Lec 2

Linear Mappings (continued)

Let L be a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$

We say that an n tuple x_1 lies in the null space of L if $L(x_1) = 0$

$$\mathcal{N}(L) = \{ x_1 : L(x_1) = 0 \}.$$

We say that an m-tuple z_1 lies in the range space of L if there exists an n vector x_2 such that $L(x_2) = z_1$.

$$\mathcal{R}(L) = \{ z_1 : \exists\, x_2 \in \mathbb{R}^n,\ L(x_2) = z_1 \}.$$

If we have a set of k linearly independent n tuples $x_1, x_2, \ldots, x_k$ such that

$$L(x_i) = 0 \qquad i = 1, 2, \ldots, k.$$

and if $\exists$ no set of $k+1$ l.i. n tuples $x_1, \ldots, x_{k+1}$: $L(x_i) = 0$ $i = 1, \ldots, k+1$

we say that

"Null space of L has dimension k and that the nullity of L is k".

The range of L and the rank of L can be likewise defined.

— x —

Matrices :—

Matrices help us do the arithmetic associated with Linear mapping. One of the common (so says the author) arithmetical operation is to compute the effect of two successive linear mappings. To do this, we need matrix multiplication.

If $A \in R^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{m \times p}$
if $C = AB$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Note that

$$AB \neq BA \text{ in general}$$

even when BA is defined.

Def:

O matrix : A matrix with all elements 0.

I identity matrix: diagonal elements 1
all other " 0.

Note that $O + A = A$
$IA = AI = A$ for all A.

- Matrices are useful in representing simultaneous equations.

- Matrices are useful in representing linear maps between two finite dimensional vector spaces.

Ex:

$$2x + 3y + 4z = 10$$
$$x + 15y + 19z = 20$$

$$\begin{pmatrix} 2 & 3 & 4 \\ 1 & 15 & 19 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$$

$$\underline{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} ; \quad Z = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$$

$$A \underline{X} = Z$$

$$L: \mathbb{R}^3 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 2x+3y+4z \\ x+15y+19z \end{pmatrix}$$

$$= A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\underline{x} \longmapsto A\underline{x}.$$

— x —

- Transpose of a matrix
- Symmetric matrices.

— x —

Adjoint of a linear transformation

Let X be an inner product space with inner product $\langle\ ,\ \rangle_X$

Let Y be an inner product space with inner product $\langle\ ,\ \rangle_Y$

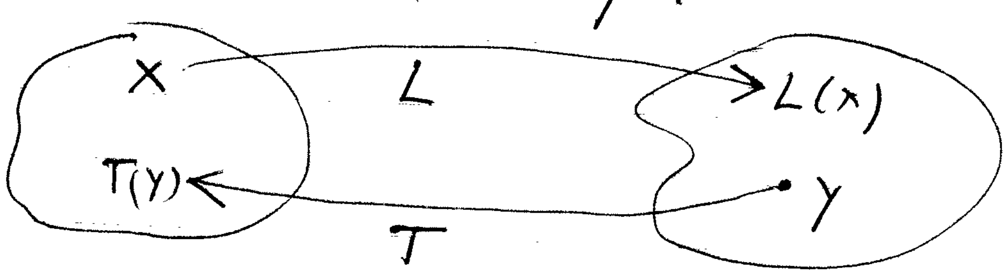
$L : X \longrightarrow Y$

$T : Y \longrightarrow X$

We will say that T is an adjoint of L if

$$\langle y, L(x) \rangle_Y = \langle T(y), x \rangle_X$$

for all x & y.



Ex:

$$L: \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 15 & 19 \end{pmatrix}$$
$$\underline{X} \longmapsto A\underline{X}$$

Let $Y \in \mathbb{R}^2$, we have

$$\langle Y, L(\underline{X}) \rangle_{\mathbb{R}^2} = Y^T A \underline{X}$$
$$= (A^T Y)^T \underline{X}$$
$$= \langle A^T Y, \underline{X} \rangle_{\mathbb{R}^3}$$

Hence

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$Y \longmapsto A^T Y \qquad A^T = \begin{pmatrix} 2 & 1 \\ 3 & 15 \\ 4 & 19 \end{pmatrix}$$

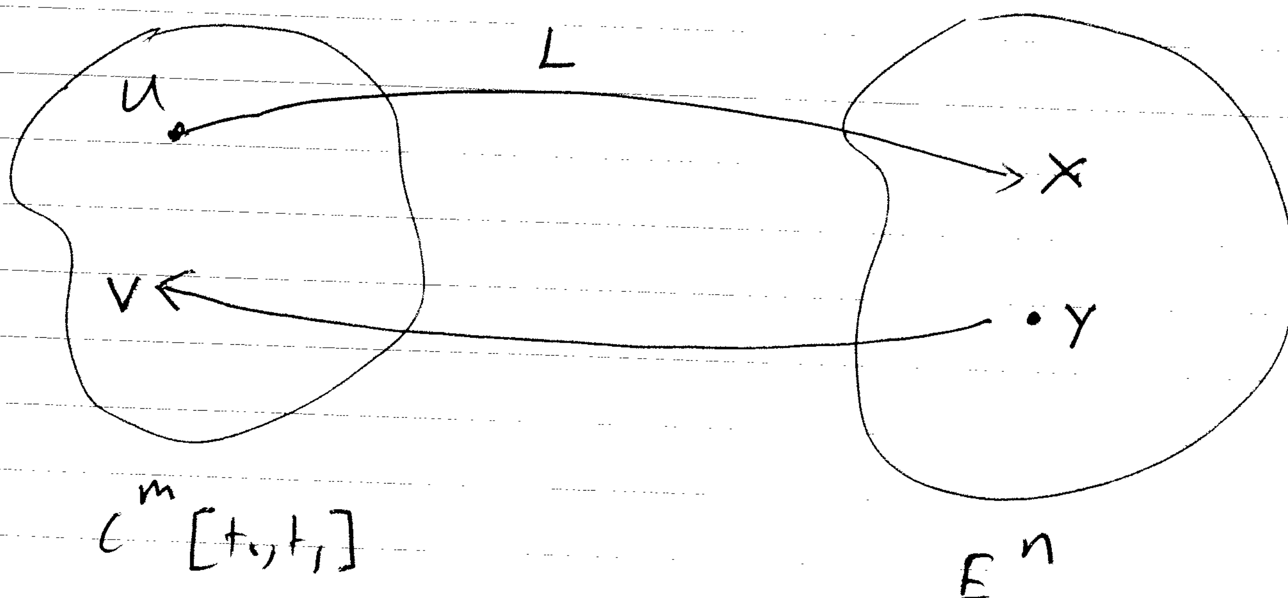
is the transpose of A.

Ex:

$$L : C^m[t_0, t_1] \longrightarrow \underline{E}^n$$

$$u \longmapsto \int_{t_0}^{t_1} B(\sigma)\, u(\sigma)\, d\sigma = x$$

B is a $n \times m$ matrix of continuous functions



$$\langle y, L(u) \rangle = y^T L(u)$$

$$= \int_{t_0}^{t_1} y^T B(\sigma)\, u(\sigma)\, d\sigma$$

$$= \int_{t_0}^{t_1} \left(B^T(\sigma)\, y\right)^T u(\sigma)\, d\sigma$$

$$= \int_{t_0}^{t_1} v(\sigma)^T u(\sigma)\, d\sigma$$

if we define

$$v(\sigma) = B^T(\sigma)\, y$$

Thus

$$T : E^n \longrightarrow C^m [t_0, t_1]$$

$$y \longmapsto B^T(\sigma)\, y$$

— X —

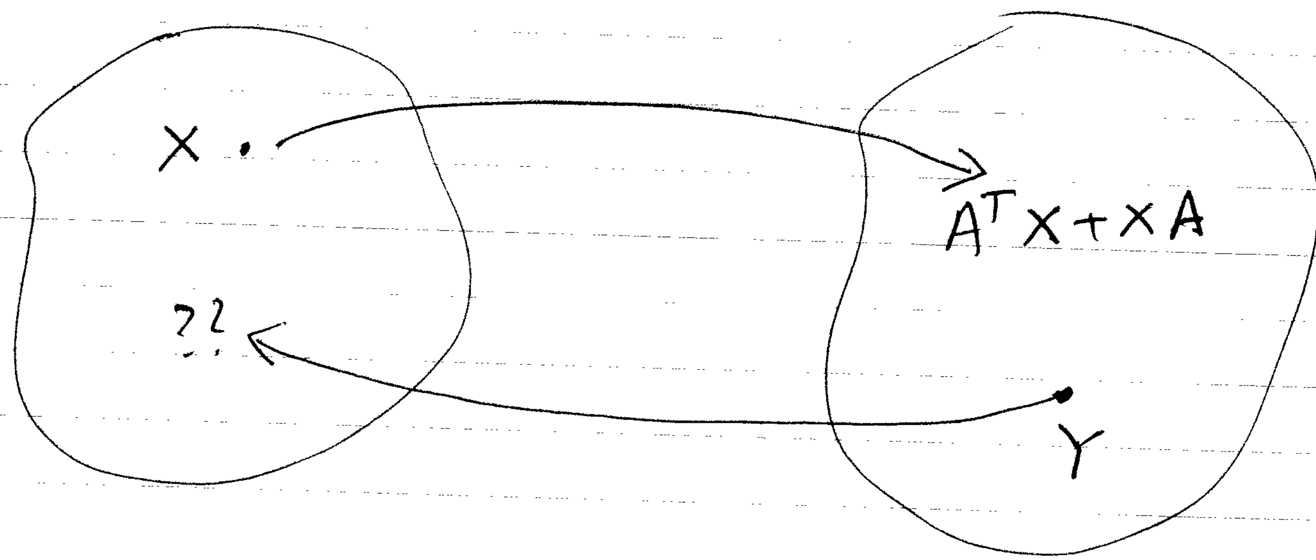
Often we shall use the notation L^* for adjoint L.

Ex:

$$L : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$$

$$X \longmapsto A^T X + XA$$

A is any fixed $n\times n$ matrix.



$$\langle Y, L(X)\rangle = \langle Y, A^T X + XA\rangle$$

$$= \langle Y, A^T X\rangle + \langle Y, XA\rangle$$

$$= tr(Y^T A^T X) + tr(Y A^T X^T)$$

$$= \operatorname{trace}(Y^T A^T X) + \operatorname{trace}(X A Y^T)$$

$$= \operatorname{trace}(Y^T A^T X) + \operatorname{trace}(A Y^T X)$$

$$= \operatorname{trace}\left(\left(AY + YA^T\right)^T X\right)$$

Hence

$$L^* Y = AY + YA^T$$

——— x ———

Theorem:

The vector equation $Ax = z$ with A and z given has a solution iff the following ~~three~~ equivalent conditions are satisfied.

(i) z lies in the range space of A

(ii) z is perpendicular to every vector in the null space of A^T.

(iii) rank A and rank $(A \mid z)$ are the same.

If x_1 is any particular sol^n then any other solution is of the form $x_1 + x_2$ where x_2 lies in the null space of A.

Proof: choose $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$

We first show that $\mathcal{R}(A) \perp \mathcal{N}(A^T)$.

Let $z \in \mathcal{R}(A) \Rightarrow \exists x : Ax = z$.

Let $w \in \mathcal{N}(A^T) \Rightarrow A^T w = 0$

Hence $\langle z, w \rangle = x^T A^T w = 0$

(i) $\Rightarrow$ (ii)

Let $z \in \mathcal{R}(A) \perp \mathcal{N}(A^T)$

$\Rightarrow z \perp \mathcal{N}(A^T)$

(ii) $\Rightarrow$ (i)

Let $U = \mathcal{N}(A^T)^\perp$

we have $\mathcal{R}(A) \subset U$

$\dim U = m - k$ where $k = $ nullity of A^T.

On the other hand

$\text{rank}(A^T) + \text{Nullity}(A^T) = m$

$\Rightarrow \text{rank}(A^T) = m - k$.

But $\text{rank}(A^T) = \text{rank}(A)$

Hence

$\dim(R(A)) = m - k$.

It follows from

$R(A) \subset U$

and $\dim R(A) = \dim U = m - k$.

that $R(A) = U$

Thus every vector in U must be in the range of A.

$$\dot{z} = B(t)\,u(t) \qquad z(t_0) = z_0 \quad (*)$$

$B(t)$ and z_0 are assumed known.

$z \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$

$u \in C^m[t_0, t_1]$

B is a $n \times m$ matrix.

Integrating $(*)$ we get

$$z(t_1) = z_0 + \int_{t_0}^{t_1} B(\sigma)\,u(\sigma)\,d\sigma.$$

Define $z(t_1) = z_1$, we have

$$z_1 - z_0 = \int_{t_0}^{t_1} B(\sigma)\,u(\sigma)\,d\sigma.$$

Often we are not given $u(\sigma)$ but the goal is to find it such that

$z_1 - z_0$ lies in the "range space" of the linear mapping

$$L(u) = \int_{t_0}^{t_1} B(\sigma) u(\sigma) d\sigma.$$

If we can find $u(\sigma)$, we say that the state is transferrable from z_0 to z_1. Otherwise not.

If we can transfer between any two states, then we say that (*) is "controllable".

Problem of controllability reduces to checking characterizing the range space of L, which is a somewhat hard problem.

Theorem

An n tuple x_1 lies in the range space of $L(u) = \int_{t_0}^{t_1} B(\sigma)u(\sigma)\,d\sigma$ iff it lies in the range space of the matrix

$$W(t_0,t_1) = \int_{t_0}^{t_1} B(\sigma)B^T(\sigma)\,d\sigma.$$

Proof: Assume that $x_1 \in$ Range space of W.

$\exists\, \eta_1 : W(t_0,t_1)\,\eta_1 = x_1$

$$\Rightarrow x_1 = \int_{t_0}^{t_1} B(\sigma) B^T(\sigma) \eta_1 \, d\sigma$$

Define $u(t) = B^T(t)\eta_1$ we have

$$x_1 = \int_{t_0}^{t_1} B(\sigma) u(\sigma) \, d\sigma$$

$\Rightarrow x_1$ is in the range of L.

Conversely assume that $x_1 \notin$ Range space of W. It follows that there exist a vector $x_2 \in$ Null space of W^T such that $\cancel{x_2^T x_2 \neq 0}$. $x_2^T x_1 \neq 0$.

ie $W^T x_2 = 0 \qquad W(t_0, t_1)^T x_2 = 0$

$\because W$ is a symmetric matrix we have

$W(t_0, t_1)\, x_2 = 0$

If we assume that

$x_1 \in$ Range of L

we have some $u_1(t)$:

$$\int_{t_0}^{t_1} B(\sigma) u_1(\sigma)\, d\sigma = x_1$$

$$\Rightarrow \int_{t_0}^{t_1} x_2^T B(\sigma) u_1(\sigma)\, d\sigma = x_2^T x_1 \neq 0.$$

But $W(t_0, t_1)\, x_2 = 0$

$$\Rightarrow x_2^T W(t_0, t_1) x_2 = 0$$

$$\parallel$$

$$\int_{t_0}^{t_1} x_2^T \underline{B}(t) \underline{B}^T(t) x_2\, dt.$$

$\because$ $\underline{B}(t)$ is continuous f^n it follows that vanishing of $\int_{t_0}^{t_1} x_2^T \underline{B}(t) \underline{B}^T(t) x_2\, dt$.

implies vanishing of

$$B^T(t)\, x_2 .$$

Hence $\int_{t_0}^{t_1} x_2^T B(\sigma) u_1(\sigma)\, d\sigma = 0$

which is a contradiction.

$\blacksquare$