Solving 2nd order o.d.e. using Laplace's Transform.

Q: Using Laplace's transform, solve

$$\ddot{y}(t) + 7\,\dot{y}(t) + 16\,y(t) = 0$$

$$y(0) = y_0 \ , \ \dot{y}(0) = v_0 .$$

Ans: Taking Laplace's Transform we get.

$$\mathcal{L}[y(t)] = Y(s)$$

$$\mathcal{L}[\dot{y}(t)] = s\,Y(s) - y(0)$$

$$\mathcal{L}[\ddot{y}(t)] = s^2 Y(s) - s y(0) - \dot{y}(0) .$$

we have

$$\left[s^2 + 7s + 16\right] Y(s) = s\,y(0) + \dot{y}(0) + 7\,y(0) .$$

$$Y(s) = \frac{s+7}{s^2+7s+16}\,y(0) + \frac{1}{s^2+7s+16}\,\dot{y}(0)$$

$$s^2+7s+16=$$
$$\left(s+\frac{7}{2}\right)^2+\left(\frac{\sqrt{15}}{2}\right)^2$$

$$s+7=\left(s+\frac{7}{2}\right)+\frac{7}{2}.$$

We have

$$Y(s)=\frac{s+7/2}{\left(s+\frac{7}{2}\right)^2+\left(\frac{\sqrt{15}}{2}\right)^2}y(0)+\frac{7/2}{\left(s+\frac{7}{2}\right)^2+\left(\frac{\sqrt{15}}{2}\right)^2}y(0)$$

$$+\frac{1}{\left(s+\frac{7}{2}\right)^2+\left(\frac{\sqrt{15}}{2}\right)^2}\dot{y}(0).$$

$$=\left[\frac{s+7/2}{\left(s+\frac{7}{2}\right)^2+\left(\frac{\sqrt{15}}{2}\right)^2}+\frac{7}{\sqrt{15}}\frac{\sqrt{15}/2}{\left(s+\frac{7}{2}\right)^2+\left(\frac{\sqrt{15}}{2}\right)^2}\right]y(0)$$

$$+\frac{2}{\sqrt{15}}\left(\frac{\sqrt{15}/2}{\left(s+\frac{7}{2}\right)^2+\left(\frac{\sqrt{15}}{2}\right)^2}\right)\dot{y}(0).$$

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$y(t) =$

$$e^{-\frac{7}{2}t}\left[\cos\frac{\sqrt{15}}{2}t + \frac{7}{\sqrt{15}}\, e^{-\frac{7}{2}t} \sin\frac{\sqrt{15}}{2}t\right] Y_0$$

$$+ \frac{2}{\sqrt{15}}\left[e^{-\frac{7}{2}t} \sin\frac{\sqrt{15}}{2}t\right]\cdot \dot{y}(0) \,.$$

Q: Using Laplace's Transform, solve (4)

$$\ddot{y} + 7\dot{y} + 3y = 0$$

$$y(0) = y_0 \quad , \quad \dot{y}(0) = v_0$$

Ans: Like in problem 1

$$Y(s) = \frac{s+7}{s^2+7s+3}\, y(0) + \frac{1}{s^2+7s+3}\, \dot{y}(0).$$

Roots of s^2+7s+3 are at

$$\frac{-7 \pm \sqrt{49-12}}{2}.$$

$$= -\frac{7}{2} \pm \sqrt{\frac{37}{4}}$$

$$= \frac{-7 \pm \sqrt{37}}{2} = \alpha, \beta.$$

$$\alpha = \frac{-7-\sqrt{37}}{2}, \quad \beta = \frac{-7+\sqrt{37}}{2}.$$

$$s^2+7s+3=(s-\alpha)(s-\beta).$$

$$\frac{s+7}{s^2+7s+3}=\frac{s+7}{(s-\alpha)(s-\beta)}=\frac{A}{s-\alpha}+\frac{B}{s-\beta}.$$

$$A=\left.\frac{s+7}{s-\beta}\right|_{s=\alpha}.=\frac{\alpha+7}{\alpha-\beta}.$$

$$B=\left.\frac{s+7}{s-\alpha}\right|_{s=\beta}.=\frac{\beta+7}{\beta-\alpha}.$$

$$\therefore\ \frac{s+7}{s^2+7s+3}=\frac{\alpha+7}{\alpha-\beta}\cdot\frac{1}{s-\alpha}-\frac{\beta+7}{\alpha-\beta}\cdot\frac{1}{s-\beta}.$$

$$\mathcal{L}^{-1}\left[\frac{s+7}{s^2+7s+3}\right]=\left[\frac{\alpha+7}{\alpha-\beta}e^{\alpha t}-\frac{\beta+7}{\alpha-\beta}e^{\beta t}\right]$$

$$\frac{1}{s^2+7s+3} = \frac{A}{s-\alpha} + \frac{B}{s-\beta}.$$

$$\frac{1}{(s-\alpha)(s-\beta)} = \frac{A}{s-\alpha} + \frac{B}{s-\beta}.$$

$$A = \left.\frac{1}{s-\beta}\right|_{s=\alpha} = \frac{1}{\alpha-\beta}$$

$$B = \left.\frac{1}{s-\alpha}\right|_{s=\beta}. = \frac{1}{\beta-\alpha}.$$

$$\frac{1}{s^2+7s+3} = \frac{1}{\alpha-\beta}\left[\frac{1}{s-\alpha} - \frac{1}{s-\beta}\right]$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+7s+3}\right) = \frac{1}{\alpha-\beta}\left(e^{\alpha t} - e^{\beta t}\right)$$

It would follow that

$$y(t) = \mathcal{L}^{-1}\left[Y(s)\right]$$

$$= \left[\frac{\alpha+7}{\alpha-\beta} e^{\alpha t} - \frac{\beta+7}{\alpha-\beta} e^{\beta t}\right] y_0 .$$

$$+ \frac{1}{\alpha-\beta}\left[e^{\alpha t} - e^{\beta t}\right] v_0 .$$