I Linear Differential Equation

By $\mathbb{R}^n$ we mean the set of objects of the form

$$(x_1 \; x_2 \; \ldots \ldots x_n)$$

with x_i real numbers. This is a specific example of a finite dimensional vector space. We define the following operations for members of $\mathbb{R}^n$ —

(a) The sum of two n tuples

$$(x_1 \ldots x_n) + (y_1 \ldots y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

(b) The product of an n tuple by a real number α is $(\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$.

With these two definitions, $\mathbb{R}^n$ is called cartesian n space. The elements may be called n-tuples or "vectors"

It is easy to verify the following standard result:

<u>Th^m 1</u> If $x, y, z \in \mathbb{R}^n$ and if $a, b \in \mathbb{R}$ then

(i) $(x+y)+z = x+(y+z)$

(ii) $0+x = x$

(iii) $x+(-x) = 0$

(iv) $x+y = y+x$

(v) $a(x+y) = ax + ay$

(vi) $(a+b)x = ax+bx$

(vii) $(ab)x = a(bx)$

(viii) $1 \cdot x = x$

It is very easy to verify this theorem.

Any set of objects V together with a definition of sum and a definition of scalar multiplication which satisfies the above eight conditions is called a Real vector space.

Examples of real vector spaces :-

(i) $\mathbb{R}^n$ as defined above.

(ii) Let $C^m[t_0, t_1]$ denote a set of m-tuples whose elements are continuous fns of time defined on the interval $t_0 \le t \le t_1$. We write the elements of $C^m[t_0, t_1]$ as column vectors i.e.

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \\ u_m \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \\ v_m \end{bmatrix}$$

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \\ \\ u_m(t) \end{bmatrix}; \quad v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \\ \\ v_m(t) \end{bmatrix}$$

It is easy to verify that the set $C^m[t_0, t_1]$ is a vector space provided we define

$$u+v = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \\ \\ u_m+v_m \end{bmatrix}, \quad au = \begin{bmatrix} au_1 \\ au_2 \\ \\ \\ au_m \end{bmatrix}$$

(iii) Let $\mathbb{R}^{m \times n}$ denote the set of all $m \times n$ arrays of real numbers arranged in the format.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

One easily verifies that $\mathbb{R}^{m \times n}$ is a "Real Vector Space" provided that addition and scalar multiplication are defined by

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2n} \\ & & \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & \cdots & a_{2n}+b_{2n} \\ \vdots & \cdots & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

$$a \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} aa_{11} & \cdots & aa_{1n} \\ aa_{21} & \cdots & aa_{2n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ aa_{m1} & \cdots & aa_{mn} \end{pmatrix}$$

Such arrays are called matrices.

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Back to $\mathbb{R}^n$

The vectors

$$(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots \ldots (0, 0, \ldots, 0, 1)$$

which we label as $e_1, e_2, \ldots, e_n$ taken together as a set of n tuples form what is called the standard basis of $\mathbb{R}^n$. It is obvious that any n-tuple x can be written as

$$x = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n .$$

Given an arbibrary collection of n or fewer vectors in $\mathbb{R}^n$, $\{x_1, \ldots, x_k\}$, we denote the subset of $\mathbb{R}^n$ which can be expressed as

$$a_1 x_1 + a_2 x_2 + \cdots + a_k x_k$$

for some choice of $a_1, \ldots, a_k$; the "subspace spanned by $\{x_1, \ldots, x_k\}$".

- If $k < n$, this subspace is not the whole space $\mathbb{R}^n$.
- If $k = n$, it may be.

If $k = n$ and the subspace is the whole space, then we say that the collection of vectors $\{x_1, \ldots, x_n\}$ forms a basis of $\mathbb{R}^n$.

<u>Def (Linear Independence)</u>

A set of n-tuples $\{x_1, x_2, \ldots, x_k\}$ is called linearly independent if

$$a_1 x_1 + a_2 x_2 + \cdots + a_k x_k = 0$$
$$\Downarrow$$
$$a_1 = 0,\ a_2 = 0, \ldots, a_k = 0 .$$

<u>Th$^{\underline{m}}$ 2</u>

Let $\{x_1, \ldots, x_k\}$, $\{y_1, \ldots, y_j\}$ be two sets of n tuples of vectors in $\mathbb{R}^n$ that span the same subspace of $\mathbb{R}^n$. Assume furthermore that the set $\{y_1, \ldots, y_j\}$ is linearly independent. It follows that $k \geq j$ and $k = j$ iff the set $\{x_1, \ldots, x_k\}$ is linearly independent.

Linear maps.
Linear transformation

Let L denote a rule which assigns to every element of $\mathbb{R}^n$ an element of $\mathbb{R}^m$.

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

We say that L is a linear transformation if for x, y in $\mathbb{R}^n$ and all scalars $a \in \mathbb{R}$ we have

(i) $L(x+y) = L(x) + L(y)$.

(ii) $L(ax) = aL(x)$.

The linear mappings of $\mathbb{R}^n$ into $\mathbb{R}^m$ can all be described by a set of simultaneous linear equations. Let $x \in \mathbb{R}^n$ & $z \in \mathbb{R}^m$, then any linear mapping

$$L : \mathbb{R}^n \to \mathbb{R}^m$$

can be described by

$$\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= z_1 \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= z_2 \\
&\cdots \\
&\cdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= z_m
\end{aligned}$$

The above notation is very clumsy but it is good to know that it exists.

Inner products & Inner product spaces.

A natural setting for a lot of what we will do in this course is an inner product space.

An inner product in a real vector space X is a mapping of

$$X \times X \longrightarrow \mathbb{R}$$

$$(x, y) \longmapsto \langle x, y \rangle$$

this is how inner products are written.

such that

(i) $\langle x, y \rangle = \langle y, x \rangle$

(ii) $\langle x, ay + bz \rangle = a \langle x, y \rangle + b \langle x, z \rangle$

(iii) $\langle x, x \rangle \geq 0$; $\langle x, x \rangle = 0$ iff $x = 0$.

A vector space with an inner product is called an "inner product space."

A finite dimensional inner product space would be called an Euclidean space.

Theorem 3

The standard dot product on $\mathbb{R}^n$ satisfies the conditions required of an inner product.

$\mathbb{R}^n$ equipped with this inner product is denoted by E^n (E for Euclidean).

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In $C^m[t_0, t_1]$ we define

$$\langle u, v \rangle = \int_{t_0}^{t_1} u^T(t)\, v(t)\, dt.$$

In $\mathbb{R}^{m\times n}$ we define

$$\langle X, Y \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} y_{ij} = \mathrm{trace}(X^T Y)$$

$\mathbb{R}^{m\times n}$ equipped with the above inner product would be denoted by $E^{m\times n}$.