H.W.5

① Solve

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 6u(t)$$

where

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$y(0) = 0 \quad , \quad \dot{y}(0) = 10$$

ⓐ using the method of homogeneous/particular solution.

ⓑ using Laplace's Transform.

Remark: Read the following notes.

(2) Repeat problem 1 for the equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 6y = 6u(t).$$

(3) Repeat problem 1 for the equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = te^{-5t}$$

Example 1

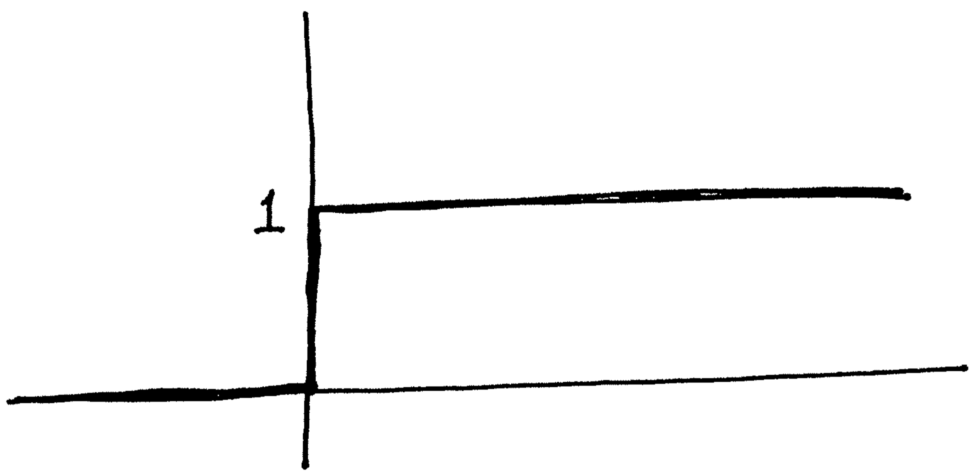
$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = u(t)$$

where

$$u(t) = 1 \quad t \geq 0$$
$$= 0 \quad t < 0$$

$$y(0) = 5 \;, \; \dot{y}(0) = 7$$

Remark: $u(t)$ has the following graph



$\therefore$ since we are only interested in solving the ordinary differential eqn for $t \geq 0$, we can think of solving the equation by treating $u(t)$ as a constant function '1'.

Sol$^{\underline{n}}$

Method of homogeneous/particular solution

step 1 (write down the characteristic polynomial)

$$\lambda^2 + 3\lambda + 1$$

step 2 (Solve the auxiliary equation)

$$\lambda^2 + 3\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{-3 \pm \sqrt{9 - 4}}{2}$$

$$= -\frac{3}{2} \pm \frac{\sqrt{5}}{2}$$

$$= -\cdot 382, \ -2\cdot 6180$$

$$\lambda_1 = -\cdot 382, \ \lambda_2 = -2\cdot 618$$

step 3 (Write down solution of the homogeneous eq$^{\underline{n}}$)

$$Y_h(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t}$$

* Dont calculate α, β yet.

<u>step 4</u> (write down particular solution)

When $u(t)$ is a constant, we choose

$$y_p(t) = A \leftarrow \text{a constant.}$$

<u>step 5</u> (calculate A)

plug $y_p(t)$ into the equation

$$\ddot{y}_p + 3\dot{y}_p + y_p = 1 \leftarrow \text{this is } u(t) \text{ for } t \geq 0.$$

$$\Rightarrow \ddot{A} + 3\dot{A} + A = 1$$

$$\Rightarrow A = 1 \quad \text{since } \dot{A} = 0,\ \ddot{A} = 0.$$

So $y_p(t) = 1$

<u>Step 6</u> (Add the homogeneous and particular solution)

$$y(t) = y_h(t) + y_p t$$

$$= \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t} + 1$$

Step 7 (Find α, β from the initial condition

$$y(0) = 5, \quad \dot{y}(0) = 7.$$

Calculate

$$y(0) = \alpha e^{\lambda_1 0} + \beta e^{\lambda_2 0} + 1 = \alpha + \beta + 1.$$

$$y(0) = 5 \Rightarrow \alpha + \beta + 1 = 5$$

$$\Rightarrow \alpha + \beta = 5 - 1 = 4.$$

Calculate

$$\dot{y}(t) = \alpha \lambda_1 e^{\lambda_1 t} + \beta \lambda_2 e^{\lambda_2 t}$$

$$\dot{y}(0) = \alpha \lambda_1 + \beta \lambda_2 = 7$$

$$\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$\alpha = \frac{\det \begin{pmatrix} 4 & 1 \\ 7 & \lambda_2 \end{pmatrix}}{\lambda_2 - \lambda_1} = \frac{4\lambda_2 - 7}{\lambda_2 - \lambda_1}.$$

$$\beta = \frac{\det\begin{pmatrix} 1 & 4 \\ \lambda_1 & 7 \end{pmatrix}}{\lambda_2 - \lambda_1} = \frac{7 - 4\lambda_1}{\lambda_2 - \lambda_1}$$

$$= \frac{4\lambda_1 - 7}{\lambda_1 - \lambda_2}$$

Step 8 (Plug α & β back into $y(t)$)

$$\boxed{y(t) = \frac{4\lambda_2 - 7}{\lambda_2 - \lambda_1} e^{\lambda_1 t} - \frac{4\lambda_1 - 7}{\lambda_2 - \lambda_1} e^{\lambda_2 t} + 1}$$

step 9 (calculate)

$$y(t) = \frac{-17\cdot472}{-2\cdot236} e^{-\cdot382t} - \frac{-8\cdot528}{-2\cdot236} e^{-2\cdot618t} + 1$$

$$= 7\cdot8139 e^{-\cdot382t} - 3\cdot8139 e^{-2\cdot618t} + 1$$

Method of Laplace's Transform.

$$\ddot{y} + 3\dot{y} + y = u(t).$$

Step 1 (Laplace Transform both left hand side and right hand side)

$$\mathcal{L}[\ddot{y} + 3\dot{y} + y] = \mathcal{L}[u(t)]$$

$$\mathcal{L}[u(t)] = \frac{1}{s}$$ ← Laplace Transform of unit step function is $\frac{1}{s}$.

Let $\mathcal{L}[y] = Y(s)$

we have

$$\mathcal{L}[3\dot{y}] = 3\mathcal{L}[\dot{y}] = 3[sY(s) - y(0)]$$

$$\mathcal{L}[\ddot{y}] = s\mathcal{L}[\dot{y}] - \dot{y}(0) = s[sY(s) - y(0)] - \dot{y}(0)$$

$$= s^2 Y(s) - s y(0) - \dot{y}(0).$$

L.H.S =

$$\mathcal{L}[\ddot{y} + 3\dot{y} + y]$$

$$= s^2 Y(s) - s y(0) - \dot{y}(0) + 3s Y(s) - 3y(0) + Y(s).$$

$$= (s^2 + 3s + 1) Y(s) - (s+3) y(0) - \dot{y}(0)$$

R.H.S. =

$$\mathcal{L}[u(t)] = \frac{1}{s}$$

We have

$$(s^2 + 3s + 1) Y(s) - (s+3) y(0) - \dot{y}(0) = \frac{1}{s}$$

$$\Rightarrow (s^2 + 3s + 1) Y(s) = (s+3) y(0) + \dot{y}(0) + \frac{1}{s}$$

$$\Rightarrow \qquad = 5(s+3) + 7 + \frac{1}{s}$$

$$= 5s + 22 + \frac{1}{s}$$

$$= \frac{5s^2 + 22s + 1}{s}$$

$$\Rightarrow Y(s) = \frac{5s^2 + 22s + 1}{s(s^2+3s+1)}$$

Step 2 (partial fraction expansion)

Factor s^2+3s+1 as.

$$(s-\lambda_1)(s-\lambda_2).$$

where $\lambda_1 = -.382$, $\lambda_2 = -2.618$
(already calculated)

write

$$\frac{5s^2+22s+1}{s(s^2+3s+1)} =$$

$$\frac{5s^2+22s+1}{s(s-\lambda_1)(s-\lambda_2)} = \frac{A}{s} + \frac{B}{s-\lambda_1} + \frac{C}{s-\lambda_2}.$$

$$A = \left.\frac{5s^2+22s+1}{(s-\lambda_1)(s-\lambda_2)}\right|_{s=0} = 1$$

$$B = \left.\frac{5s^2 + 22s + 1}{s(s-\lambda_2)}\right|_{s=\lambda_1}$$

$$= \frac{5\lambda_1^2 + 22\lambda_1 + 1}{\lambda_1(\lambda_1 - \lambda_2)} = 7.814$$

$$C = \left.\frac{5s^2 + 22s + 1}{s(s-\lambda_1)}\right|_{s=\lambda_2}$$

$$= \frac{5\lambda_2^2 + 22\lambda_2 + 1}{\lambda_2(\lambda_2 - \lambda_1)} = -3.814$$

$$\therefore \frac{5s^2 + 22s + 1}{s(s-\lambda_1)(s-\lambda_2)} = \frac{1}{s} + \frac{7.814}{s-\lambda_1} - \frac{3.814}{s-\lambda_2}.$$

step 3 (Take the Laplace inverse)

$$y(t) = \mathcal{L}^{-1}\left[Y(s)\right]$$

$$= \mathcal{L}^{-1}\left[\frac{5s^2+22s+1}{s(s^2+3s+1)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{A}{s}\right] + \mathcal{L}^{-1}\left[\frac{B}{s-\lambda_1}\right] + \mathcal{L}^{-1}\left[\frac{C}{s-\lambda_2}\right]$$

$$= A + Be^{\lambda_1 t} + Ce^{\lambda_2 t}$$

$$\boxed{y(t) = 1 + 7.814e^{-.382t} - 3.814e^{-2.618t}}$$

Example 2

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + \frac{13}{4}y = u(t)$$

$$u(t) = 1 \quad t \geq 0$$
$$= 0 \quad t < 0$$

$$y(0) = 5 \; ; \; \dot{y}(0) = 7$$

Sol^n

Method of homogeneous/particular sol^n

Step 1 $\quad \lambda^2 + 3\lambda + \frac{13}{4}$

Step 2 $\quad \lambda^2 + 3\lambda + \frac{13}{4} = 0$

$$\Rightarrow \lambda = \frac{-3 \pm \sqrt{9 - 13}}{2}$$

$$= \frac{-3 \pm 2i}{2} = -\frac{3}{2} \pm i$$

Step 3

$$y_h(t) = \alpha e^{-\frac{3}{2}t} \cos t + \beta e^{-\frac{3}{2}t} \sin t$$

$$= e^{-\frac{3}{2}t}\left[\alpha \cos t + \beta \sin t\right]$$

Step 4 $\quad y_p(t) = A$

Step 5 $\quad \ddot{A} + 3\dot{A} + \frac{13}{4}A = 1$

$$\Rightarrow \frac{13}{4}A = 1$$

$$\Rightarrow A = \frac{4}{13}$$

Step 6

$$y(t) = \frac{4}{13} + e^{-\frac{3}{2}t}\left[\alpha \cos t + \beta \sin t\right]$$

Step 7

$$y(0) = \frac{4}{13} + \alpha = 5$$

$$\alpha = 5 - \frac{4}{13} = \frac{65-4}{13} = \frac{61}{13}$$

$$\dot{y}(t) = \frac{d}{dt}\left[e^{-\frac{3}{2}t}(\alpha \cos t + \beta \sin t)\right]$$

$$= e^{-\frac{3}{2}t}\left[-\alpha \sin t + \beta \cos t\right]$$

$$+ \left(-\frac{3}{2}\right)e^{-\frac{3}{2}t}(\alpha \cos t + \beta \sin t)$$

$$\dot{y}(0) = \beta - \frac{3}{2}\alpha = 7$$

$$\beta = 7 + \frac{3}{2} \cdot \frac{61}{13} = 7 + \frac{183}{26}$$

$$= \frac{182+183}{26} = \frac{365}{26}$$

Step 8

$$y(t) = \frac{4}{13} + e^{-\frac{3}{2}t}\left[\frac{61}{13}\cos t + \frac{365}{26}\sin t\right]$$

— x —

Method of Laplace's Transform.

$$\mathcal{L}\left[\ddot{y} + 3\dot{y} + \frac{13}{4}y\right] = \mathcal{L}[u]$$

$$\Rightarrow s^2 Y(s) - s\,y(0) - \dot{y}(0)$$
$$+ 3s\,Y(s) - 3\,y(0) + \frac{13}{4}Y(s) = \frac{1}{s}$$

$$\Rightarrow \left[s^2 + 3s + \frac{13}{4}\right]Y(s) = (s+3)\,y(0) + \dot{y}(0) + \frac{1}{s}$$

$$= \frac{5s^2 + 22s + 1}{s}.$$

$$\Rightarrow Y(s) = \frac{5s^2 + 22s + 1}{\left(s^2 + 3s + \frac{13}{4}\right)s}.$$

Partial Fraction Expansion.

$$\frac{5s^2+22s+1}{\left(s^2+3s+\frac{13}{4}\right)s} = \frac{As+B}{s^2+3s+\frac{13}{4}} + \frac{C}{s}. \quad (*)$$

$$C = \left.\frac{5s^2+22s+1}{s^2+3s+\frac{13}{4}}\right|_{s=0} = \frac{4}{13}$$

Adding the R.H.S. of (*) we get

$$\frac{As^2+Bs+Cs^2+3Cs+\frac{13}{4}C}{\left(s^2+3s+\frac{13}{4}\right)s}.$$

Equating the numerator of L.H.S & R.H.S in (*)
we get

$$5 = A + C$$

$$22 = B + 3C$$

$$1 = \frac{13}{4}C$$

Hence

$$C = \frac{4}{13}$$

$$A = 5 - \frac{4}{13} = \frac{65-4}{13} = \frac{61}{13}$$

$$B = 22 - 3C = 22 - \frac{12}{13}$$

$$= \frac{286-12}{13} = \frac{274}{13}$$

$$\therefore \frac{5s^2 + 22s + 1}{\left(s^2 + 3s + \frac{13}{4}\right)s} = \frac{\frac{61}{13}s + \frac{274}{13}.}{s^2 + 3s + \frac{13}{4}.} + \frac{4/13}{s}.$$

Calculating Laplace Inverse

$$\mathcal{L}^{-1}\left[\frac{4/13}{s}\right] = \frac{4}{13}.$$

Q: How to calculate

$$\mathcal{L}^{-1}\left[\frac{\frac{61}{13}s + \frac{274}{13}.}{s^2+3s+\frac{13}{4}}\right]$$

write

$$s^2+3s+\frac{13}{4} = \left(s+\frac{3}{2}\right)^2 - \frac{9}{4} + \frac{13}{4}$$

$$= \left(s+\frac{3}{2}\right)^2 + 1$$

write

$$\frac{61}{13}s + \frac{274}{13} = \frac{61}{13}\left[s + \frac{274}{61}\right]$$

$$= \frac{61}{13}\left[s+\frac{3}{2}\right] - \frac{61}{13}\cdot\frac{3}{2} + \frac{61}{13}\,\frac{274}{61}$$

$$= \frac{61}{13}\left[s+\frac{3}{2}\right] - \frac{183}{26} + \frac{274}{13}$$

$$= \frac{61}{13}\left(s+\frac{3}{2}\right) + \frac{365}{26}$$

It follows that

$$\frac{\frac{61}{13}s + \frac{274}{13}}{s^2+3s+\frac{13}{4}} = \frac{\frac{61}{13}\left(s+\frac{3}{2}\right)+\frac{365}{26}}{\left(s+\frac{3}{2}\right)^2+1}$$

$$= \frac{61}{13}\,\frac{s+3/2}{\left(s+\frac{3}{2}\right)^2+1} + \frac{365}{26}\cdot\frac{1}{\left(s+\frac{3}{2}\right)^2+1}$$

$$\mathcal{L}^{-1}\left(\frac{s+3/2}{\left(s+\frac{3}{2}\right)^2+1}\right) = e^{-3/2\,t}\cos t.$$

$$\mathcal{L}^{-1}\left(\frac{1}{\left(s+\frac{3}{2}\right)^2+1}\right) = e^{-3/2\,t}\sin t.$$

Combining terms we get

$$\mathcal{L}^{-1}\left(\frac{\frac{61}{13}s+\frac{274}{13}}{s^2+3s+\frac{13}{4}}\right)$$

$$= \frac{61}{13}e^{-\frac{3}{2}t}\cos t + \frac{365}{26}e^{-\frac{3}{2}t}\sin t.$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{5s^2+22s+1}{\left(s^2+3s+\frac{13}{4}\right)s}\right)$$

$$= \frac{4}{13} + \frac{61}{13}e^{-\frac{3}{2}t}\cos t + \frac{365}{26}e^{-\frac{3}{2}t}\sin t$$

Example 3

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = te^{-4t}$$

$$y(0) = 5 \;;\; \dot{y}(0) = 7$$

Method of homogeneous/particular sol^n

Step 1

Homogeneous sol^n is same as Example 1

$$y_h(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t}$$

Step 2

$$y_p(t) = e^{-4t}(A + Bt)$$

Step 3 calculate A and B

$$\dot{y}_p(t) = e^{-4t}B + -4e^{-4t}(A + Bt)$$

$$= e^{-4t}\left[(B - 4A) - 4Bt\right]$$

$$\ddot{y}_p = e^{-4t}\left[\qquad -4B\right]$$
$$-4e^{-4t}\left[(B-4A)-4Bt\right]$$
$$= e^{-4t}\left[-4B-4B+16A+16Bt\right]$$
$$= e^{-4t}\left[(16A-8B)+16Bt\right]$$

$$\ddot{y}_p + 3\dot{y}_p + y_p =$$
$$= e^{-4t}\left[(16A-8B)+16Bt\right.$$
$$(3B-12A)-12Bt$$
$$\left.(A) \qquad + Bt\right]$$
$$= e^{-4t}\left[(5A-5B)+5Bt\right] = te^{-4t}$$

$$\Rightarrow A = B, \quad 5B = 1$$

$$B = \frac{1}{5}, \; A = \frac{1}{5}$$

$$y_p(t) = \frac{1}{5}e^{-4t}(1+t).$$

Step 4 (Add the homogeneous & particular solution)

$$y(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t} + \frac{1}{5} e^{-4t}(1+t).$$

Step 5 (Find α, β from initial condition)

$$y(0) = 5, \quad \dot{y}(0) = 7$$

$$y(0) = \alpha + \beta + \frac{1}{5} = 5$$

$$\alpha + \beta = 5 - \frac{1}{5} = \frac{24}{5}$$

— x —

$$\dot{y}(t) = \alpha \lambda_1 e^{\lambda_1 t} + \beta \lambda_2 e^{\lambda_2 t} + \frac{1}{5} e^{-4t} + \frac{1}{5}(1+t)(-4) e^{-4t}$$

$$\dot{y}(0) = \alpha \lambda_1 + \beta \lambda_2 + \frac{1}{5} + \frac{1}{5}(-4)$$

$$= \alpha \lambda_1 + \beta \lambda_2 - \frac{3}{5} = 7$$

$$\alpha \lambda_1 + \beta \lambda_2 = 7 + \frac{3}{5} = \frac{35}{5} + \frac{3}{5} = \frac{38}{5}$$

$$\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 24/5 \\ 38/5 \end{pmatrix}$$

$$\alpha = \frac{\det \begin{pmatrix} 24/5 & 1 \\ 38/5 & \lambda_2 \end{pmatrix}}{\lambda_2 - \lambda_1} = \frac{1}{5} \frac{24\lambda_2 - 38}{\lambda_2 - \lambda_1}$$

$$= 9.019$$

$$\beta = \frac{\det \begin{pmatrix} 1 & 24/5 \\ \lambda_1 & 38/5 \end{pmatrix}}{\lambda_2 - \lambda_1} = \frac{1}{5} \frac{38 - 24\lambda_1}{\lambda_2 - \lambda_1}.$$

$$= -4.219$$

Step 6 (write $y(t)$)

$$\boxed{y(t) = 9.019 e^{-.382t} - 4.219 e^{-2.618t} + \frac{1}{5} e^{-4t}(1+t).}$$

Final Answer

Method of Laplace's Transform.

Step 1

LHS =

$$(s^2+3s+1)Y(s) - (s+3)y(0) - \dot{y}(0)$$

(From Example 1).

RHS =

$$\mathcal{L}\left[te^{-4t}\right] = \frac{1}{(s+4)^2}$$

Equating LHS = RHS, we get

$$(s^2+3s+1)Y(s) = (s+3)y(0) + \dot{y}(0) + \frac{1}{(s+4)^2}$$

$$= 5(s+3) + 7 + \frac{1}{(s+4)^2}$$

$$= 5s + 22 + \frac{1}{(s+4)^2}$$

$$= \frac{(5s+22)(s^2+8s+16)+1}{(s+4)^2}$$

$$= \frac{5s^3+62s^2+256s+353}{(s+4)^2}$$

$$Y(s) = \frac{5s^3 + 62s^2 + 256s + 353}{(s-\lambda_1)(s-\lambda_2)(s+4)^2}$$

Step 2

Partial fraction expansion

$$\frac{5s^3 + 62s^2 + 256s + 353}{(s-\lambda_1)(s-\lambda_2)(s+4)^2} = \frac{A}{s-\lambda_1} + \frac{B}{s-\lambda_2} + \frac{C}{s+4} + \frac{D}{(s+4)^2} \quad \text{(A)}$$

$$A = \left. \frac{5s^3 + 62s^2 + 256s + 353}{(s-\lambda_2)\quad(s+4)^2} \right|_{s=\lambda_1} = 9.019$$

$$B = \left. \frac{5s^3 + 62s^2 + 256s + 353}{(s-\lambda_1)\ (s+4)^2} \right|_{s=\lambda_2} = -4.219$$

$$D = \left. \frac{5s^3 + 62s^2 + 256s + 353}{(s-\lambda_1)(s-\lambda_2)} \right|_{s=-4} = \frac{1}{5}$$

Step 3 (Partial fraction Expansion)

We still need C and now the trick cannot be applied.

Adding the RHS of ★ we get

$$\frac{A(s-\lambda_2)(s+4)^2 + B(s-\lambda_1)(s+4)^2 + C(s-\lambda_1)(s-\lambda_2)(s+4) + D(s-\lambda_1)(s-\lambda_2)}{(s-\lambda_1)(s-\lambda_2)(s+4)^2}$$

The constant in the numerator is given by

$$A(-\lambda_2)\,16 + B(-\lambda_1)\,16 + C\lambda_1\lambda_2 4 + D\lambda_1\lambda_2$$

$$= -16\left[\lambda_2 A + \lambda_1 B\right] + (4C + D)\lambda_1\lambda_2$$

(where $\lambda_1\lambda_2 = 1$)

$$= -16\left[\lambda_2 A + \lambda_1 B\right] + 4C + D$$

$$= 352 \cdot 2 + 4C$$

Equating the constant in the numerator of ✪ we get

$$352\cdot 2 + 4c = 353$$

$$\Rightarrow 4c = \cdot 8$$

$$\Rightarrow c = \cdot 2 = \frac{1}{5}.$$

Step 4 (calculating inverse Laplace's Transform)

$$y(t) = \mathcal{L}^{-1}\left(\frac{A}{s-\lambda_1}\right) + \mathcal{L}^{-1}\left(\frac{B}{s-\lambda_2}\right) + \mathcal{L}^{-1}\left(\frac{C}{s+4}\right) + \mathcal{L}^{-1}\frac{D}{(s+4)^2}.$$

$$= 9\cdot 019e^{-\cdot 382t} - 4\cdot 219e^{-2\cdot 618t} + \frac{1}{5}e^{-4t} + \frac{1}{5}te^{-4t}.$$

Final Answer:

$$y(t) = 9.019\, e^{-.382t} - 4.219\, e^{-2.618t} + \frac{1}{5} e^{-4t}(1+t).$$