Lec 4

In lec 3 we have introduced state space system of the form

$$\dot{\underline{X}} = A\underline{X} + Bu \qquad (4.1)$$

we considered two specific examples — the satellite problem and the spread of an epidemic disease problem. Assuming $u \equiv 0$, we solved the autonomous equation

$$\dot{\underline{X}} = A\underline{X}, \quad \underline{X}(0) = X_0 \qquad (4.2)$$

We also discussed how the autonomous system can be reduced to a canonical form by defining

$$\underline{X} = TZ \qquad (4.3)$$

and $\dot{Z} = T^{-1}ATZ + T^{-1}Bu$

The purpose of this lecture is to formally solve (4.1), (4.2) using matrix exponentials.

Exponential of a $n \times n$ matrix A:

Let A be a $n \times n$ matrix. We define

$$e^{At} \triangleq I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots\cdots \qquad (4.4)$$

The above power series converges uniformly for any choice of A. to a matrix f^{n} of t.

Example:

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

can be verified by directly applying (4.4)

Example:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Verify by actual multiplication that

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \& \quad A^j = 0 \quad \text{for } j > 2.$$

$$e^{At} = \underline{I} + At + \frac{A^2 t^2}{2!}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & t^2/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Remark:

In general if A & B are $n \times n$ matrices.

$$e^{(A+B)t} \neq e^{At} e^{Bt} \qquad (4.5)$$

unless A & B commute

i.e $AB = BA$.

Example

$$A = \begin{pmatrix} -5 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & -5 \end{pmatrix}$$

write

$$A = C + D$$

where

$$C = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

verify that $CD = DC$, hence

$$e^{At} = e^{Ct} e^{Dt}$$

$$e^{Ct} = \begin{pmatrix} e^{-5t} & 0 & 0 \\ 0 & e^{-5t} & 0 \\ 0 & 0 & e^{-5t} \end{pmatrix}; \quad e^{Dt} = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Thus

$$e^{At} = \begin{pmatrix} e^{-5t} & te^{-5t} & \frac{t^2}{2}e^{-5t} \\ 0 & e^{-5t} & te^{-5t} \\ 0 & 0 & e^{-5t} \end{pmatrix}$$

Example:

If A is $m \times m$, B is $p \times p$ and

(A) $C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$

then

$$e^{Ct} = \begin{pmatrix} e^{At} & 0 \\ 0 & e^{Bt} \end{pmatrix}$$

(B) $C = \begin{pmatrix} A & I \\ 0 & A \end{pmatrix}$

then

$$e^{Ct} = \begin{pmatrix} e^{At} & te^{At} \\ 0 & e^{At} \end{pmatrix}$$

Example:

$$A = \begin{pmatrix} -5 & 1 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & -6 \end{pmatrix}$$

$$e^{At} = \left(\begin{array}{cc|cc} e^{-5t} & te^{-5t} & 0 & 0 \\ 0 & e^{-5t} & 0 & 0 \\ \hline 0 & 0 & e^{-6t} & te^{-6t} \\ 0 & 0 & 0 & e^{-6t} \end{array}\right)$$

Example (non trivial)

$$A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}$$

Example

$$A = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$$

writing

$$A = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

we obtain

$$e^{At} = \begin{pmatrix} e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t \\ -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{pmatrix}$$

Remark:

Let A and B be two $n \times n$ matrices such that $\exists$ a $n \times n$ matrix T:

$$A = T^{-1} B T$$

then

$$\boxed{e^{At} = T^{-1} e^{Bt} T} \qquad (4\cdot 6)$$

<u>Theorem</u>

$$\frac{d}{dt} e^{At} = A e^{At}$$

can be verified by differentiating (4·4) term by term.

<u>Solving autonomous eq$^{\underline{n}}$ (4·2)</u>

Consider

$$\dot{x} = A x, \quad x(0) = x_0 \qquad (4\cdot 2)$$

An unique sol$^{\underline{n}}$ of (4·2) is given by

$$x(t) = e^{At} x_0 .$$

Note that

$$\dot{x}(t) = \frac{d}{dt}\left[e^{At}x_0\right]$$

$$= Ae^{At}x_0$$

$$= Ax(t)$$

Moreover

$$x(0) = e^{A0}x_0 = x_0 \quad \because e^{0} = I$$

Theorem

Consider the state space system (4·1)

i.e

$$\dot{\underline{x}} = A\underline{x} + Bu \quad (4·1)$$

A is a constant $n \times n$ matrix

B is a constant $n \times p$ matrix.

where

$$\underline{x}(0) = \underline{x}_0 \in \mathbb{R}^n$$

Then an unique sol$\underline{n}$ of (4·1) is given by

$$\underline{x}(t) = e^{At}\underline{x}_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau. \quad (4·7)$$

Proof:

$$\dot{\underline{x}}(t) = \frac{d}{dt}\left[e^{At}\left(\underline{x}_0 + \int_0^t e^{-A\tau} B u(\tau)\, d\tau\right)\right]$$

$$= \left[\frac{d}{dt} e^{At}\right]\left(\underline{x}_0 + \int_0^t e^{-A\tau} B u(\tau)\, d\tau\right)$$

$$+ e^{At}\left(\frac{d}{dt}\int_0^t e^{-A\tau} B u(\tau)\, d\tau\right)$$

$$= A e^{At}\left(\underline{x}_0 + \int_0^t e^{-A\tau} B u(\tau)\, d\tau\right)$$

$$+ e^{At}\left(e^{-At} B u(t)\right)$$

$$= A\left[e^{At}\underline{x}_0 + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau\right] + B u(t)$$

$$= A\,\underline{x}(t) + B u(t).$$

$$\underline{X}(0) = e^{0}\,\underline{X}_0 + \int_0^0 e^{-A\tau} B u(\tau)\, d\tau.$$

$$= \underline{X}_0.$$

Ex:

$$\dot{x} = ax + bu(t)$$

where $x(0) = 10$, $a = -5$, $b = 1$

and $u(t) = 2 \quad t \geqslant 0$

From (4.7) we have

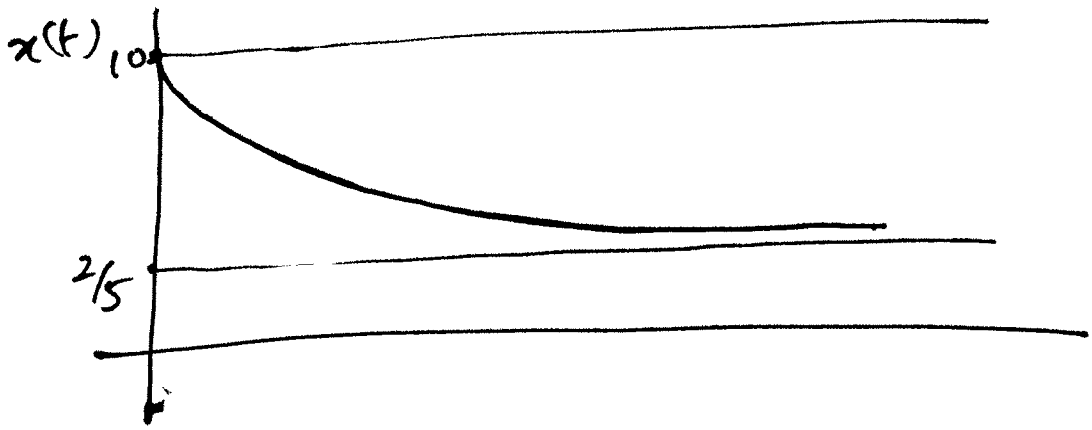
$$x(t) = e^{-5t}(10) + \int_0^t e^{-5(t-\tau)}\, 2\, d\tau.$$

$$= 10e^{-5t} + 2e^{-5t}\int_0^t e^{5\tau}\, d\tau.$$

$$= 10e^{-5t} + \frac{2}{5}e^{-5t}\left[e^{5t} - 1\right]$$

$$= 10e^{-5t} + \frac{2}{5}\left[1 - e^{-5t}\right]$$

Thus $x(t) = \frac{2}{5} + \frac{48}{5}e^{-5t}$



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Remark:

If $\dot{\underline{x}} = A\underline{x}(t) + B(t)u(t)$

$\underline{x}(0) = \underline{x}_0$

$$\underline{x}(t) = e^{At}\underline{x}_0 + \int_0^t e^{A(t-\tau)}B(\tau)u(\tau)\,d\tau. \quad (4.7a)$$

Thus (4.7) is valid even when B is a non-constant matrix

Ex:

$$\dot{x} = ax + bt\,u(t)$$

$$x(0) = 10, \quad a = -5, \quad b = 1$$

and

$$u(t) = 2 \quad t \geq 0$$

we have

$$x(t) = e^{-5t}\,10 + \int_0^t e^{-5(t-\tau)}\,\tau\,2\,d\tau.$$

$$= 10e^{-5t} + 2e^{-5t}\int_0^t \tau e^{5\tau}\,d\tau$$

can you please complete this calculation.

Controllability

1 Consider a simple linear dynamical system

$$\dot{X}(t) = b(t)u(t)$$ ← Note that b is also a f^{n} of 't'.

Where
$x(t_0) = X_0$
and the 'A' matrix is zero. We assume that b and X_0 are known.

The problem is to find, if possible, a $u(t)$ such that $x(t_1) = X_f$ at some $t_1 > t_0$.

We can write

$$x(t_1) = x(t_0) + \int_{t_0}^{t_1} b(\tau)u(\tau)\,d\tau .$$

If $x_f - x_0$ lies in the range space of the linear mapping

$$L(u) = \int_{t_0}^{t_1} b(\tau)\, u(\tau)\, d\tau .$$

then the desired transfer is possible.

<u>Lemma</u>:

$x_f - x_0$ lies in the range space of L

iff it lies in the range space of

$$W = \int_{t_0}^{t_1} b(\tau)\, b^T(\tau)\, d\tau$$

Proof:

If $x_f - x_0$ lines in the range of W then

$\exists\, \eta \in \mathbb{R}^n$:

$$x_f - x_0 = W \eta$$

$$\Rightarrow x_f - x_0 = \int_{t_0}^{t_1} b(\tau)\, b^T(\tau)\, d\tau\, \eta$$

$$= \int_{t_0}^{t_1} b(\tau)\, u(\tau)\, d\tau$$ where we define $u(\tau) = b^T(\tau)\eta$.

Hence

$x_f - x_0$ lies in the range space of L.

Conversely

If $x_f - x_0$ does not lie in the range of W it follows that $\exists\, \rho \in \mathbb{R}^n$:

$$W\rho = 0 \quad \text{and} \quad \rho^T(x_f - x_0) \neq 0$$

$\Uparrow$

The above fact is a consequence of W being a symmetric matrix. "If W is symmetric matrix then Range of W $\perp$ Null space of W

We would like to show that

$x_f - x_0$ does not lie in the range

space of L.

Assume it does, it would follow that

$$x_f - x_0 = \int_{t_0}^{t_1} b(\tau) u_1(\tau)\, d\tau \qquad (*)$$

for some $u_1(\tau)$.

$(*)$ would imply that

$$p^T (x_f - x_0) = \int_{t_0}^{t_1} p^T b(\tau) u_1(\tau)\, d\tau \neq 0 \qquad (\triangle)$$

where as

$$p^T W p = 0 \Rightarrow \int_{t_0}^{t_1} p^T b(\tau) b^T(\tau) p \, d\tau = 0 \qquad (**)$$

It follows from ** that

$$\rho^T b(\tau) \equiv 0$$

which contradicts $\triangle$

This completes the proof

$\blacksquare$

Corollary

There exists a control $u(t)$ which transfers the state of the system $\dot{x} = b(t)u(t)$ from x_0 at $t = t_0$ to x_1 at $t = t_1$ iff $x_1 - x_0$ lies in the range of W. If this transfer is possible, then one particular control which actually drives the state from x_0 to x_1 is

$$u(t) = b^T(t)\eta$$

where η is any solution of $W\eta = x_1 - x_0$.

Ex 1

consider

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix} u(t)$$

Q. Is the above system controllable?

A. $$W = \int_{t_0}^{t_1} \begin{pmatrix} \tau \\ \tau^2 \end{pmatrix} (\tau \quad \tau^2)\, d\tau$$

$$= \int_{t_0}^{t_1} \begin{pmatrix} \tau^2 & \tau^3 \\ \tau^3 & \tau^4 \end{pmatrix} d\tau .$$

$$= \left.\begin{pmatrix} \frac{\tau^3}{3} & \frac{\tau^4}{4} \\ \frac{\tau^4}{4} & \frac{\tau^5}{5} \end{pmatrix}\right|_{t_0}^{t_1}$$

Rank $W = 2$ Hence controllable.

If we want to drive this system from 7

$$\underline{X}(0)=\begin{pmatrix}0\\0\end{pmatrix} \quad \text{to} \quad \underline{X}(1)=\begin{pmatrix}4\\5\end{pmatrix}$$

We have

$$W=\begin{pmatrix}\frac{1}{3} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{5}\end{pmatrix}$$

Need to solve

$$W\eta = \begin{pmatrix}4\\5\end{pmatrix}$$

$$\eta = \begin{pmatrix}1/3 & 1/4\\ 1/4 & 1/5\end{pmatrix}^{-1}\begin{pmatrix}4\\5\end{pmatrix}$$

$$u(t) = \begin{pmatrix}t & t^2\end{pmatrix}\begin{pmatrix}1/3 & 1/4\\ 1/4 & 1/5\end{pmatrix}^{-1}\begin{pmatrix}4\\5\end{pmatrix}$$

$\boxed{\text{End of Example}}$

(2) We would now generalize our results to the dynamical system

$$\dot{x} = Ax + b(t)u(t) \qquad \begin{matrix} x(t_0) = x_0 \\ x(t_1) = x_1 \end{matrix}$$

we assume A is not a f^n of 't'.

We have

$$x_1 = x(t_1) = e^{A(t_1 - t_0)} x_0 + \int_{t_0}^{t_1} e^{A(t_1 - \tau)} b(\tau) u(\tau) \, d\tau$$

Multiplying by

$$e^{A(t_0 - t_1)}$$

on both sides, we have

$$e^{A(t_0 - t_1)} x_1 = x_0 + \int_{t_0}^{t_1} e^{A(t_0 - \tau)} b(\tau) u(\tau) d\tau$$

$$\Rightarrow \underbrace{e^{A(t_0 - t_1)} x_1 - x_0}_{\triangleq \bar{x}} = \int_{t_0}^{t_1} e^{A(t_0 - \tau)} b(\tau) u(\tau) d\tau.$$

For controllability

$\bar{x}$ must be in the range of $\bar{L}$

where

$$\bar{L}(u) = \int_{t_0}^{t_1} e^{A(t_0-\tau)} b(\tau) u(\tau)\, d\tau .$$

Define

$$z(t) = e^{A(t_0-t)} x(t)$$

It follows that

$$\dot{z} = e^{A(t_0-t)}(-Ax) +$$

$$e^{A(t_0-t)}(Ax + bu)$$

$$= e^{A(t_0-t)} b(t)\, u(t) .$$

Define $b_1(t) = e^{A(t_0-t)} b(t)$

we have

$$\dot{Z} = b_1(t)\, u(t) \quad ☆☆$$

Which is a control system in the form discussed before. We state the following

__Lemma__

$\bar{x}$ lies in the range space of $\bar{L}$ iff it lies in the range space of.

$$\bar{W} = \int_{t_0}^{t_1} b_1(\tau) b_1^T(\tau)\, d\tau$$

$$= \int_{t_0}^{t_1} e^{A(t_0-\tau)} b(\tau) b^T(\tau) e^{A^T(t_0-\tau)}\, d\tau .$$

If $\bar{x}$ is in the range space of $\bar{L}$, then we can choose $u(t)$ given by

$$u(t) = b_1^T(t)\,\eta$$

$$= e^{A(t_0 - t)}\, b(t)\,\eta$$

where η is any solution of

$$\bar{W}\eta = \bar{x}.$$

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Remark:

1. $\overline{W}$ is called the controllability gramian. It depends on both t_0 & t_1 and is written as

$$\overline{W}(t_0, t_1) = \int_{t_0}^{t_1} e^{A(t_0-\tau)} b(\tau) b^T(\tau) e^{A^T(t_0-\tau)} d\tau$$

2. If A is also a $f^{\underline{n}}$ of t, we replace

$e^{A(t_0-\tau)}$ by $\phi(t_0, \tau)$

where ϕ is the transition matrix

If $\dot{x} = A(t)x(t)$, $x(t_0) = x_0$

then $\phi(t, t_0)$ is sol$^{\underline{n}}$ to the matrix diff eq$^{\underline{n}}$.

$\dot{\phi} = A(t)\phi$

$\phi(t_0) = I$

$\dot{\phi}(t, t_0) = A(t)\phi(t, t_0)$

$\phi(t_0, t_0) = I$

3 $\overline{W}(t_0, t_1)$ is symmetric and non-negative definite for $t_1 \geqslant t_0$

$\overline{W}(t, t_1)$ satisfies the linear matrix differential equation

$$\frac{d}{dt}\overline{W}(t, t_1) = A(t)\overline{W}(t, t_1) + \overline{W}(t, t_1)A^T(t) - b(t)b^T(t)$$

where $\overline{W}(t_1, t_1) = 0$

When A & b are time invariant

$$\dot{x} = Ax + bu$$

$$\bar{W}(t_0,t_1) = \int_{t_0}^{t_1} e^{A(t_0-\tau)}\, b\, b^T e^{A^T(t_0-\tau)}\, d\tau .$$

Define controllability matrix

$$\mathcal{C} = \left(b \quad Ab \quad A^2 b \cdots A^{n-1} b\right)$$

and

$$W_I = \mathcal{C}\,\mathcal{C}^T.$$

<u>Theorem</u>: Range and null space of $\bar{W}$ coincides with the range and null space of W_I

$$\text{Rank of } \bar{W} = \text{Rank of } \mathcal{C}$$

Range space and null space of $\bar{W}$ does not depend on the choice of t_0 and t_1

Proof of the Theorem

Let $x_1 \in$ Null space of $\overline{W}$

then

$$0 = x_1^T \overline{W} x_1 =$$

$$\int_{t_0}^{t_1} x_1^T e^{A(t_0-\tau)} b b^T e^{A^T(t_0-\tau)} x_1 \, d\tau .$$

It follows that

$$b^T e^{A^T(t_0-\tau)} x_1 = 0 \quad \forall \tau .$$

Expanding by Taylor's series we obtain

$$b^T x_1 = 0$$

$$b^T A^T x_1 = 0 \qquad \Rightarrow x_1 \in \text{Null space of } e^T$$

$$\Rightarrow x_1 \in \text{Null space of } W_I .$$

$$\vdots$$

$$b^T A^{n-1\,T} x_1 = 0 .$$

Conversely let

$$x_1 \in \text{Null space of } W_I .$$

it follows that

$$e e^T x_1 = 0$$

$$\Rightarrow x_1^T e e^T x_1 = 0$$

$$\Rightarrow x_1^T e = 0$$

$$\Rightarrow x_1^T A^i b = 0 \quad i = 0, 1, \ldots, n-1$$

using Cayley Hamilton Theorem

Writing

$$e^{A(t_0-\tau)} = \sum_{i=0}^{n-1} \alpha_i(t_0-\tau) A^i$$

α_i is a fn of $t_0 - \tau$.

We have

$$x_1^T \bar{W} = \int_{t_0}^{t_1} x_1^T e^{A(t_0-\tau)} b b^T e^{A^T(t_0-\tau)} d\tau$$

$$= \int_{t_0}^{t_1} \underbrace{\left[\sum_{i=0}^{n-1} \alpha_i(t_0-\tau) x_1^T A^i b \right]}_{=0} b^T e^{A^T(t_0-\tau)} d\tau$$

$\therefore$ $\overline{W}$ is symmetric we have

$$\overline{W} x_1 = 0$$

Hence
$x_1 \in$ Null space of $\overline{W}$.

Thus we have:

Null space of $\overline{W}$ coincides with the null space of W_I.

Moreover since $\overline{W}$ and W_I are symmetric their range spaces are orthogonal to the null spaces. It follows that

Range space of $\overline{W}$ coincides with the Range space of W_I.

Finally to show that

$$\text{Rank of } \overline{W} = \text{Rank of } e$$

note that

Null space of W_I coincides with the null space of e^T

Hence

$$\text{rank of } W_I = \text{rank of } e^T$$

However

$$\text{rank } e^T = \text{rank } e$$

Hence $\text{rank of } W_I = \text{rank of } e$.

Finally Range and null spaces of $\overline{W}$ and W_I are same, hence

$$\text{rank } \overline{W} = \text{rank } W_I = \text{rank } e. \quad ▨$$

Problem

Consider the 2nd order system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t \\ t^2 \end{pmatrix} u(t).$$

Find if possible a $u(t)$ that will drive the system from $x(0) = 0$ to $x(1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

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Problem:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ 1 & \beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$

$$\beta = -3, \alpha = -2$$

Repeat the above problem.

Back to the satellite problem.

Recall from Lec 3 that the linearized motion is given by

$$\dot{x} = Ax + Bu$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

An easy calculation shows that

$$(B \,|\, AB \,|\, A^2B \,|\, A^3B) =$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 2\omega & -\omega^2 & 0 \\ 1 & 0 & 0 & 2\omega & -\omega^2 & 0 & 0 & -2\omega^3 \\ 0 & 0 & 0 & 1 & -2\omega & 0 & 0 & -4\omega^2 \\ 0 & 1 & -2\omega & 0 & 0 & -4\omega^2 & 2\omega^3 & 0 \end{pmatrix}$$

which is of rank 4. So the linearized motion is controllable.

A) If one of the input u_2 is inoperative ie if $u_2 = 0$ we choose

$$B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and obtain

$$(B \ AB \ A^2B \ A^3B) = \begin{pmatrix} 0 & 1 & 0 & -\omega^2 \\ 1 & 0 & -\omega^2 & 0 \\ 0 & 0 & -2\omega & 0 \\ 0 & -2\omega & 0 & 2\omega^3 \end{pmatrix}$$

which is of rank 3.

B) If u_1 is inoperative ie if $u_1 = 0$

we choose

$$B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and obtain

$$(B \quad AB \quad A^2B \quad A^3B) =$$

$$\begin{pmatrix} 0 & 0 & 2\omega & 0 \\ 0 & 2\omega & 0 & -2\omega^3 \\ 0 & 1 & 0 & -4\omega^2 \\ 1 & 0 & -4\omega^2 & 0 \end{pmatrix}$$

which is of rank 4.

$\therefore$ u_1 was radial thrust u_2 was tangential thrust we see that loss of radial thrust does not destroy controllability whereas loss of tangential thrust does.

Problem:

Discuss the controllability properties of the state space eqn on page 29 Lec 3 The eqn for "spread of an epidemic disease"