Midterm II (solutions)

2) Ans:

Define

$$\mathcal{C} = (B \mid AB \mid A^2B \mid \dots \mid A^{n-1}B)$$

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

$$\& \quad \mathcal{H}_n = \mathcal{O}\cdot\mathcal{C} = \begin{pmatrix} CB & CAB & \dots & CA^{n-1}B \\ CAB & CA^2B & \dots & CA^nB \\ \vdots & & & \vdots \\ CA^{n-1}B & CA^nB & \dots & CA^{2n-2}B \end{pmatrix}$$

Also let $\mathcal{H}$ be the infinite Hankel matrix as defined in the question paper.

$(\mathrm{I} \Rightarrow \mathrm{II})$

Assume that both $\mathcal{C}$ and $\mathcal{O}$ have rank n. Note that $\mathcal{C}$ is a $n \times nm$ matrix hence $\exists$ a nonsingular permutation matrix P_1:

$$\cancel{P_1 \mathcal{C} = \begin{pmatrix} x & x & \cdots & x & I & x & x & \cdots & x \end{pmatrix}}$$

$$\mathcal{C} P_1 = \begin{pmatrix} M & x & x & x & \cdots & x \end{pmatrix}$$

where M is a $n \times n$ non-singular matrix.

Likewise since $\mathcal{O}$ is a $np \times n$ matrix $\exists$ a non-singular permutation matrix P_2:

$$P_2 \mathcal{O} = \begin{pmatrix} N \\ x \\ x \\ \vdots \\ x \end{pmatrix}$$

where N is a $n \times n$ nonsingular matrix.

It follows that

$P_2 \mathcal{O} \mathcal{C} P_1 =$

$$\begin{pmatrix} NM & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & & & & \\ \times & \times & \times & \times & \times \end{pmatrix}$$

The r.h.s. matrix has rank $\geq n$,

hence

$\operatorname{rank}(\mathcal{O}\mathcal{C}) \geq n$ since P_1 & P_2 are square and non-singular.

However

$$\operatorname{rank}(\mathcal{O}\, \mathcal{C}) \leq \min[\operatorname{rank}\mathcal{O}, \operatorname{rank}\mathcal{C}]$$

$$\leq n$$

because $\mathcal{O}$ is of size $np \times n$

$\mathcal{C}$ is of size $n \times nm$.

$\therefore \operatorname{rank}(\mathcal{O}\mathcal{C}) = n. \Rightarrow \operatorname{rank} \mathcal{H}_n = n.$

By Cayley Hamilton's Thm

$\operatorname{rank} \mathcal{H} = \operatorname{rank} \mathcal{H}_n = n$.

(II => I)

Assume $\operatorname{rank} H = n$.

It follows that $\operatorname{rank} \mathcal{H}_n = n$. for otherwise if $\operatorname{rank} \mathcal{H}_n = n_1 < n$, by Cayley Hamilton Thm we have $\operatorname{rank} \mathcal{H} = n_1 < n$. which will violate the assumption.

$\because \mathcal{H}_n = \mathcal{O} \cdot \mathcal{C}$ and $\operatorname{rank} \mathcal{H}_n = n$

we have

$$n = \operatorname{rank}(\mathcal{O} \cdot \mathcal{C}) \leq \min[\operatorname{rank}(\mathcal{O}), \operatorname{rank}(\mathcal{C})]$$

$\therefore \operatorname{rank}(\mathcal{O}) \geq n$, $\operatorname{rank}(\mathcal{C}) \geq n$.

Because of the sizes of $\mathcal{O}$ and $\mathcal{C}$

$\operatorname{rank}(\mathcal{O}) \leq n$ & $\operatorname{rank}(\mathcal{C}) \leq n$

Hence $\operatorname{rank}(\mathcal{O}) = n$, $\operatorname{rank}(\mathcal{C}) = n$.

3 Ans:

Consider a new dynamical system.

$$\dot{x} = A^T x + c^T u$$
$$y = b^T x.$$
(*)

By assumption, (*) is not controllable but observable. It follows that

$$\text{rank}\left[c^T, A^T c^T, A^{T^2} c^T, \ldots, A^{T^{n_1-1}} c^T\right] = n_1 < n$$

Define

$$P_1 = \left[c^T, A^T c^T, \ldots, A^{T^{n_1-1}} c^T, v_1, v_2, \ldots, v_{n-n_1}\right]$$

where v_i-s are defined in such a way that P_1 is non-singular.

It follows that

$$P_1^{-1} A^T P_1 = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \qquad P_1^{-1} c^T = \begin{pmatrix} * \\ 0 \end{pmatrix} \qquad (**)$$

$$b^T P_1 = \begin{pmatrix} * & * \end{pmatrix}.$$

By taking transposes in $(**)$ we obtain

$$P_1^T A (P_1^{-1})^T = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

$$c (P_1^{-1})^T = \begin{pmatrix} * & 0 \end{pmatrix}$$

$$P_1^T b = \begin{pmatrix} * \\ * \end{pmatrix}$$

Define $P^{-1} = P_1^T$

ie $P = (P_1^T)^{-1}$

The result follows.