H. W. 6 (answers)

1 Ans:

$$b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ; \quad Ab = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} ; \quad A^2 b = \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}$$

Define

$$P = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 7 & -3 & 1 \end{pmatrix} ; \quad P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ +2 & 3 & 1 \end{pmatrix}$$

$P^{-1} A P =$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 7 & -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} -3 & 1 & 0 \\ 7 & -3 & 1 \\ -16 & 7 & -3 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$P^{-1}b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus

$$P = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 7 & -3 & 1 \end{pmatrix} \qquad \underbrace{F = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \quad g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{This is the controllable canonical form.}}$$

—— x ——

2 $A = \begin{pmatrix} -5 & 1 & 0 \\ 4 & 0 & 1 \\ 39 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$Ab = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $A^2 b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Define

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

2 Ans.

i Characteristic polynomial of A is given by

$$\det(\lambda I - A) =$$

$$\lambda^3 + 4\lambda^2 - 9\lambda + 43.$$

Thus the controllable canonical form is

$$A_c = \begin{pmatrix} -4 & 1 & 0 \\ 9 & 0 & 1 \\ -43 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = b$$

Moreover the feedback canonical form is

$$A_f = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -43 & 9 & -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = b.$$

(ii) The matrix P_1 that will transform

$$(A, b) \xmapsto{P_1} (A_c ; b)$$

ie $P_1^{-1} A P_1 = A_c$, $P_1^{-1} b = b$

is given by $P_1 = (A^2 b, Ab, b)$

P_1 is computed as

$$\begin{array}{c} P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \\ \qquad\;\; \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ A^2 b & Ab & b \end{array} \end{array}$$

(iii) The matrix Q_2 that will transform

$(A_f, b) \xmapsto{Q_2} (A_c, b)$ [as was done in problem 1]

ie $Q_2^{-1} A_f Q_2 = A_c$, $Q_2^{-1} b = b$

is given by

$$Q_2 = \left(A_f^2 b, A_f b, b\right)$$

$$Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 25 & -4 & 1 \end{pmatrix}$$

(iv) Define $P_2 = Q_2^{-1}$, we conclude that the matrix P_2 will transform

$$(A_c, b) \xmapsto{P_2} (A_f, b)$$

ie $\quad P_2^{-1} A_c P_2 = A_f, \; P_2^{-1} b = b$

(v) Define $P = P_1 P_2$. It follows that

$$P^{-1} A P = A_f, \quad P^{-1} b = b.$$

ie the matrix P will transform

$$(A, b) \xmapsto{P} (A_f, b)$$

(Note that this is a new and perhaps easier method to calculate P not discussed in my notes)

(vi) Calculate P_2

$$Q_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -9 & 4 & 1 \end{pmatrix} = P_2$$

(vi)

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -9 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -4 & 5 & 1 \end{pmatrix}$$

(vi) Starting with

$$\dot{x} = Ax + bu$$

if we define

$$x = Pz$$

we obtain

$$\dot{z} = A_f z + bu$$

Define F_1 such that $u = F_1 z$ would place the characteristic polynomial of

$$A_f + bF_1$$

at $(\lambda+1)^3 = \lambda^3 + 3\lambda^2 + 3\lambda + 1$

We would have

$$A_f + b F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix}$$

If $F_1 = (f_1 \quad f_2 \quad f_3)$ it follows that

$$f_1 - 43 = -1 \qquad f_1 = 42$$
$$f_2 + 9 = -3 \qquad f_2 = -12$$
$$f_3 - 4 = -3 \qquad f_3 = 1$$

$\therefore \quad F_1 = (42 \quad -12 \quad 1)$ will do the job.

(vi) Writing

$$u = F_1 z = F_1 P^{-1} x$$

$F = (131 \quad -17 \quad 1)$
wow, I finally have the answer.

we would have

$$F = F_1 P^{-1} = (42 \quad -12 \quad 1) \begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 29 & -5 & 1 \end{pmatrix}$$

$$= (131 \quad -17 \quad 1)$$

3 Ans

(i) $$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 3 \\ 0 & 4 \\ 0 & -1 \\ 0 & -7 \end{pmatrix}$$

$\text{rank}\,[B,\ AB] = 4$

Hence the system is controllable

(ii)

b_1	b_2
Ab_1	Ab_2
2	2

← they are all independent

$(2, 2)$ are the Kronecker indices.

(iii) The Kronecker controllable canonical form is given by

$$A_c = \left(\begin{array}{cc|cc} \times & 1 & \times & 0 \\ \times & 0 & \times & 0 \\ \hline \times & 0 & \times & 1 \\ \times & 0 & \times & 0 \end{array}\right) \qquad B_c = \left(\begin{array}{cc} 0 & 0 \\ 1 & \times \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right)$$

(iv) Define

$$P = \left(Ab_1,\ b_1,\ Ab_2,\ b_2 \right)$$

$$= \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -7 & 0 \end{pmatrix}$$

(v) $$A^2 b_1 = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 9 \end{pmatrix} = \frac{41}{7} Ab_1 + \overset{57/7}{\downarrow} b_1 - \frac{9}{7} Ab_2 + \frac{26}{7} b_2$$

First column of A_c is

$$\begin{pmatrix} 41/7 \\ 57/7 \\ -9/7 \\ 26/7 \end{pmatrix}$$

(vi) $A^2 b_2 =$

$$\begin{pmatrix} 2 & 1 & 3 & 0 \\ 3 & 0 & 4 & 0 \\ 5 & 0 & -1 & 1 \\ 9 & 0 & -7 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ -1 \\ -7 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 9 \\ 34 \end{pmatrix}$$

$$= \frac{151}{7} A b_1 + \frac{171}{7} b_1 - \frac{34}{7} A b_2 + \frac{29}{7} b_2$$

3rd column of A_c is $\begin{pmatrix} 151/7 \\ 171/7 \\ -34/7 \\ 29/7 \end{pmatrix}$

(vii)

$$\underbrace{A_c = \begin{pmatrix} 41/7 & 1 & 151/7 & 0 \\ 57/7 & 0 & 171/7 & 0 \\ -9/7 & 0 & -34/7 & 1 \\ 26/7 & 0 & 29/7 & 0 \end{pmatrix}, B_c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Kronecker Controllable canonical form.}}$$

(viii)

We want

$$A_f = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \times & \times & \times & \times \\ 0 & 0 & 0 & 1 \\ \times & \times & \times & \times \end{pmatrix}, \quad B_f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Q: Here is a problem for you.

Find P_1:

$$P_1^{-1} A_c P_1 = A_f, \quad P_1^{-1} B_c = B_f$$

Note that

$$A_f = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 57/7 & 41/7 & 171/7 & 151/7 \\ 0 & 0 & 0 & 1 \\ 26/7 & -9/7 & 29/7 & -34/7 \end{pmatrix}, \quad B_f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$\boxed{\text{Look at pages 33 \& 44 of Lec10}}$

(ix) (a) Let F_1 be described as

$$F_1 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ f_5 & f_6 & f_7 & f_8 \end{pmatrix}$$

Choose $f_1 = -57/7$, $f_2 = -41/7$, $f_3 = -\frac{171}{7} + 1$

$f_4 = -\frac{151}{7}$, $f_5 = -\frac{26}{7} - 44$

$f_6 = 9/7 + 15$, $f_7 = -29/7 - 29$, $f_8 = \frac{34}{7} - 19$

We calculate

$$A_f + B_f F_1 =$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -44 & 15 & -29 & -19 \end{pmatrix}$$

which has char. polynomial at

$$\lambda^4 + 19\lambda^3 + 29\lambda^2 - 15\lambda + 44$$

$\underline{X}$ $(A, B) \xmapsto{P} (A_c, B) \xmapsto{P_1} (A_f, B)$

$$(A, B) \xmapsto[PP_1]{} (A_f\ B)$$

Require

$$F = F_1 (PP_1)^{-1}$$

I have P. You calculate P_1 and finish the job.

4 Ans:

I yes because $\mathcal{C}$ has rank 6.

II

b_1	b_2	b_3
Ab_1	Ab_2	
$A^2 b_1$		
3	2	1

Kronecker indices are $(3, 2, 1)$.

III Kronecker controllable canonical form

$$A_c = \left(\begin{array}{ccc|cc|c} \times & 1 & 0 & \times & 0 & \times \\ \times & 0 & 1 & \times & 0 & \times \\ \times & 0 & 0 & \times & 0 & \times \\ \hline \times & 0 & 0 & \times & 1 & \times \\ \times & 0 & 0 & \times & 0 & \times \\ \hline \times & 0 & 0 & \times & 0 & \times \end{array}\right), \quad B_c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \times & \times \\ 0 & 0 & 0 \\ 0 & 1 & \times \\ 0 & 0 & 1 \end{pmatrix}$$

(IV) Define

$$P = \left(A^2 b_1, Ab_1, b_1, Ab_2, b_2, b_3\right)$$

$$A^3 b_1 = 0\,A^2 b_1 + 1\,Ab_1 + 0\,b_1 + 1\,Ab_2 + 0\,b_2 - 1\,b_3$$

co-ordinates are $(0 \;\; 1 \;\; 0 \;\; 1 \;\; 0 \;\; -1)$

$$A^2 b_2 = -7\,A^2 b_1 + 0\,Ab_1 + 0\,b_1 + 6\,Ab_2 + 4\,b_2 + 0\,b_3.$$

co-ordinates are $(-7 \;\; 0 \;\; 0 \;\; 6 \;\; 4 \;\; 0)$

$$Ab_3 = 0\,A^2 b_1 + 0\,Ab_1 + 3\,b_1 - 5\,Ab_2 + 2\,b_2 + 0\,b_3$$

co-ordinates are $(0 \;\; 0 \;\; 3 \;\; -5 \;\; 2 \;\; 0)$.

$$\therefore A_c = \left(\begin{array}{ccc|cc|c} 0 & 1 & 0 & -7 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ \hline 1 & 0 & 0 & 6 & 1 & -5 \\ 0 & 0 & 0 & 4 & 0 & 2 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \end{array}\right); \quad B_c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Q: Can you transform

$$(A_c, B_c) \longmapsto (A_f, B_f)$$

similar to what we did in problem 3.

—— x ——