$$\boxed{\text{Lec } 5}$$

$$\boxed{\text{Story of } \dot{x}(t) = A(t)\, x(t)}$$

On the story of

$$\dot{x}(t) = A(t)\, x(t)$$

Let us assume that A is a square $n \times n$ matrix whose elements are continuous fns of time on the interval $0 \le t \le T$.

We define a sequence of matrices M_k as follows:

$$M_0 = I$$

$$M_1 = I + \int_0^t A(\sigma) M_0(\sigma)\, d\sigma$$

. . . . .

. . . . .

$$M_k = I + \int_0^t A(\sigma) M_{k-1}(\sigma)\, d\sigma .$$

Theorem:

The sequence of matrices $M_0, M_1, \ldots$ converges uniformly on the interval $[0, T]$. If we denote the limit function by $\phi(t)$ then for $0 \leq t \leq T$, we have

$$\dot{\phi}(t) = A(t)\,\phi(t)$$

& $\phi(0) = I$.

One can rewrite M_k as follows.

$$M_0 = I$$

$$M_1 = M_0 + \int_0^t A(\sigma_1)\, d\sigma_1 .$$

$$M_2 = M_1 + \int_0^t A(\sigma_1) \left[\int_0^{\sigma_1} A(\sigma_2)\, d\sigma_2 \right] d\sigma_1 .$$

$$\vdots$$

$$M_k = M_{k-1} + \underbrace{\int_0^t A(\sigma_1) \int_0^{\sigma_1} A(\sigma_2) \int_0^{\sigma_2} A(\sigma_3) \cdots \cdots \int_0^{\sigma_{k-1}} A(\sigma_k)\, d\sigma_k\, d\sigma_{k-1} \cdots d\sigma_1}_{P_k(t)}$$

The series

$$I + P_1(t) + P_2(t) + \cdots\cdots$$

is called the $\boxed{\text{Peano-Baker series}}$

The theorem on page 2 says that the P-B series converges uniformly in the interval $[0, T]$. We define

$$\phi(t) = \sum_{j=0}^{\infty} P_j(t) \qquad P_0 = I.$$

<u>Lemma:</u>

If $A(t)$ is a scaler continuous $f^{\underline{n}}$ of t, i.e if we write $A(t) = a(t)$, then

$$\phi(t) = \exp\left[\int_0^t a(\sigma)\,d\sigma\right].$$

Proof:

$$P_0 = 1$$

$$P_1 = \int_0^t a(\sigma)\,d\sigma$$

$$P_2 = \int_0^t a(\sigma_1)\int_0^{\sigma_1} a(\sigma_2)\,d\sigma_2\,d\sigma_1 .$$

$$= \left[\int_0^{\sigma_1} a(\sigma_2)\,d\sigma_2\right]\Big|_0^t P_1 - \underbrace{\int_0^t a(\sigma_1)\int_0^{\sigma_1} a(\sigma_2)\,d\sigma_2\,d\sigma_1}_{P_2}$$

$$\Rightarrow P_2 = P_1^2 - P_2$$

$$\Rightarrow P_2 = \frac{1}{2}P_1^2 = \frac{1}{2}\left[\int_0^t a(\sigma)\,d\sigma\right]^2$$

Assume that

$$P_k(t) = \frac{1}{k!}\left[\int_0^t a(\sigma)\,d\sigma\right]^k$$

We write

$$P_{k+1}(t) = \int_0^t a(\sigma)\,P_k(\sigma)\,d\sigma$$

$$= P_k(\sigma)\Big|_0^t P_1(t)$$

$$- \int_0^t \frac{1}{(k-1)!}\left[\int_0^{\sigma_2} a(\sigma)\,d\sigma\right]^{k-1} a(\sigma_2)\,P_1(\sigma_2)\,d\sigma_2$$

$$= P_k(t)\,P_1(t) - \int_0^t \frac{1}{(k-1)!}\,a(\sigma_2)\left[\int_0^{\sigma_2} a(\sigma)\,d\sigma\right]^k d\sigma_2$$

$$= P_k(t)\,P_1(t) - k\int_0^t a(\sigma_2)\,P_k(\sigma_2)\,d\sigma_2$$

$$= P_k(t)\,P_1(t) - k\,P_{k+1}(t).$$

Hence

$$(k+1)\, p_{k+1}(t) = p_k(t)\, p_1(t)$$

$$\Rightarrow p_{k+1}(t) = \frac{1}{k+1} \cdot \frac{1}{k!} \left[p_1(t)\right]^k p_1(t)$$

$$= \frac{1}{(k+1)!} \left[p_1(t)\right]^{k+1}.$$

$\blacksquare$

Proof of the theorem on page 2

In order to prove that the sequence $M_k(t)$ converges, we need to show that the scalar sequence consisting of $(i,j)^{th}$ element converges for $i=1,\dots,n$; $j=1,2,\dots,n$. etc.

Recall that a series of continuous scalar $f^{n}s$ $x_1(t)+x_2(t)+\cdots$ defined on $[0,T]$ converges absolutely and uniformly on the interval $[0,T]$ if $\exists$ sequence of positive constants c_i : $\forall t\in[0,T]$ we have $|x_i(t)|\le c_i$ and the series $c_1+c_2+\cdots$ converges.

"This is called Weierstrass M-test"

Define $\eta(t)$ as the maximum absolute values of any entry in $A(t)$. ie

$$\eta(t) = \max_{i,j} |a_{ij}(t)|$$

Let $\gamma(t) = \int_0^t \eta(\sigma)\, d\sigma$.

Let us denote

$$E_{ij}(Q)$$

to be the i,j^{th} element of the matrix Q.

Let $P_k(t)$ be the k^{th} term of the Peano-Baker series (page 4). We have

$$E_{ij}(P_k(t)) = E_{ij}\left(\int_0^t \int_0^{\sigma_1} \cdots \int_0^{\sigma_{k-1}} A(\sigma_1) \cdots A(\sigma_k)\, d\sigma_1 \cdots d\sigma_k \right)$$

If Q_1 & Q_2 are two $n \times n$ matrices, we have

$$\left| E_{ij}(Q_1 Q_2) \right| \le n \max_{ij} \left| E_{ij}(Q_1) \right| \cdot \max_{ij} \left| E_{ij}(Q_2) \right| .$$

It would follow that

$$E_{ij}\left(P_k(t)\right) \le \int_0^t \int_0^{\sigma_1} \cdots \int_0^{\sigma_{k-1}} n^{k-1} \eta(\sigma_1)\,\eta(\sigma_2) \cdots \eta(\sigma_k)\, d\sigma_1 \cdots d\sigma_k .$$

$$= n^{k-1} \frac{r^k(t)}{k!}$$

If we define

$$c_k(t) = n^{k-1} \frac{r^k(t)}{k!}$$

it follows that

$$E_{ij}(P_k(t)) \leq c_k(t)$$

Moreover

$$c_0 + c_1 + c_2 + \cdots\cdots$$

$$= 1 + r(t) + \frac{n r^2(t)}{2!} + \cdots$$

$$= 1 + \frac{1}{n}\left(nr + \frac{n^2 r^2}{2!} + \cdots\cdots\right)$$

$$= 1 + \frac{1}{n}\left(e^{nr} - 1\right)$$

Hence $E_{ij}(P_k(t))$ converges for every i, j.

To show that

$$\dot{\phi}(t) = A(t)\,\phi(t)$$

recall that

$$\phi(t) = I + \int_0^t A(\sigma_1)\,d\sigma_1 + \int_0^t \int_0^{\sigma_1} A(\sigma_1)\,A(\sigma_2)\,d\sigma_1\,d\sigma_2 \,. + \cdots \cdots$$

It follows that

$$\dot{\phi}(t) = 0 + A(t) + A(t)\int_0^t A(\sigma_1)\,d\sigma_1 \,. + A(t)\int_0^t \int_0^{\sigma_1} A(\sigma_1)\,A(\sigma_2)\,d\sigma_1\,d\sigma_2$$

$$= A(t)\,\phi(t).$$

Consider the o.d.e

$$\dot{x} = A(t)x$$
$$x(0) = x_0.$$

We have

$$x(t) = \phi(t)x_0.$$

To see this note that

$$\dot{x}(t) = \dot{\phi}(t)x_0 = A(t)\phi(t)x_0 = A(t)x(t).$$

Moreover

$$x(0) = \phi(0)x_0 = I x_0 = x_0.$$

The matrix $\phi(t)$ is called the transition matrix.

<u>Corollary</u>:

If A is a real constant $n \times n$ matrix

then $\phi(t) = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots\cdots$

$$\triangleq e^{At}.$$

Proof:

$$P_1 = \int_0^t A \, d\sigma = A \int_0^t d\sigma = At.$$

$$P_2 = \int_0^t \int_0^{\sigma_1} A^2 \, d\sigma_2 \, d\sigma_1$$

$$= A^2 t^2 / 2!$$

— ———

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$\blacksquare$

Example

consider the 2^{nd} order eqn

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} \sigma(t) & -\omega(t) \\ +\omega(t) & \sigma(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Calculate the transition matrix.

writing $z(t) = x_1(t) + i\, x_2(t)$

$a(t) = \sigma(t) + i\,\omega(t)$

we have

$$\begin{aligned} \dot{z}(t) &= \dot{x}_1(t) + i\,\dot{x}_2(t) \\ &= (\sigma x_1 - \omega x_2) + i(\omega x_1 + \sigma x_2) \\ &= (\sigma + i\omega)\, x_1 + i\,(\sigma + i\omega)\, x_2 \\ &= a(t)\, z(t). \end{aligned}$$

$\because$ $a(t)$ is scalar, it follows that

$$z(t) = e^{\int_0^t a(\tau)\,d\tau}\, z(0).$$

Hence

$$x_1(t) + i x_2(t) = e^{\int_0^t [\sigma(\tau) + i\omega(\tau)]\,d\tau}\,.\, \left[x_1(0) + i x_2(0)\right]$$

$$= e^{\int_0^t \sigma(\tau)\,d\tau} \left[\cos\int_0^t \omega(\tau)\,d\tau + i\sin\int_0^t \omega(\tau)\,d\tau\right] \left(x_1(0) + i\, x_2(0)\right).$$

We write

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{\int_0^t \sigma(\tau)\,d\tau} \left(\begin{array}{c|c} \cos\int_0^t \omega(\tau)\,d\tau & -\sin\int_0^t \omega(\tau)\,d\tau \\ \hline +\sin\int_0^t \omega(\tau)\,d\tau & \cos\int_0^t \omega(\tau)\,d\tau \end{array}\right) \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

The transition matrix

$$\phi(t) =$$

$$e^{\int_0^t \sigma(\tau)d\tau} \cdot \begin{pmatrix} \cos\int_0^t \omega(\tau)d\tau & -\sin\int_0^t \omega(\tau)d\tau \\ \sin\int_0^t \omega(\tau)d\tau & \cos\int_0^t \omega(\tau)d\tau \end{pmatrix}.$$

— x —

Remark: If σ and ω are constants we have

$$\phi(t) = \begin{pmatrix} e^{\sigma t}\cos\omega t & -e^{\sigma t}\sin\omega t \\ e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{pmatrix}.$$

Example

Consider a 2^{nd} order o.d.e

$$\ddot{y} + 3\dot{y} + \omega(t)^2 y = 0$$

where we assume that

$$\dot{\omega} = -3\omega \qquad \omega(0) = \omega_0$$
$$y(0) = y_0$$
$$\dot{y}(0) = y_1 .$$

Problem: Calculate $y(t)$.

Define $x_2(t) = y(t)$

$$x_1(t) = \frac{\dot{y}(t)}{\omega(t)}$$

We have

$$\dot{x}_2 = \dot{y} = \omega x_1$$

$$\dot{x}_1 = \frac{\omega \ddot{y} - \dot{y}\dot{\omega}}{\omega^2} = \frac{\ddot{y}}{\omega} + \frac{\dot{y}}{\omega} 3$$

$$= \frac{-3\dot{y} - \omega^2 y}{\omega} + 3\frac{\dot{y}}{\omega} = -\omega y \equiv -\omega x_2$$

$\therefore$ we have

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$\omega(t) = e^{-3t}\omega_0$$

Rotation with an exponentially decaying angular velocity.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) = \begin{bmatrix} \cos\int_0^t e^{-3\tau}\omega_0\,d\tau & -\sin\int_0^t e^{-3\tau}\omega_0\,d\tau \\ \sin\int_0^t e^{-3\tau}\omega_0\,d\tau & \cos\int_0^t e^{-3\tau}\omega_0\,d\tau \end{bmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$\int_0^t e^{-3\tau}\,d\tau = \left.\frac{e^{-3\tau}}{-3}\right|_0^t = -\frac{1}{3}\left(e^{-3t}-1\right)$$

$$= \frac{1}{3}\left(1-e^{-3t}\right).$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) =$$

$$\begin{pmatrix} \cos\left[\frac{\omega_0}{3}(1-e^{-3t})\right] & -\sin\left[\frac{\omega_0}{3}(1-e^{-3t})\right] \\ \sin\left[\frac{\omega_0}{3}(1-e^{-3t})\right] & \cos\left[\frac{\omega_0}{3}(1-e^{-3t})\right] \end{pmatrix}$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$Y(t) = x_2(t) =$$

$$x_1(0)\sin\left[\frac{\omega_0}{3}(1-e^{-3t})\right] +$$

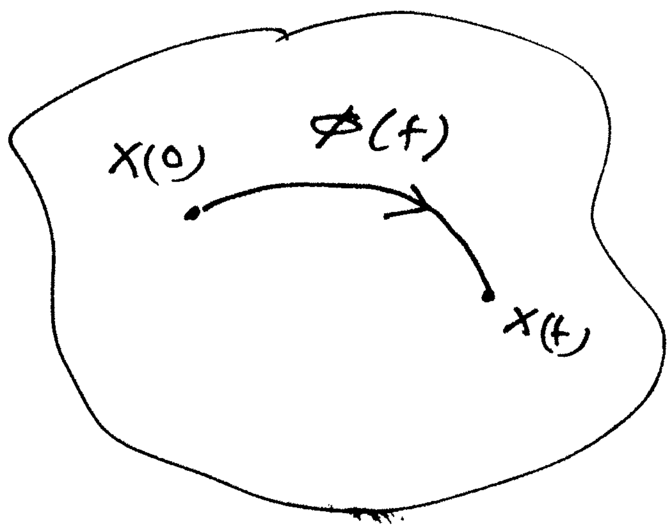
$$x_2(0)\cos\left[\frac{\omega_0}{3}(1-e^{-3t})\right]$$

$$= \frac{Y_1}{\omega_0}\sin\left[\frac{\omega_0}{3}(1-e^{-3t})\right] + Y_0\cos\left[\frac{\omega_0}{3}(1-e^{-3t})\right]$$

What does $\phi(t)$ mean?!

Recall

$$x(t) = \phi(t)\, x(0)$$



$\phi(t)$ maps the vector $x(0)$ at $t=0$ to the vector $x(t)$ at $t=t$

$$\phi(t) : \mathbb{R}^n \to \mathbb{R}^n$$
$$x(0) \longmapsto x(t) = \phi(t)\, x(0).$$

Thus $\phi(t)$ describes how the state evolves in time starting from $t=0$.

Often, we are interested in the transition of state from $t = t_0$ instead of $t = 0$. It becomes necessary to define $\phi(t, t_0)$ as the state transition matrix which satisfies the matrix differential eqn

$$\dot{\phi}(t, t_0) = A(t)\,\phi(t, t_0)$$

where

$$\phi(t_0, t_0) = I.$$

Existence of $\phi(t, t_0)$ will follow from the Peano-Baker series where the lower limit of 0 on page 3 has to be replaced by t_0.

Here onwards, we shall denote by $\phi(t, 0)$, what we had denoted by $\phi(t)$ earlier.

Remark:

1. What does $\phi(t_1, t_2)$ mean ??

Solve matrix o.d.e.

$$\dot{M}(t) = A(t) M(t)$$

$$M(t_2) = I.$$

Calculate $M(t_1)$. This would be $\phi(t_1, t_2)$.

2. Consider the vector o.d.e

$$\dot{x}(t) = A(t) x(t)$$

$$x(t_2) = x_0.$$

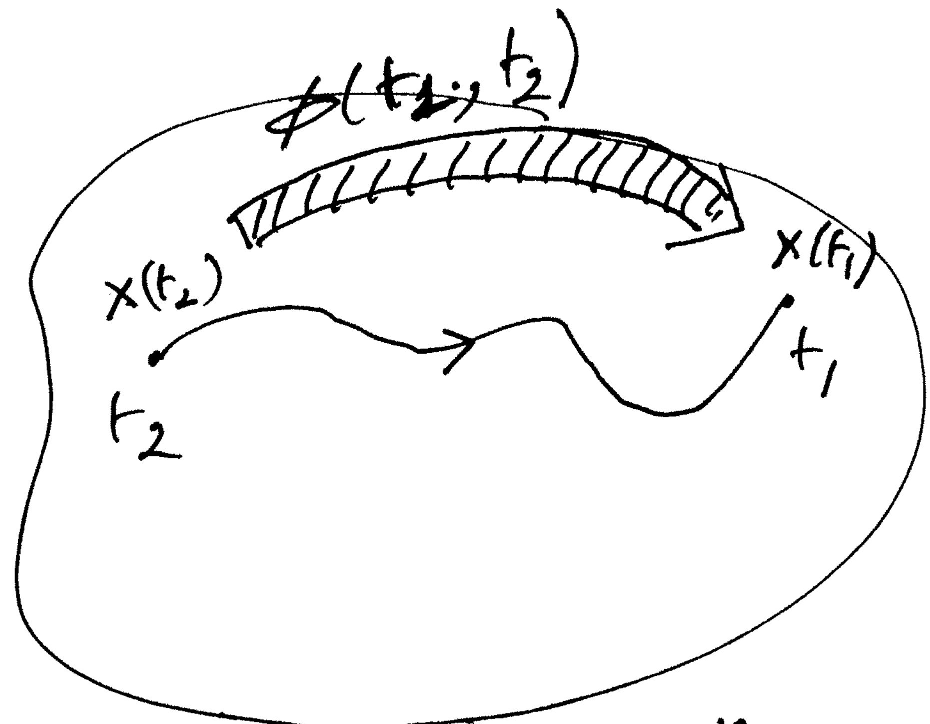
It would follow that

$$x(t) = \phi(t, t_2)\, x_0 .$$

and

$$x(t_1) = \phi(t_1, t_2)\, x_0 .$$

$\phi(t_1, t_2)$ maps the state from x_0 at $t = t_2$ to $x(t_1)$ at $t = t_1$.



$$\phi(t_1, t_2) : \mathbb{R}^n \to \mathbb{R}^n .$$

$$x(t_2) \longmapsto x(t_1) = \phi(t_1, t_2)\, x(t_2)$$

Theorem:

If t_1, t_2, t_3 are 3 instances of time where $t_1 \leq t_2 \leq t_3$ we have

$$\phi(t_3, t_1) = \phi(t_3, t_2)\phi(t_2, t_1)$$

Theorem

If t_1, t_2, t_3 are any 3 instances of time (not necessarily ordered)

$$\phi(t_3, t_1) = \phi(t_3, t_2)\, \phi(t_2, t_1).$$

Cor: If $t_3 = t_1$ we have

$$\phi(t_1, t_1) = \phi(t_1, t_2)\, \phi(t_2, t_1) = I$$

Hence $\phi(t_1, t_2) = \phi(t_2, t_1)^{-1}$!

Choosing $t_1 = 0,\ t_2 = t$ we have

$$\phi(0,t) = \phi(t,0)^{-1}$$

$$\begin{tikzcd}
\underset{t=0}{\mathbb{R}^n} \arrow[r, bend left, "\phi(t,0)"] & \underset{t=t}{\mathbb{R}^n} \arrow[l, bend left, "\phi(0,t)"]
\end{tikzcd}$$

$\therefore\ \phi(0,t)\,\phi(t,0) = I$

It follows that

$$\frac{d}{dt}\left[\phi(0,t)\right]\phi(t,0) + \phi(0,t)\frac{d}{dt}\left[\phi(t,0)\right] = 0$$

$$\Rightarrow \frac{d}{dt}\left[\phi(0,t)\right]\cdot\phi(t,0) + \phi(0,t)A\phi(t,0) = 0$$

$$\Rightarrow \left[\frac{d}{dt}\phi(0,t) + \phi(0,t)A\right] \phi(t,0) = 0.$$

$$\forall\ \phi(t,0).$$

$$\Rightarrow \frac{d}{dt}\phi(0,t) = -\phi(0,t)A.$$

$$\Rightarrow \boxed{\frac{d}{dt}\phi(t,0)^{-1} = -\phi(t,0)^{-1}A}$$

Th$\underline{m}$ (Abel-Jacobi-Liouville)

If $\phi(t, t_0)$ is the transition matrix for $\dot{x}(t) = A(t)\,x(t)$ then

$$\det\left[\phi(t, t_0)\right] = e^{\int_{t_0}^{t} \text{trace}\, A(\sigma)\, d\sigma}$$

Hence $\phi(t, t_0)$ is non-singular if

$$\int_{t_0}^{t} \text{trace}\, A(\sigma)\, d\sigma \quad \text{is finite.}$$

Proof: The theorem follows easily from this following fact:

Let $M(t)$ be a square $n \times n$ matrix

then

$$\frac{d}{dt}\left[\det M(t)\right] = \text{trace}\left[C^T(t)\, \dot{M}(t)\right] \quad *$$

where $C^T(t)$ is the transpose of the cofactor matrix.

Replacing $M(t)$ by $\phi(t, t_0)$ we obtain

$$\frac{d}{dt}\left[\det \phi(t, t_0)\right] =$$

$$\operatorname{trace}\left[C^T(t, t_0)\, A(t)\phi(t, t_0)\right]$$

$$= \operatorname{trace}\left[C^T(t, t_0)\, \phi(t, t_0) \cdot A(t)\right]$$

However

$$C^T(t, t_0)\, \phi(t, t_0)$$

$$= \det \phi(t, t_0) \cdot I$$

Hence

$$\frac{d}{dt}\left[\det \phi(t, t_0)\right] = \operatorname{trace}\left[\det \phi(t, t_0)\, A(t)\right]$$

$$= \det \phi(t, t_0) \operatorname{trace} A(t)$$

It follows that

$$\phi(t, t_0) = e^{\int_{t_0}^{t} \mathrm{trace}\, A(\sigma)\, d\sigma}$$

∎

Remark:

The determinant of an $n \times n$ matrix can be interpreted as the volume in E^n contained in the parallelopiped generated by the columns in the matrix

Thus Abel-Jacobi-Liouville's theorem gives us an expression for how this volume evolves in time. In particular if

$$\operatorname{trace} A(t) = 0 \quad \forall t$$

we have

$$\det \phi(t, t_0) = 1 \quad \forall t,$$

Hence the volume remains constant. "Volume preserving flow".

$\dot{x}(t) = A(t)\,x(t)$ $\qquad\qquad$ $\dot{z}(t) = B(t)\,z(t)$

$x(t_0) = x_0$ $\qquad\qquad$ $z(t_0) = z_0$

Assume

$$z(t) = P(t)\,x(t)$$

$$\Rightarrow \dot{z}(t) = \dot{P}x + P\dot{x}$$

$$= \dot{P}x + PAx$$

$$= (\dot{P} + PA)\,x$$

$$= \underbrace{(\dot{P} + PA)\,P^{-1}}_{B}\,z$$

$$\dot{z} = B z$$

$$z_0 = z(t_0) = P(t_0)\,x(t_0) = P(t_0)\,x_0 .$$

Define

$\phi_{\underline{x}}(t, t_0) \leftarrow$ transition for x system.

$\phi_z(t, t_0) \leftarrow$ " " z "

Theorem

$$\Phi_X(t,t_0) = P^{-1}(t)\,\Phi_Z(t,t_0)\,P(t_0)$$

$$\begin{tikzcd}
x\ \text{space}\ \mathbb{R}^n \arrow[r, "\Phi_x(t,t_0)"] \arrow[d, "P(t_0)"'] & \mathbb{R}^n \\
z\ \text{space}\ \mathbb{R}^n \arrow[r, "\Phi_z(t,t_0)"'] & \mathbb{R}^n \arrow[u, "P^{-1}(t)"']
\end{tikzcd}$$

Proof:

$$\dot{\Phi}_x(t,t_0) = \left[\frac{d}{dt}P^{-1}(t)\right]\Phi_z(t,t_0)P(t_0)$$

$$+\ P^{-1}(t)\,\frac{d}{dt}\left[\Phi_z(t,t_0)\right]P(t_0)$$

$$= -P(t)^{-1}\,\dot{P}\,P(t)^{-1}\,\Phi_z(t,t_0)\,P(t_0)$$

$$+\ P^{-1}(t)\,B(t)\,\Phi_z(t,t_0)\,P(t_0).$$

$$= -P^{-1}\dot{P}P^{-1}\phi_z(t,t_0)P(t_0)$$

$$+P^{-1}(\dot{P}+PA)P^{-1}\phi_z(t,t_0)P(t_0).$$

$$= P^{-1}PAP^{-1}\phi_z(t,t_0)P(t_0).$$

$$= AP^{-1}\phi_z(t,t_0)P(t_0)$$

$$= A\phi_x(t,t_0)$$

If we assume that $B(t)=B$ a constant matrix we find a $P(t)$:

$$\left[\dot{P}(t)+P(t)A(t)\right]P^{-1}(t)=B$$

then
$$\dot{x}=A(t)x(t)$$
$$x(t_0)=x_0$$
can be transformed to a time invariant system

$$\dot{z}(t) = B\,z(t)$$
$$z(t_0) = P(t_0)\,x_0 .$$

This transformation is not very useful since $P(t)$ needs to be computed.

Remark:

We are often interested in placing additional constraints on $P(t)$ as described below:

Def: A matrix $P(t)$ is called a Lyapunov transformation if

1. $P(t), \dot{P}(t)$ are continuous and bounded functions on $[t_0, \infty)$
2. $\exists$ a constant m!
$$0 < m < |\det(P(t))| .$$

Linear Time Varying Dynamical Eqns with periodic $A(t)$.

consider

$$\dot{x} = A(t)x(t) \qquad \text{where } A(t+T) = A(t).$$

$$x(t_0) = x_0$$

Let $\Phi_{\underline{x}}(t,0)$ be the transition matrix.

Claim:

$\Phi_{\underline{x}}(t+T, T)$ is also a transition matrix.

ie $$\frac{d}{dt}\Phi_{\underline{x}}(t+T,T) = A(t)\,\Phi_{\underline{x}}(t+T,T).$$

proof: $$\frac{d}{dt}\Phi_{\underline{x}}(t+T,T) = A(t+T)\,\Phi_{\underline{x}}(t+T,T)$$
(from def)

$$= A(t)\,\Phi_{\underline{x}}(t+T,T).$$

$\blacksquare$

Hence $$\Phi_x(t+T,T) = \Phi_x(t,0)$$
$\forall t$.

Define $Q = \Phi_{\underline{X}}(T, 0)$.

Claim:

$$\Phi_{\underline{X}}(t+T, 0) = \Phi_{\underline{X}}(t, 0)\, Q .$$

Proof:

$$\phi_{\underline{X}}(T, 0) = Q \quad (\text{Def of } Q)$$

$$\Rightarrow \Phi_{\underline{X}}(t+T, T)\, \phi_{\underline{X}}(T, 0) = \Phi_{\underline{X}}(t+T, T)\, Q .$$

$$= \Phi_{\underline{X}}(t, 0)\, Q .$$

$$\Rightarrow \Phi_{\underline{X}}(t+T, 0) = \Phi_{\underline{X}}(t, 0)\, Q .$$

$\blacksquare$

A very confusing and possibly nontrivial

Lemma :-

Let Q be a non-singular matrix

$\exists$ a possibly complex matrix F :

$$e^{FT} = Q .$$

$$P(t) = e^{F(t-t_0)} P(t_0) \phi_{\underline{x}}^{-1}(t, t_0)$$

We define $P(t)$ as above

or

$$P(t) = e^{Ft} \phi_{\underline{x}}^{-1}(t, 0)$$

Starting from

$$\dot{x}(t) = A(t)x(t)\,,\quad x(t_0) = x_0$$

$$A(t+T) = A(t) \quad \forall t.$$

Define

$$\begin{aligned} z(t) &= P(t)\,x(t) \\ &= \cancel{e^{Ft}\,\Phi_x^{-1}(t,0)\,x(t)} \\ &= e^{F(t-t_0)}\,P(t_0)\,\phi_x^{-1}(t,t_0)\,x(t) \end{aligned}$$

$$\begin{tikzcd} x \text{ space} \quad \mathbb{R}^n \arrow[r, "\phi_x(t,t_0)"] \arrow[d, "P(t_0)"'] & \mathbb{R}^n \arrow[d, "P(t)."] \\ z \text{ space} \quad \mathbb{R}^n \arrow[r, "e^{F(t-t_0)}"'] & \mathbb{R}^n \end{tikzcd}$$

We obtain $\dot{z} = F z\,,\ z(t_0) = z_0.$

Claim

$$P^{-1}(t+T) = P^{-1}(t) \quad \forall t.$$

Proof:

$$P^{-1}(t+T) =$$

$$\Phi_{\underline{X}}(t+T, 0)\, e^{-F(t+T)}$$

$$= \Phi_{\underline{X}}(t+T, T)\, \Phi_{\underline{X}}(T, 0)\, e^{-FT} e^{-Ft}.$$

$$= \Phi_{\underline{X}}(t, 0)\, e^{FT} e^{-FT} e^{-Ft}$$

$$= \Phi_{\underline{X}}(t, 0)\, e^{-Ft}$$

$$= P^{-1}(t)$$

Remark:

It follows from this claim that

$$z(t) = P(t)\, x(t)$$

is a Liapunov transformation.

Theorem (Floquet - Liapunov)

The dynamical system

$\dot{x}(t) = A(t)x(t)$ $A(t)$ is continuous and differentiable in $[0,\infty)$

$A(t+T) = A(t)$

is reducible to a dynamical system with constant co-efficient

$\dot{z}(t) = F z(t)$

via a Liapunov transformation

$z(t) = P(t)\, x(t)$

where

$$P(t) = e^{Ft}\, \Phi_X^{-1}(t,0).$$

and where F is defined as follows

$$e^{FT} \triangleq \Phi_X(T,0).$$