Lec 11

Transfer Functions of SISO systems

I Let A is a $n \times n$ matrix

b a $n \times 1$ vector

c a $1 \times n$ vector

We define a rational function

$$g(s) = c(sI - A)^{-1} b$$

The $f^{\underline{n}}$ $g(s)$ can be written as a ratio of two polynomials $\frac{n(s)}{d(s)}$ where

$n(s)$ is of degree $\leq n-1$

$d(s)$ is a monic polynomial of degree n.

Ex: 1

$$A = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$c = \begin{pmatrix} c_1 & c_2 \end{pmatrix}$$

Q: What is $g(s)$?

Ans: $sI - A = \begin{pmatrix} s & -1 \\ -\alpha & s-\beta \end{pmatrix}$;

$$(sI - A)^{-1} = \begin{pmatrix} s-\beta & 1 \\ \alpha & s \end{pmatrix} \Big/ \det(sI - A).$$

$$\det(sI - A) = s^2 - \beta s - \alpha$$

$$c(sI - A)^{-1} b =$$

$$\frac{c_2 s + c_1}{s^2 - \beta s - \alpha}$$

Thus
$$g(s) = \frac{c_2 s + c_1}{s^2 - \beta s - \alpha}$$

Def: A pair of polynomials $n(s)$, $d(s)$ is said to be coprime if

$$\text{rank} \underbrace{\begin{pmatrix} n(s) & d(s) \end{pmatrix}}_{1 \times 2 \text{ vector}} = 1 \quad \forall s.$$

Remark: Two coprime polynomials do not have a common root.

Ex: 2 $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $c = \begin{pmatrix} 1 & 1 \end{pmatrix}$

$$g(s) = \frac{s+1}{s^2 + 3s + 2} = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}$$

Controllability matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}$$ which is of rank 2

Observability matrix is

$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}$$ which is of rank 1

The dynamical system in example 2 is controllable but not observable.

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Def: Let $g(s) = \frac{n(s)}{d(s)}$ where

1. $n(s)$ & $d(s)$ are coprime
2. degree $n(s) \leq$ degree $d(s)$

We shall define

degree of $g(s)$ = degree of $d(s)$.

Degree of $g(s)$ in Ex 2 is 1.

II When are two polynomials co-prime??

Let us consider two quadratic polynomials

$$n(s) = as^2 + bs + c$$
$$d(s) = ds^2 + es + f.$$

We want to find out when are the polynomials $n(s)$ and $d(s)$ co-prime.

If $n(s)$ and $d(s)$ are not co-prime, then

$$\frac{n(s)}{d(s)} = \frac{\alpha s + \beta}{\gamma s + \delta}$$ i.e. there will be a cancellation.

writing

$$(as^2 + bs + c)(\gamma s + \delta) = (ds^2 + es + f)(\alpha s + \beta)$$

$$\Rightarrow (a\gamma - d\alpha)s^3 + (a\delta + b\gamma - d\beta - e\alpha)s^2 + (c\gamma + b\delta - f\alpha - e\beta)s + (c\delta - f\beta) = 0$$

$\Rightarrow$

$$a\gamma - d\alpha = 0$$
$$a\delta + b\gamma - d\beta - e\alpha = 0$$
$$c\gamma + b\delta - f\alpha - e\beta = 0$$
$$c\delta - f\beta = 0$$

$$\begin{pmatrix} a & -d & 0 & 0 \\ b & -e & a & -d \\ c & -f & b & -e \\ 0 & 0 & c & -f \end{pmatrix} \begin{pmatrix} \gamma \\ \alpha \\ \delta \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (*)$$

If $n(s)$ and $d(s)$ are not co-prime then $(*)$ can be solved for a nontrivial vector $\begin{pmatrix} \gamma \\ \alpha \\ \delta \\ \beta \end{pmatrix}$ ie if

$$\det \begin{pmatrix} a & -d & 0 & 0 \\ b & -e & a & -d \\ c & -f & b & -e \\ 0 & 0 & c & -f \end{pmatrix} = 0$$

Theorem 1

$n(s)$ and $d(s)$ are co-prime iff

$$\det\begin{pmatrix} a & d & 0 & 0 \\ b & e & a & d \\ c & f & b & e \\ 0 & 0 & c & f \end{pmatrix} \neq 0$$

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Corollary: 1

The t.f. $g(s)$ in example 1 is of degree 2 iff

$$\underbrace{\det\begin{pmatrix} 0 & 1 & 0 & 0 \\ c_2 & -\beta & 0 & 1 \\ c_1 & -\alpha & c_2 & -\beta \\ 0 & 0 & c_1 & -\alpha \end{pmatrix}}_{\begin{array}{c} \parallel \\ -\det\begin{pmatrix} c_2 & 0 & 1 \\ c_1 & c_2 & -\beta \\ 0 & c_1 & -\alpha \end{pmatrix} \end{array}} \neq 0$$

The determinant is given by

$$(-1)\left[c_2(-c_2\alpha + c_1\beta)\right]$$

$$+ (-1)\left[c_1^2\right]$$

$$= c_2^2\alpha - c_1c_2\beta - c_1^2$$

Thus $g(s)$ in example 1 is of degree 2 iff

$$\boxed{c_1^2 + c_1c_2\beta - c_2^2\alpha \neq 0}$$

$$\Updownarrow$$

$$\det\begin{pmatrix} c_1 & c_2 \\ c_2\alpha & c_1 + c_2\beta \end{pmatrix} \neq 0$$

$$\Updownarrow$$

$$\det\begin{pmatrix} c \\ cA \end{pmatrix} \neq 0 \qquad \text{where } c = (c_1 \;\; c_2)$$

$$A = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$$

Ex 1 (continued)

consider

$$\dot{x} = Ax + bu \qquad \text{**}$$
$$y = cx$$

where

$$A = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}; \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad c = (c_1 \quad c_2)$$

Define $g(s) = c(sI - A)^{-1}b$

We have seen so far that

$g(s)$ is of degree 2 iff ** is observable

Note that ** is always controllable.

III Using the matrices A, b, C we can define a sequence of real numbers

$$h_1, h_2, h_3, \ldots$$

as follows

$$h_j = CA^{j-1}b \quad , j = 1, 2, 3, \ldots$$

The sequence defined is given by

$$Cb, CAb, CA^2b, CA^3b, \ldots$$

Using the above sequence we construct a hankel matrix

$$H = \begin{pmatrix} h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & h_4 & \cdots \\ h_3 & h_4 & h_5 & \cdots \\ \vdots & \vdots & & \cdots \\ \vdots & \vdots & & \cdots \end{pmatrix}$$

and the submatrices

$$H_j = \begin{pmatrix} h_1 & h_2 & \cdots & h_j \\ h_2 & h_3 & \cdots & h_{j+1} \\ \vdots & & & \vdots \\ h_j & h_{j+1} & \cdots & h_{2j-1} \end{pmatrix}$$

We define

$$\text{rank } H \triangleq \sup_j \left[\text{rank}(H_j)\right].$$

$\underline{\text{IV}}$ Connection between the sequence

$cb, cAb, cA^2b, \ldots\ldots$

and the rational $f^{\underline{n}}$

$$g(s) = c(sI - A)^{-1} b$$

is the following:

write $(sI - A) = s\left(I - \frac{A}{s}\right)$, we compute

$$(sI - A)^{-1} = \frac{1}{s}\left[I + \frac{A}{s} + \frac{A^2}{s^2} + \cdots\cdots\right]$$

$$g(s) = \frac{cb}{s} + \frac{cAb}{s^2} + \frac{cA^2b}{s^3} + \cdots\cdots\cdots$$

Example 2 (continued)

$$g(s) = \frac{1}{s+2}$$

$$s+2 \,\big)\, 1 \quad \left(\frac{1}{s} - \frac{2}{s^2} + \frac{4}{s^3} \cdots\cdots\right.$$

$$\underline{\overset{-}{1} + \overset{-}{\frac{2}{s}}}$$

$$-\cancel{\frac{2}{s}}$$

$$\underline{\overset{+}{-\cancel{\frac{2}{s}}} \overset{+}{-} \frac{4}{s^2}}$$

$$\cancel{\frac{4}{s^2}}$$

$$\underline{\overset{-}{\cancel{\frac{4}{s^2}}} \overset{-}{+} \frac{8}{s^3}}$$

$$-\frac{8}{s^3}$$

$$\vdots$$

$$\therefore\; g(s) = \frac{1}{s} - \frac{2}{s^2} + \frac{4}{s^3} \cdots\cdots$$

We obtain a sequence

$$1, -2, 4, \ldots\ldots$$

Every rational fn $g(s)$ can be associated with a unique sequence by successive long division.

The Hankel matrix is given by

$$\begin{pmatrix} 1 & -2 & 4 & \cdots \\ -2 & 4 & -8 & \cdots \\ 4 & -8 & 16 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is of rank 1

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Question:

Given a sequence

$$h_1, h_2, h_3, \ldots$$

define

$$g(s) = \sum_{j=1}^{\infty} \frac{h_j}{s^j} = \frac{h_1}{s} + \frac{h_2}{s^2} + \frac{h_3}{s^3} + \cdots$$

When is $g(s)$ a strictly proper rational function of degree n.

Ans:

Precisely when

$$H = \begin{pmatrix} h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & h_4 & \cdots \\ h_3 & h_4 & h_5 & \cdots \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \end{pmatrix}$$ has rank n.

<u>Let us try to understand this for $n=2$</u>

If $g(s)$ is a strictly proper rational $f^{\underline{n}}$ of degree 2 we obtain

$$g(s) = \frac{h_1}{s} + \frac{h_2}{s^2} + \frac{h_3}{s^3} + \cdots = \frac{bs+c}{s^2+es+f}$$

for some b, c, e, f.

Cross multiplying and equating the co-efficients we obtain

$$h_1 = b$$

$$h_1 e + h_2 = c$$

$$h_1 f + h_2 e + h_3 = 0$$

$$h_2 f + h_3 e + h_4 = 0$$

$$\vdots$$

$$h_j f + h_{j+1} e + h_{j+2} = 0 \qquad j = 1, 2, 3, \ldots\ldots$$

i.e.

$$\begin{pmatrix} h_3 \\ h_4 \\ h_5 \\ \vdots \end{pmatrix} = -f \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \end{pmatrix} - e \begin{pmatrix} h_2 \\ h_3 \\ h_4 \\ \vdots \end{pmatrix} \qquad \star$$

Thus the hankel matrix H has rank 2.

<u>Conversely</u>, if H has rank 2 one can uniquely solve for f and e using the recursion $\star$. One can now

solve for b and c using

$$b = h_1$$
$$c = h_1 e + h_2 .$$

This way one constructs $g(s)$ uniquely from the sequence $h_1, h_2, \ldots$.

Moreover $\deg g(s) = 2$ for otherwise H would not have rank 2 violating the assumption that it does have rank 2.

<u>Conclusion:</u>

There is a 1-1 correspondence between strictly proper rational functions of degree 2 and Hankel matrices of rank 2.

Ex 3

Consider the following sequence

$$1, 1, 2, 3, 5, 8, 13, \cdots\cdots$$

The corresponding Hankel matrix is

$$H = \begin{pmatrix} 1 & 1 & 2 & \cdot & \cdot & \cdot \\ 1 & 2 & 3 & \cdot & \cdot & \cdot \\ 2 & 3 & 5 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

which is of rank 2.

Define

$$g(s) = \frac{1}{s} + \frac{1}{s^2} + \frac{2}{s^3} + \frac{3}{s^4} + \frac{8}{s^5} + \cdots\cdots$$

As claimed earlier, we can write

$$g(s) = \frac{bs + c}{s^2 + es + f}$$

where

$$b = h_1 = 1$$

$$c = h_1 e + h_2 = e + 1$$

$$f = e = -1 \text{ (from the recursion in } \star\text{)}.$$

$$\therefore c = e + 1 = 0.$$

$$\therefore \quad g(s) = \frac{s}{s^2 - s - 1}.$$

If we define

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c = (0 \quad 1)$$

$$g(s) = c(sI - A)^{-1} b = \frac{s}{s^2 - s - 1}.$$

Thus we have established a connection between

1. Fibbonaci sequence

   1, 1, 2, 3, 5, 8, - - - - .

2. Rational function.

$$g(s) = \frac{s}{s^2 - s - 1}$$

3. Dynamical system.

$$\dot{x} = Ax + bu, \quad y = cx$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

(V) We are now ready to write down the main theorem of this lecture.

Consider matrices A, b, c as defined on page 1.

Main Theorem

The following statements are equivalent.

(I) $\dot{x} = Ax + bu,\ y = cx$
is controllable and observable.

(II) The hankel matrix

$$H = \begin{pmatrix} cb & cAb & cA^2b & \cdot & \cdot & \cdot \\ cAb & cA^2b & cA^3b & \cdot & \cdot & \cdot \\ cA^2b & cA^3b & cA^4b & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$ has rank n.

(III) $g(s) = c(sI - A)^{-1}b$ is of degree n.

Proof of the Main Theorem.

(and you thought that we shall never prove anything)

I $\Rightarrow$ II

controllability $\Rightarrow$ $\begin{pmatrix} b & Ab & A^2b & \cdots & A^{n-1}b \end{pmatrix}$ has rank n.

observability $\Rightarrow$ $\begin{pmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{pmatrix}$ has rank n.

Hence

$$H_n = \begin{pmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{pmatrix} \begin{pmatrix} b & Ab & A^2b & \cdots & A^{n-1}b \end{pmatrix}$$

has rank n

$$= \begin{pmatrix} cb & cAb & cA^2b & \cdots & cA^{n-1}b \\ cAb & cA^2b & cA^3b & \cdots & cA^n b \\ \vdots & \vdots & \vdots & & \vdots \\ cA^{n-1}b & cA^n b & cA^{n+1}b & \cdots & cA^{2n-2}b \end{pmatrix}$$ has rank n.

It follows that

$$\text{rank } H \geq n.$$

But Cayley Hamilton's Theorem would tell us that

$$\text{rank } H \leq n.$$

Hence $\text{rank } H = n$.

II => III

Assume that degree $g(s) = n_1 < n$, we write

$$g(s) = \frac{\alpha_1 s^{n_1-1} + \alpha_2 s^{n_1-2} + \cdots + \alpha_{n_1}}{s^{n_1} + \beta_1 s^{n_1-1} + \beta_2 s^{n_1-2} + \cdots + \beta_{n_1}}.$$

Equating $g(s)$ to

$$\frac{cb}{s} + \frac{cAb}{s^2} + \frac{cA^2b}{s^3} + \cdots$$

cross multiplying and comparing the co-efficients $\frac{1}{s}$, $\frac{1}{s^2}$ etc (as was done on pages 13 & 14 we see that

"$(n_1+1)^{th}$ column of H is a linear combination of the first n_1 columns of H."

Thus rank $H \leq n_1$.

III $\Rightarrow$ I

Let us assume that the dynamical system

$$\dot{x} = Ax + bu, \; y = cx$$

is not controllable. Let P be a $n \times n$ invertible matrix where

$$x = Pz$$

and

$$\dot{z} = P^{-1}\dot{x} = P^{-1}Ax + P^{-1}bu$$
$$= P^{-1}APz + P^{-1}bu.$$

$$y = cPz.$$

If

$$\operatorname{rank}\left(b \;\; Ab \; \cdots \; A^{n-1}b\right) = n_1 < n$$

we can find a P such that

$$P^{-1}AP = \left(\begin{array}{c|c} F_{11} & F_{12} \\ \hline 0 & F_{22} \end{array}\right); \quad P^{-1}b = \left(\begin{array}{c} g_1 \\ \hline 0 \end{array}\right)$$

$$CP = \left(h_1 \quad h_2 \right)$$

where F_1 is $n_1 \times n_1$, g_1 is $n_1 \times 1$, h_1 is $1 \times n_1$

the other matrices have compatible sizes.

It is easy to check that

$$g(s) = C(sI - A)^{-1}b = [CP]\, P^{-1}(sI - A)^{-1}P\, [P^{-1}b]$$

$$= CP\left[sI - P^{-1}AP\right]^{-1}P^{-1}b$$

Thus

$$g(s) = \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} (sI - F_{11})^{-1} & * \\ 0 & (sI - F_{22})^{-1} \end{pmatrix} \begin{pmatrix} g_1 \\ 0 \end{pmatrix}$$

$$= h_1 (sI - F_{11})^{-1} g_1$$

It follows that

$$\deg g(s) \leq n_1 .$$

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If the dynamical system

$$\dot{x} = Ax + bu, \quad y = cx$$

is not observable, we assume that

$$\text{rank}\begin{pmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{pmatrix} = n_1 < n$$

and find a P such that

$$P^{-1}AP = \begin{pmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{pmatrix}; \quad P^{-1}b = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$cP = \begin{pmatrix} h_1 & 0 \end{pmatrix}$$

where F_{11} is $n_1 \times n_1$, h_1 is $1 \times n_1$ and other matrices are of compatible sizes.

Thus

$$g(s) = h_1 (sI - F_{11})^{-1} g_1$$

$$\deg g(s) \leq n_1 .$$

$\blacksquare$

VI Poles and Zeros.

Let us write

$$g(s) = \frac{n(s)}{d(s)} \qquad \deg n(s) < \deg d(s)$$

where $n(s)$ and $d(s)$ are co-prime.

Define

Finite Zeros of $g(s)$ = Zeros of $n(s)$.

~~Zeros~~ Poles of $g(s)$ = Zeros of $d(s)$.

If $r = \deg d(s) - \deg n(s)$, we say that $g(s)$ has r infinite zeros.

Ex 4:

Calculate the poles and zeros of

$$g(s) = \frac{s}{s^2 - s - 1}.$$

$n(s) = s$

$d(s) = s^2 - s - 1$

$d(s) = 0 \Rightarrow$

$$s = \frac{1 \pm \sqrt{1 + 4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

poles at $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$

1 finite zero at $s = 0$.

1 infinite zero.