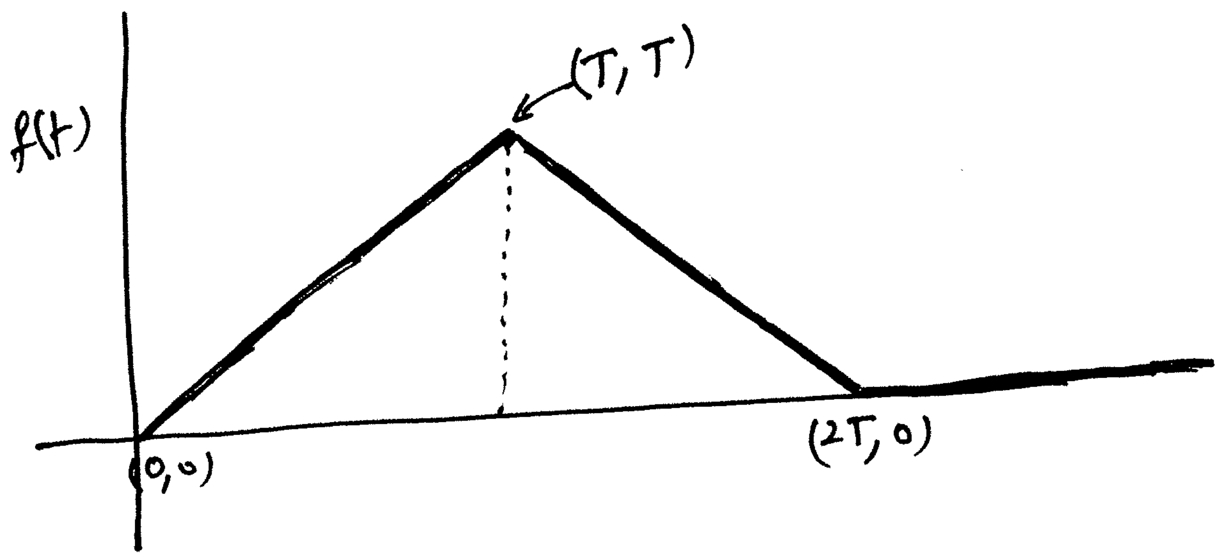
Short homework:

1. Solve the problem

$$\ddot{y} + 5\dot{y} + 6y = f(t)$$

$y(0) = \dot{y}(0) = 0$ using convolution function

Take $f(t)$ to be the following

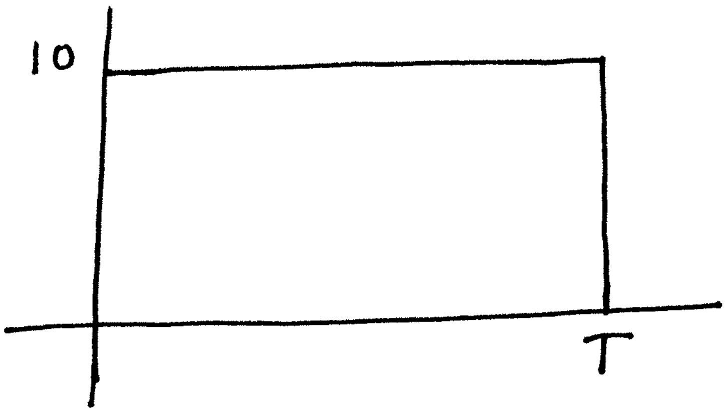


$$f(t) = \begin{cases} t & 0 \leq t < T \\ 2T - t & T \leq t < 2T \\ 0 & \text{elsewhere.} \end{cases}$$

We are solving the problem

$$\ddot{y} + 3\dot{y} + 2y = f(t)$$

$$y(0) = 0, \quad \dot{y}(0) = 0$$



$$f(t) = 10 \quad 0 \le t \le T$$
$$= 0 \quad \text{elsewhere}$$

Take $T = \ln 75$
$= 4.3175$

Of course, I know that you can solve this problem using

- particular sol^n/homogeneous sol^n trick.
- Laplace Transform

with a little bit of luck.

We would like to solve this problem using convolution.

If

$h_1(t)$ and $h_2(t)$ are two functions defined for $t \geqslant 0$, we define

$$h_1 ** h_2 = \int_0^t h_1(t-\tau)\, h_2(\tau)\, d\tau .$$

called the convolution of h_1 and h_2.

If $\mathcal{L}(h_1) = H_1(s)$

$\mathcal{L}(h_2) = H_2(s)$

One can show that

$$\mathcal{L}(h_1 ** h_2) = H_1(s)\, H_2(s).$$

Taking the Laplace's Transform of the differential equation we get.

$$\mathcal{L}\left(\ddot{y}+3\dot{y}+2y\right)=\mathcal{L}\left(f(t)\right)=F(s).$$

$$\Rightarrow (s^2+3s+2)\,Y(s)=F(s).$$

$$\Rightarrow Y(s)=\left(\frac{1}{s^2+3s+2}\right)\cdot F(s).$$

If we define

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+3s+2}\right)=h(t)$$

we have

$$\boxed{y(t)=h(t)**f(t)}$$

as the solution to the eqn which combines both particular and homogeneous.

Note: $y(0)=\dot{y}(0)=0$ is assumed.

To calculate $h(t)$ we write

$$\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}.$$

$$\Rightarrow \boxed{h(t) = e^{-t} - e^{-2t}}$$

Thus.

$$y(t) = \int_0^t e^{-(t-\tau)} - e^{-2(t-\tau)} f(\tau)\,d\tau.$$

$$= e^{-t}\int_0^t e^{\tau} f(\tau)\,d\tau.$$

$$- e^{-2t}\int_0^t e^{2\tau} f(\tau)\,d\tau.$$

For $t < T$, $f(t) = 10$

$$y(t) = e^{-t}10\int_0^t e^{\tau}d\tau - e^{-2t}10\int_0^t e^{2\tau}d\tau$$

$$= 10e^{-t}[e^{t}-1]$$
$$-5e^{-2t}[e^{2t}-1]$$

$$y(t) = 5 - 10e^{-t} + 5e^{-2t}.$$

for $0 \leq t < T$

For $t \geqslant T$, we have

$$y(t) = 10e^{-t}\int_0^T e^{\tau}\,d\tau - 10e^{-2t}\int_0^T e^{2\tau}\,d\tau$$

$$= 740e^{-t} - 28120\,e^{-2t}$$

$$t \geqslant T = \ln 75 = 4{\cdot}3175$$

At $t = 4{\cdot}3175$

$y(4{\cdot}3175) = 4{\cdot}8676$.