatrix}$$

$\Rightarrow u_2 + u_3 = 0, \; u_3 = 0 \Rightarrow u_2 = u_3 = 0$

Hence

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

To find gen. eigenvector V_2 we solve

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -u_1 \\ -u_2 \\ -u_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{matrix} u_2 + u_3 = 1 \\ u_3 = 0 \end{matrix} \quad \Rightarrow \quad u_2 = 1, u_3 = 0$$

Hence

$$V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

To find gen. eigenvector V_3 we solve

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -u_1 \\ -u_2 \\ -u_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{matrix} u_2 + u_3 = 0 \\ u_3 = 1 \end{matrix} \quad \Rightarrow \quad \begin{matrix} u_2 = -1 \\ u_3 = 1. \end{matrix}$$

Hence $V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

Matrix P is defined as

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

To compute P^{-1} we write the augmented matrix

$$\left(\overbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix}}^{P} \left| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right.\right)$$

and reduce it to

$$\left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \left| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}}_{P^{-1}} \right.\right)$$

Finally we have

$$P^{-1} A P =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = B.$$

$\blacksquare$

2 Ans:

(i) Write

$$e^{At^2} = \alpha_0 I + \alpha_1 A.$$

A has repeated eigenvalues at

$$\lambda_1 = -3 \quad \lambda_2 = -3.$$

Replace A by λ we get

$$e^{\lambda t^2} = \alpha_0 + \alpha_1 \lambda$$

&

$$t^2 e^{\lambda t^2} = \alpha_1 \quad \left(\text{Taking derivative w.r.t. } \lambda\right)$$

Substituting $\lambda = -3$, we obtain.

$$\boxed{\begin{aligned} e^{-3t^2} &= \alpha_0 - 3\alpha_1 \\ t^2 e^{-3t^2} &= \alpha_1 \end{aligned}}$$

We compute

$$e^{At^2} = \alpha_0 I + \alpha_1 A$$

$$= \begin{pmatrix} e^{-3t^2} & t^2 e^{-3t^2} \\ 0 & e^{-3t^2} \end{pmatrix}$$

(ii) If $\dot{\underset{\sim}{X}} = 2tA\underset{\sim}{X}$

we have

$$\underset{\sim}{X}(t) = \exp\left[\left(\int_0^t 2\sigma \, d\sigma\right) A\right] \underset{\sim}{X}(0)$$

because $2tA$ & $\int_0^t 2\sigma A \, d\sigma$ commutes as a matrix pair.

Thus

$$\underset{\sim}{X}(t) = \exp\left[t^2 A\right] \underset{\sim}{X}(0).$$

since

$$\underline{X}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

we have

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{-3t^2} & t^2 e^{-3t^2} \\ 0 & e^{-3t^2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} (t^2+1)e^{-3t^2} \\ e^{-3t^2} \end{pmatrix}$$

3 Ans

(i) $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b} u$$

$$(b \mid Ab) = \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix}$$

$$\text{rank}(b \mid Ab) = 2$$

Hence $\circledast$ is controllable.

(ii)

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$e^{-At} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$$

$$e^{-At} b = \begin{pmatrix} 2-3t \\ 3 \end{pmatrix}.$$

Denote by W the controllability gramian matrix

$$W = \int_0^1 e^{-A\tau} b b^T e^{-A^T\tau} d\tau .$$

$W =$

$$\int_0^1 \begin{pmatrix} 2-3\tau \\ 3 \end{pmatrix} \begin{pmatrix} 2-3\tau & 3 \end{pmatrix} d\tau$$

$$= \int_0^1 \begin{pmatrix} (2-3\tau)^2 & 6-9\tau \\ 6-9\tau & 9 \end{pmatrix} d\tau .$$

$$= \left. \begin{pmatrix} 4\tau-6\tau^2+3\tau^3 & 6\tau-\frac{9}{2}\tau^2 \\ 6\tau-\frac{9}{2}\tau^2 & 9\tau \end{pmatrix} \right|_0^1$$

$$= \begin{pmatrix} 1 & 3/2 \\ 3/2 & 9 \end{pmatrix}$$

W is of full rank indicating that $*$ is controllable.

(iii) Using variation of constants formula we write

$$\underline{x}(t) = e^{At}\,\underline{x}(0) + \int_0^t e^{A(t-\tau)}\, b\, u(\tau)\, d\tau .$$

$$\Rightarrow e^{-At}\,\underline{x}(t) - \underline{x}(0) = \int_0^t e^{-A\tau}\, b\, u(\tau)\, d\tau .$$

for $t=1$, $\underline{x}(0)=0$, $\underline{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we write

$$e^{-A1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \int_0^1 e^{-A\tau}\, b\, u(\tau)\, d\tau .$$

$$e^{-A1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \Rightarrow e^{-A1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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For $u(\tau) = b^T e^{-A^T\tau}\,\eta$ we have

$$W\eta = \begin{pmatrix}1\\0\end{pmatrix}$$

$\Rightarrow$

$$\begin{pmatrix}1 & 3/2\\ 3/2 & 9\end{pmatrix}\eta = \begin{pmatrix}1\\0\end{pmatrix}$$

If $\eta = \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$ we use cramer's rule

and write

$$\eta_1 = \frac{9}{9 - 9/4} = 4/3$$

$$\eta = \begin{pmatrix}4/3\\ -2/9\end{pmatrix}$$

∎

$$\eta_2 = \frac{-3/2}{9 - 9/4} = -\frac{\cancel{3}}{\cancel{2}}\,\frac{1}{9}\,\frac{\overset{2}{\cancel{4}}}{\cancel{3}} = -\frac{2}{9}.$$