H. W. 4 (Solutions)

1 Ans.

a Given a square $n \times n$ matrix A,
$\exists$ a $n \times n$, nonsingular matrix P:

$$P^{-1} A P = J$$

where J is a $n \times n$ matrix in the Jordan Canonical form. Define

$z = P^{-1} x$, we obtain.

$$\dot{z} = P^{-1} A x = P^{-1} A P z = J z$$

Hence

$$z(t) = e^{Jt} z(0)$$

Since eigenvalues of A have negative real parts, it follows that $\|z(t)\| < M$ $\forall t$. for some $M > 0$.

$\because\ x = Pz$

we have

$$\|x\| \leq \|P\| \, \|z\|$$

$$\Rightarrow \exists N > 0 : \|x\| < N.$$

$$\forall t \geq 0.$$

$\blacksquare$

(b) $\dot{x}_1 = -4x_1$ $\Rightarrow$ $x_1(t) = e^{-4t} x_1(0) = 2e^{-4t}$

$\dot{x}_2 = x_2$ $\quad$ $x_2(t) = e^{t} x_2(0) = 3e^{t}$

$$\|x\| = \sqrt{4e^{-8t} + 9e^{2t}}.$$

which is not bounded.

(c) $$e^{Bt} = \begin{pmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{pmatrix}$$

$\|z\| = \|x\|$ $\leftarrow$ vector z is obtained by rotating x around $(0,0)$ as the center.

Thus we have (i).

$$\|z\| = \sqrt{4e^{-8t} + 9e^{2t}}$$

Hence $\|z\|$ is not bounded

(d) $z = e^{Bt} x$

$$\Rightarrow \dot{z} = e^{Bt} B x + e^{Bt} \dot{x}$$

$$= e^{Bt} B x + e^{Bt} A x.$$

$$= e^{Bt}(A+B) x$$

$$= \underbrace{e^{Bt}(A+B)e^{-Bt}}_{R(t).} z.$$

$$\therefore \dot{z} = R(t) z.$$

$$z(0) = e^{0} x(0) = x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

(e) At any t, the eigenvalues of $R(t)$ is given by the char. poly

$$p(\lambda) = |\lambda I - R(t)|$$
$$= \det\left[\lambda I - e^{Bt}(A+B)e^{-Bt}\right]$$
$$= \det\left\{e^{Bt}\left[\lambda I - (A+B)\right]e^{-Bt}\right\}$$
$$= \det(e^{Bt})\det\left[\lambda I - (A+B)\right]\det(e^{-Bt})$$
$$= \det\left[\lambda I - (A+B)\right]$$

Eigenvalues of $R(t)$ are precisely the eigenvalues of $A+B$.

f $A+B = \begin{pmatrix} -4 & 3 \\ -3 & 1 \end{pmatrix}$

Char poly of $A+B$ is given by

$$\det \begin{pmatrix} \lambda+4 & -3 \\ 3 & \lambda-1 \end{pmatrix}$$

$$= (\lambda+4)(\lambda-1) + 9$$

$$= \lambda^2 + 3\lambda + 5$$

Eigenvalues at $\dfrac{-3 \pm \sqrt{9-20}}{2}$

$$= -\frac{3}{2} \pm i\sqrt{\frac{11}{4}}.$$

2 Ans

A (i)

$$\phi^{-1}(t,0)\,\phi(t,0) = I.$$

$$\Rightarrow \frac{d}{dt}\left[\phi^{-1}(t,0)\right]\phi(t,0) + \phi^{-1}(t,0)\,\dot{\phi}(t,0) = 0$$

$$\Rightarrow \frac{d}{dt}\left[\phi^{-1}(t,0)\right] = -\phi^{-1}(t,0)\,\dot{\phi}(t,0)\,\phi(t,0)^{-1}.$$

$$= -\phi^{-1}(t,0)\,A(t)\,\phi(t,0)\,\phi(t,0)^{-1}.$$

$$= -\phi^{-1}(t,0)\,A(t).$$

(ii) From Abel-Jacobi-Liouville's Theorem we know that

$$\det(e^{Rt}) = \exp\left(\int_0^t (\text{trace } R)\, d\sigma\right)$$

$$= \exp[\text{trace } R \cdot t].$$

$$\det[\Phi(t,0)] = \exp\left(\int_0^t \text{trace}[A(\sigma)]\, d\sigma\right).$$

$\therefore \det P(t) =$

$$\exp[\text{trace } R \cdot t]\, \exp\left[-\int_0^t \text{trace}[A(\sigma)]\, d\sigma\right]$$

$\neq 0$ for any $t \geq 0$.

$\therefore P(t)$ is nonsingular.

(iii)

$$P(t) = e^{Rt} \phi(t,0)^{-1}.$$

$$\dot{P}(t) = R e^{Rt} \phi(t,0)^{-1}$$

$$+ e^{Rt} \frac{d}{dt} \left[\phi(t,0)^{-1} \right]$$

$$= R P(t) + e^{Rt} (-1) \phi^{-1}(t,0) A(t).$$

$$= R P(t) - P(t) A(t).$$

(iv). $z(t) = P(t) \underline{x}(t).$

$$\Rightarrow \dot{z}(t) = \dot{P}(t) \underline{x}(t) + P(t) \dot{\underline{x}}(t).$$

$$= \dot{P}(t) \underline{x}(t) + P(t) A(t) \underline{x}(t).$$

$$= \left[\dot{P}(t) + P(t) A(t) \right] \underline{x}(t).$$

$$= \left[\dot{P}(t) + P(t) A(t) \right] P^{-1}(t) z(t).$$

(v) $\dot{z}(t) = [\dot{P} + PA] P^{-1} z(t)$

$= (RP - PA + PA) P^{-1} z.$

$= RPP^{-1} z$

$= Rz.$

(vi). In general $P(t)$ & $\dot{P}(t)$ are not necessarily bounded. on the interval $(-\infty, \infty)$. Hence the transformation

$z(t) = P(t) X(t)$

is not necessarily a Lyapunov. Transformation

(vii) $P^{-1}(t+T) = \phi(t+T, T) e^{-Rt}.$

$= \phi(t, 0) e^{-Rt} = P^{-1}(t).$

(viii) In the interval $[0,T]$, $P(t)$, $\dot{P}(t)$ and $\det P(t)$ satisfy the assumptions of a Liapunov Transformation. This follows from continuity of $P(t)$, $\dot{P}(t)$ & $\det P(t)$. Finally $\because$ $P(t)$ is periodic, the assumptions are also satisfied in the interval $[nT,(n+1)T]$ for $n=0,\pm1,\pm2,\ldots$

(B) $\dot{x}=\alpha(t)Ax$

$$\Rightarrow x(t)=e^{A\int_0^t \alpha(\sigma)\,d\sigma}x(0).$$

$$\phi(t,0)=e^{A\int_0^t \alpha(\sigma)\,d\sigma}$$

$$\phi(T,0)=e^{A\int_0^T \alpha(\sigma)\,d\sigma}=e^{A\frac{1}{T}\int_0^T \alpha(\sigma)\,d\sigma\, T}$$

$$\therefore R=\frac{1}{T}\int_0^T \alpha(\sigma)\,d\sigma\, A.$$

$$P(t) = e^{Rt}\,\phi(t,0)^{-1}.$$

$$\therefore\ e^{\frac{1}{T}\int_0^T \alpha(\sigma)\,d\sigma\, A t} \cdot e^{-A\int_0^t \alpha(\sigma)\,d\sigma}.$$

If $z = Px$, we have

$$\dot{z} = Rz = \frac{1}{T}\int_0^T \alpha(\sigma)\,d\sigma\, A z.$$

3 Ans:

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$A^2B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \end{pmatrix}$$

rank $\mathcal{C} = 2$

Hence $*$ is not controllable.

$$ee^T =$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 3 & 8 \\ 3 & 3 & 3 \\ 8 & 3 & 8 \end{pmatrix}$$

Range of $ee^T =$

$$\text{span}\left[\begin{pmatrix} 8 \\ 3 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right]$$

Range of ee^T is a 2-plane in $\mathbb{R}^3$.

To find the eqn of the plane find the cross product of $(8,3,8)$ & $(1,1,1)$.

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 8 & 3 & 8 \end{vmatrix}$$

$$= i\,5 - j\,0 + k(-5)$$

$\begin{pmatrix} 5 \\ 0 \\ -5 \end{pmatrix}$ is a vector $\perp$ $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 8 \\ 3 \\ 8 \end{pmatrix}$.

$\therefore$ Eqn of the plane is $x_1 - x_3 = 0$.

(c) $$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$e^{-At} x(t) - x(0) = \int_0^t e^{-A\tau} B u(\tau)\, d\tau.$$

At $t = T = 10$ we have

$$e^{-10A} x(10) - x(0) = \int_0^{10} e^{-A\tau} B u(\tau)\, d\tau.$$

$$\Rightarrow e^{-10A} x(10) - x(0) \in \text{plane } P$$

where $P = \{ \underline{x} : x_1 - x_3 = 0 \}$.

$$\Rightarrow e^{-10A} x(10) \in P_1$$

where $P_1 = \{ \underline{x} : x_1 - x_3 = d \}$.

$d = 1 \quad \because x_1(0) = 2 \quad x_2(0) = x_3(0) = 1$.

$$\therefore P_1 = \{ \underline{x} : x_1 - x_3 = 1 \}.$$

$\therefore$

$$x(10) \in e^{10A} P_1$$

where $P_1 = \{ \underline{x} : x_1 - x_3 = 1 \}$.

(d) for $t = T$

$$x(T) \in e^{AT} P_1$$

where $P_1 = \{ \underline{x} : x_1 - x_3 = 1 \}$.

Set of all points is

$$\{ \xi : \xi = e^{AT} \eta,\ \eta \in \{ \underline{x} : x_1 - x_3 = 1 \},\ T > 0 \}.$$

(e)

$$P^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} = P.$$

$$\dot{z} = PAP^{-1}z + PBu.$$

We can verify that

$$PAP^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad PB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(f)

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

controllable

$$\dot{z}_3 = z_3.$$

not controllable

(g) $\dot{z}_3 = z_3$

$\Rightarrow z_3(t) = e^{t} z_3(0)$

$z_3 = x_1 - x_3$

$\Rightarrow z_3(0) = x_1(0) - x_3(0) = 1.$

$\therefore z_3(t) = e^{t}.$

$\Rightarrow x_1(t) - x_3(t) = e^{t}. \quad \forall t \geq 0.$

(h) At $t = T = 10$ we have

$x_1(10) - x_3(10) = e^{10}.$

We have a plane

$\{ \underline{x} : x_1 - x_3 = e^{10} \} = P_1.$

At $t = 10$, since $z_3(10) = e^{10}$ the set of controllable pts at $t = 10$ must be contained in P_1.

On the other hand since (z_1, z_2) is controllable, one can reach any pt in the plane P_1 at $t = T = 10$.

(i) The set of all points that can be reached at $t=T$ is given by

$$\{ \underline{x} : x_1 - x_3 = e^T \}.$$

For any $T > 0$ these pts are described by

$$\{ \underline{x} : x_1 - x_3 > 0 \}.$$

(j) $z_3(0) = 1 \quad z_3(T) = 2$

$$z_3(t) = e^t z_3(0) = e^t.$$

At $t = T$ we have

$$z_3(T) = 2 = e^T \quad T = \ln 2.$$

$$Z(0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \dot{X}(0).$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$Z(T) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}(T) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$e^{At} = e^{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t} \, e^{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} t}$$

$$= \begin{pmatrix} e^{t} & 0 \\ 0 & e^{t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{t} & 0 \\ te^{t} & e^{t} \end{pmatrix}$$

$$e^{At} B = e^{At}$$

$$W(0,T) = \int_0^T \begin{pmatrix} e^{-t} & 0 \\ -te^{-t} & e^{-t} \end{pmatrix} \begin{pmatrix} e^{-t} & -te^{-t} \\ 0 & e^{-t} \end{pmatrix} dt$$

$$= \int_0^T \begin{pmatrix} e^{-2t} & -te^{-2t} \\ -te^{-2t} & t^2 e^{-2t} + e^{-2t} \end{pmatrix} dt$$

$$\int_0^T e^{-2t}\,dt = \left.\frac{e^{-2t}}{-2}\right|_0^T = -\frac{e^{-2T}}{2} + \frac{1}{2}$$
$$= \frac{1}{2}\left[1 - e^{-2T}\right]$$

At $T = \ln 2$

$$e^{-2T} = \frac{1}{e^{2T}} = \left(\frac{1}{e^T}\right)^2 = \frac{1}{4}.$$

$$\therefore \int_0^T e^{-2t}\,dt = \frac{1}{2}\cdot\frac{3}{4} = \frac{3}{8}$$

— x —

$$\int_0^T t e^{-2t}\,dt = \left. t\frac{e^{-2t}}{-2}\right|_0^T + \int_0^T \frac{e^{-2t}}{2}\,dt$$
$$= \frac{Te^{-2T}}{-2} + \frac{1}{2}\int_0^T e^{-2t}\,dt.$$
$$= -\frac{1}{8}\ln 2 + \frac{1}{2}\,\frac{3}{8} = \frac{1}{8}\left[\frac{3}{2} - \ln 2\right]$$

$$-\int_0^T t e^{-2t}\,dt = \frac{1}{8}\left(\ln 2 - \frac{3}{2}\right)$$

— x —

$$\int_0^T t^2 e^{-2t}\,dt = \left.\frac{t^2 e^{-2t}}{-2}\right|_0^T + \int_0^T \cancel{2}t\,\frac{e^{-2t}}{\cancel{2}}\,dt.$$
$$= -\frac{1}{2}T^2 e^{-2T} + \int_0^T t e^{-2t}\,dt.$$
$$= -\frac{1}{8}(\ln 2)^2 + \frac{1}{8}\,\frac{3}{2} - \frac{1}{8}\ln 2 = \frac{3}{16} - \frac{1}{8}\ln 2\left[1 + \ln\right.$$

$$W(0,T) =$$

$$\begin{pmatrix} \frac{3}{8} & \frac{1}{8}\left(\ln 2 - \frac{3}{2}\right) \\ \frac{1}{8}\left(\ln 2 - \frac{3}{2}\right) & \frac{3}{16} - \frac{1}{8}\ln 2 - \frac{1}{8}(\ln 2)^2 + \frac{3}{8} \end{pmatrix}$$

$$\begin{pmatrix} \frac{3}{8} & -\frac{3}{16} + \frac{1}{8}\ln 2 \\ -\frac{3}{16} + \frac{1}{8}\ln 2 & \frac{9}{16} - \frac{1}{8}\ln 2 - \frac{1}{8}(\ln 2)^2 \end{pmatrix}$$

We have

$$e^{-AT}x(T) = x(0) + \int_0^T e^{-A\tau}Bu(\tau)d\tau.$$

$$\text{ie } \left[e^{-AT}x(T) - x(0)\right] = \int_0^T e^{-A\tau}Bu(\tau)d\tau.$$

choose

$$u(\tau) = B^T e^{-A^T\tau}\xi.$$

and write

$$W(0,T)\xi = \begin{pmatrix} e^{-T} & 0 \\ -Te^{-T} & e^{-T} \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{-T} - 1 \\ -2Te^{-T} + 2e^{-T} - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -\cancel{2}(\ln 2)\frac{1}{\cancel{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ -\ln 2 \end{pmatrix}$$

$$\xi = W(0,T)^{-1}\begin{pmatrix} 0 \\ -\ln 2 \end{pmatrix}$$

$$W(0,T)^{-1} = \frac{\begin{pmatrix} * & 3/16 - \frac{1}{8}\ln 2 \\ * & 3/8 \end{pmatrix}}{\det W(0,T)}.$$

$$\therefore \xi = \begin{pmatrix} -\frac{3}{16}\ln 2 + \frac{1}{8}(\ln 2)^2 \\ -\frac{3}{8}\ln 2 \end{pmatrix} \Big/ \det W(0,T)$$

$$u(\tau) = e^{-A^T\tau}\xi.$$

$$= \begin{pmatrix} e^{-\tau} & -\tau e^{-\tau} \\ 0 & e^{-\tau}. \end{pmatrix}\xi.$$

k

Char poly of A is given by

$$\det\begin{pmatrix} \lambda-1 & -1 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & -1 & \lambda-1 \end{pmatrix}$$

$$= (\lambda-1)^3$$

$\therefore$ Eigenvalues at 1, 1, 1.

For $\lambda=1$

$$(\lambda I - A \mid B) = \left(\begin{array}{ccc|cc} 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{array}\right)$$

$\operatorname{rank}(\lambda I - A \mid B) = 2$ for $\lambda = 1$.

$\blacksquare$