$$\boxed{\text{Lec } 6}$$

Story of

$$\dot{x}(t) = A(t)\,x(t) + f(t).$$

We are looking at eqns of the form

$$\dot{x}(t) = A(t)\,x(t) + f(t). \quad (*)$$

Problem is to solve $(*)$

Special case 1 $(A(t) \equiv 0)$

$$\dot{x}(t) = f(t)$$

$$x(t) = \cancel{x(0)}\ x(t_0) + \int_{t_0}^{t} f(\sigma)\,d\sigma$$

initial condition
at $t = t_0$.

Special case 2 $(f(t) = 0)$

$$\dot{x}(t) = A(t)\,x(t).$$

$$x(t) = \phi(t, t_0)\,x(t_0).$$

transition matrix.

Define a new variable

$$z(t) = \phi(t_0, t)\, x(t)$$

Remark: $\phi(t, t_0) = \phi(t_0, t)^{-1}$

Lemma: $\dot{\phi}(t_0, t) = -\phi(t_0, t) A(t)$.

$$\phi(t, t_0)\, \phi(t_0, t) = I$$

Taking the derivative w.r.t. t we have

$$\frac{d}{dt}\phi(t, t_0)\, \phi(t_0, t) + \phi(t, t_0) \frac{d}{dt}\phi(t_0, t) = 0$$

$$\Rightarrow A(t) \overbrace{\phi(t, t_0)\, \phi(t_0, t)}^{= I} + \phi(t, t_0) \frac{d}{dt}\phi(t_0, t) = 0$$

$$\Rightarrow \frac{d}{dt}\phi(t_0, t) = -\phi(t, t_0)^{-1} A(t).$$

$$= -\phi(t_0, t) A(t).$$

It follows that

$$\dot{z}(t) = \dot{\phi}(t_0, t)\, x(t) + \phi(t_0, t)\, \dot{x}(t)$$

$$= -\phi(t_0, t) A(t) x(t) + \phi(t_0, t) A(t) x(t) + \phi(t_0, t) f(t)$$

$$= \phi(t_0, t) f(t).$$

Thus

$$\dot{z}(t) = \phi(t_0, t) f(t)$$

This implies

$$z(t) = z(t_0) + \int_{t_0}^{t} \phi(t_0, \sigma) f(\sigma)\, d\sigma.$$

$$\Rightarrow \phi(t_0, t)\, x(t) = x(t_0) + \int_{t_0}^{t} \phi(t_0, \sigma) f(\sigma)\, d\sigma$$

Thus we have

$$x(t) = \phi(t_0,t)^{-1} x(t_0) + \int_{t_0}^{t} \phi(t_0,t)^{-1} \phi(t_0,\sigma) f(\sigma)\, d\sigma$$

$$= \phi(t,t_0) x(t_0) + \int_{t_0}^{t} \phi(t,t_0) \phi(t_0,\sigma) f(\sigma)\, d\sigma .$$

Theorem 6.1

(Variation of constants formula)

Let $\phi(t, t_0)$ be the state transition matrix for $\dot{x}(t) = A(t)x(t)$, then the unique sol$^{\underline{n}}$ of

$$\dot{x}(t) = A(t)\,x(t) + f(t)$$

$$x(t_0) = x_0$$

is given by

$$x(t) = \phi(t, t_0)\,x_0 + \int_{t_0}^{t} \phi(t, \sigma)\,f(\sigma)\,d\sigma.$$

Corollary 6.2

The solution of the inhomogeneous linear constant equation $\dot{x} = Ax + f(t)$; $x(0) = x_0$ is given by

$$x(t) = e^{At}x_0 + \int_{t_0}^{t} e^{A(t-\sigma)} f(\sigma)\,d\sigma.$$

Example

(Newtons 2^{nd} law)

For a unit mass

$$\underset{\text{acc}^n}{\ddot{x}} = \underset{\text{force}}{f(t)}.$$

Define $x_1 = x$, $x_2 = \dot{x}$, we have

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & t-\sigma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ f(\sigma) \end{pmatrix} d\sigma$$

Hence

$$x_1(t) = x_1(0) + t\,x_2(0) + \int_0^t (t-\sigma) f(\sigma)\, d\sigma$$

$$x_2(t) = x_2(0) + \int_0^t f(\sigma)\, d\sigma$$

In the original variables we write

$$x(t) = x(0) + t\,\dot{x}(0) + \int_0^t (t-\sigma)\, f(\sigma)\, d\sigma \,.$$

Integral form of Newton's Law

Example (The satellite problem)

$$\dot{\underline{X}} = A\underline{X} + Bu$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{pmatrix} ; \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The state transition matrix $\phi(t, t_0)$ is given by

$$\begin{pmatrix} 4 - 3\cos\omega(t-t_0) & \dfrac{\sin\omega(t-t_0)}{\omega} & 0 & \dfrac{2(1-\cos\omega(t-t_0))}{\omega} \\ 3\omega\sin\omega(t-t_0) & \cos\omega(t-t_0) & 0 & 2\sin\omega(t-t_0) \\ 6(-\omega(t-t_0) + \sin\omega(t-t_0)) & \dfrac{-2(1-\cos\omega(t-t_0))}{\omega} & 1 & \dfrac{-3\omega(t-t_0) + 4\sin\omega(t-t_0)}{\omega} \\ 6\omega(-1+\cos\omega(t-t_0)) & -2\sin\omega(t-t_0) & 0 & -3+4\cos\omega(t-t_0) \end{pmatrix}$$

Starting from $x(0) = x_0$, we have

$$x(t) = \phi(t,0)\,x_0 + \int_0^t \phi(t,\sigma)\,B(\sigma)\,d\sigma$$

where

$$\phi(t,\sigma)\,B(\sigma) =$$

$$\begin{bmatrix} \dfrac{\sin\omega(t-\sigma)}{\omega} & \dfrac{2(1-\cos\omega(t-\sigma))}{\omega} \\ \cos\omega(t-\sigma) & 2\sin\omega(t-\sigma) \\ \dfrac{-2(1-\cos\omega(t-\sigma))}{\omega} & \dfrac{-3\omega(t-\sigma)+4\sin\omega(t-\sigma)}{\omega} \\ -2\sin\omega(t-\sigma) & -3+4\cos\omega(t-\sigma) \end{bmatrix}$$

An example that we cannot live without

The Euler Equations for the angular velocities of a rigid body are

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\,\omega_2 \omega_3 + u_1$$
$$I_2 \dot{\omega}_2 = (I_3 - I_1)\,\omega_1 \omega_3 + u_2$$
$$I_3 \dot{\omega}_3 = (I_1 - I_2)\,\omega_1 \omega_2 + u_3$$

$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$ is the angular velocity in a body fixed co-ordinate system coinciding with the principal axes

$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ is the applied torque.

I_1, I_2, I_3 are the principal moment of inertia.

If $I_1 = I_2$ we call the body symmetrical. In this case, for $u = 0$ we have

$$\dot{\omega}_3 = 0 \implies \omega_3 = \omega_0 \text{ (a constant)}.$$

Hence

$$\dot{\omega}_1 = \left[\left(\frac{I_2 - I_3}{I_2}\right)\omega_0\right]\omega_2$$

$$\dot{\omega}_2 = \left[\left(\frac{I_3 - I_2}{I_2}\right)\omega_0\right]\omega_1$$

Define $\alpha = \left(\frac{I_2 - I_3}{I_2}\right)\omega_0$, it follows that

$$\boxed{\begin{aligned}\dot{\omega}_1 &= \alpha\,\omega_2\\ \dot{\omega}_2 &= -\alpha\,\omega_1\end{aligned}} \qquad \begin{bmatrix}\omega_1(t)\\ \omega_2(t)\end{bmatrix} = \begin{pmatrix}\cos\alpha t & \sin\alpha t\\ -\sin\alpha t & \cos\alpha t\end{pmatrix}\begin{pmatrix}\omega_1(0)\\ \omega_2(0)\end{pmatrix}$$

Choose $\omega_1(0)=1$, $\omega_2(0)=0$, we have

$$\boxed{\omega_3 = \omega_0,\ \omega_2 = -\sin\alpha t,\ \omega_1 = \cos\alpha t}$$

as a sol$^{\underline{n}}$ to the symmetrical Euler eq$^{\underline{n}}$

Define

$$\bar\omega_1 = \omega_1 - \cos\alpha t$$

$$\bar\omega_2 = \omega_2 + \sin\alpha t$$

$$\bar\omega_3 = \omega_3 - \omega_0$$

we have

$$\dot{\bar\omega}_3 = \dot\omega_3 = 0$$

$$\dot{\bar\omega}_1 = \dot\omega_1 + \alpha\sin\alpha t = \alpha\omega_2 + \alpha\sin\alpha t = \alpha\bar\omega_2$$

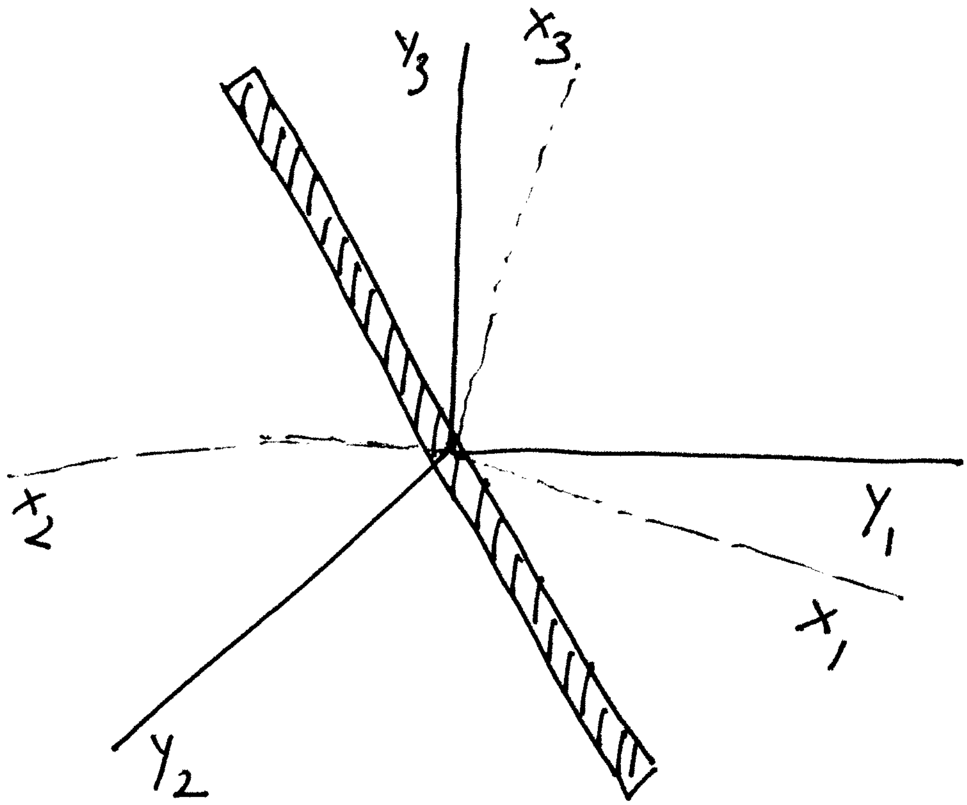
$$\dot{\bar\omega}_2 = \dot\omega_2 + \alpha\cos\alpha t = -\alpha(\omega_1 - \cos\alpha t) = -\alpha\bar\omega_1$$

Hence

$$\begin{pmatrix} \dot{\bar\omega}_1 \\ \dot{\bar\omega}_2 \\ \dot{\bar\omega}_3 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar\omega_1 \\ \bar\omega_2 \\ \bar\omega_3 \end{pmatrix}$$

$\blacksquare$

Consider the problem of orienting one set of co-ordinate axes w.r.t. a second set



Say that the projection on the y_j axis of a unit vector along the x_i axis is a_{ij}. These a_{ij}-s are called direction cosines and we can put them in the form of a matrix A.

There are nine such direction cosines.

If the x-system is rotating about

its x_1 axis with an angular velocity ω_1

x_2 " " " " " ω_2

x_3 " " " " " ω_3

then $A(t)$ changes with time as follows:

$$\dot{A}(t) = \Omega(t) A(t)$$

$$\Omega = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

— x —

Thus the orientation of a symmetrical rigid body in torque free spinning motion is equivalent to finding the transition matrix for the system

$$\dot{x}(t) = \Omega(t)\, x(t) \qquad (*)$$

where

$$\Omega(t) = \begin{pmatrix} 0 & \omega_0 & \omega_0' \sin \alpha t \\ -\omega_0 & 0 & \omega_0' \cos \alpha t \\ -\omega_0' \sin \alpha t & -\omega_0' \cos \alpha t & 0 \end{pmatrix}$$

where $\omega_0, \omega_0', \alpha$ are all constants and

$$\boxed{\alpha = \frac{I_2 - I_3}{I_2}\, \omega_0}$$

Problem: Find the state transition matrix of $(*)$.