Sol$^{\underline{n}}$ to the Short homework problem.

1 Ans:

$$(s^2+5s+6)Y(s) = F(s).$$

$$s^2+5s+6 = (s+3)(s+2)$$

$$Y(s) = \frac{1}{(s+3)(s+2)} F(s).$$

$$\frac{1}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2}$$

$$A = \left.\frac{1}{s+2}\right|_{s=-3} = \frac{1}{2-3} = -1$$

$$B = \left.\frac{1}{s+3}\right|_{s=-2} = \frac{1}{1} = 1$$

$$\therefore \frac{1}{(s+3)(s+2)} = \frac{1}{s+2} - \frac{1}{s+3}$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s+3)(s+2)}\right) = e^{-2t} - e^{-3t}.$$

(2)

$$\therefore\; y(t) = \left[e^{-2t} - e^{-3t}\right] ** f(t)$$

$$= \int_0^t \left[e^{-2(t-\tau)} - e^{-3(t-\tau)}\right] f(\tau)\, d\tau$$

$$= e^{-2t} \int_0^t f(\tau) e^{2\tau}\, d\tau$$

$$- e^{-3t} \int_0^t f(\tau) e^{3\tau}\, d\tau .$$

when $0 < t < T$ we have

$$y(t) = e^{-2t} \int_0^t \tau e^{2\tau}\, d\tau \;\; = A(t)$$

$$- e^{-3t} \int_0^t \tau e^{3\tau}\, d\tau \;\; = B(t).$$

Let us calculate $A(t)$.

$$\int \tau e^{2\tau} d\tau = \frac{\tau e^{2\tau}}{2} - \int \frac{e^{2\tau}}{2} d\tau.$$

$$= \frac{1}{2}\tau e^{2\tau} - \frac{1}{4} e^{2\tau}$$

$$\int_0^t \tau e^{2\tau} d\tau = \left[\frac{1}{2} t e^{2t} - \frac{1}{4} e^{2t}\right] - \left[0 - \frac{1}{4}\right]$$

$$\boxed{A(t) = \frac{1}{2} t e^{2t} - \frac{1}{4} e^{2t} + \frac{1}{4}}$$

Let us calculate $B(t)$

$$\int \tau e^{3\tau} d\tau = \tau \frac{e^{3\tau}}{3} - \int \frac{e^{3\tau}}{3} d\tau .$$

$$= \frac{1}{3} \tau e^{3\tau} - \frac{1}{9} e^{3\tau}$$

$$\int_0^t \tau e^{3\tau} d\tau = \frac{1}{3} t e^{3t} - \frac{1}{9} e^{3t} + \frac{1}{9} .$$

$$\boxed{B(t) = \frac{1}{3} t e^{3t} - \frac{1}{9} e^{3t} + \frac{1}{9} .}$$

When $0 < t < T$

$$y(t) = e^{-2t} A(t) - e^{-3t} B(t)$$

When $T < t < 2T$

$$y(t) = e^{-2t}\left[\overbrace{\int_0^T \tau e^{2\tau}\,d\tau}^{=A(T)} + \overbrace{\int_T^t (2T-\tau)e^{2\tau}\,d\tau}^{=C(t)}\right] - e^{-3t}\left[\underbrace{\int_0^T \tau e^{3\tau}\,d\tau}_{=B(T)} + \underbrace{\int_T^t (2T-\tau)e^{3\tau}\,d\tau}_{=D(t)}\right]$$

$$y(t) = e^{-2t}\left[A(T) + C(t)\right] - e^{-3t}\left[B(T) + D(t)\right]$$

Let us calculate $C(t)$ & $D(t)$ 6

$$\int (2T-\tau)e^{\alpha\tau}\,d\tau$$

$$= (2T-\tau)\frac{e^{\alpha\tau}}{\alpha} - \int (-1)\frac{e^{\alpha\tau}}{\alpha}\,d\tau.$$

$$= \frac{2T-\tau}{\alpha}e^{\alpha\tau} + \frac{1}{\alpha^2}e^{\alpha\tau}$$

$$\int_T^t (2T-\tau)e^{\alpha\tau}\,d\tau =$$

$$\left[\frac{2T-t}{\alpha}e^{\alpha t} + \frac{1}{\alpha^2}e^{\alpha t}\right] -$$

$$\left[\frac{T}{\alpha}e^{\alpha T} + \frac{1}{\alpha^2}e^{\alpha T}\right]$$

⑦

$$C(t) = \left[\frac{2T-t}{2} e^{2t} - \frac{1}{4} e^{2t}\right] - \left[\frac{T}{2} e^{2T} + \frac{1}{4} e^{2T}\right]$$

$$D(t) = \left[\frac{2T-t}{3} e^{3t} + \frac{1}{9} e^{3t}\right] - \left[\frac{T}{3} e^{3T} + \frac{1}{9} e^{3T}\right]$$

when $t > 2T$:

$$y(t) = e^{-2t}\left[\int_0^T \tau e^{2\tau}\,d\tau + \int_T^{2T} (2T-\tau)e^{2\tau}\,d\tau\right] - e^{-3t}\left[\int_0^T \tau e^{3\tau}\,d\tau + \int_T^{2T} (2T-\tau)e^{3\tau}\,d\tau\right]$$

$$= e^{-2t}\left[A(T) + C(T)\right] - e^{-3t}\left[B(T) + D(T)\right]$$

In summary

$$y(t) = e^{-2t} A(t) - e^{-3t} B(t)$$

when $0 < t < T$

$$e^{-2t}\left[A(T) + C(t)\right] - e^{-3t}\left[B(T) + D(t)\right]$$

when $T < t < 2T$

$$e^{-2t}\left[A(T) + C(T)\right] - e^{-3t}\left[B(T) + D(T)\right]$$

when $2T \leq t$