Home Work 4.
Solutions.

1 Ans:

a In all these problems we assume

$y(0) = \dot{y}(0) = 0$.

char polynomial is

$\lambda^2 + 15\lambda + 50$.

$= (\lambda+5)(\lambda+10)$.

$\therefore\ y_h(t) = Ae^{-5t} + Be^{-10t}$.

$y_p(t) = Ce^{-15t} + D\sin t + E\cos t$.

$\dot{y}_p(t) = -15Ce^{-15t} - E\sin t + D\cos t$

$\ddot{y}_p(t) = 225Ce^{-15t} - D\sin t - E\cos t$

$$225C - 225C + 50C = 1$$

$$\therefore \boxed{C = \frac{1}{50}}$$

$$\boxed{\begin{aligned} -D - 15E + 50D &= 1 \\ -E + 15D + 50E &= 0 \end{aligned}}$$

$$49D - 15E = 1.$$

$$15D + 49E = 0.$$

$$\begin{pmatrix} 49 & -15 \\ 15 & 49 \end{pmatrix} \begin{pmatrix} D \\ E \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$D = \frac{49}{49^2 + 15^2} \ ; \ E = \frac{-15}{49^2 + 15^2}.$$

$$y_p(t) = \frac{1}{50} e^{-15t} + \frac{49 \sin t - 15 \cos t}{49^2 + 15^2}$$

$$y_h(t) = A e^{-5t} + B e^{-10t}.$$

$$y(t) = y_p(t) + y_h(t).$$

To find A, B we use initial conditions.

$$y(0) = y_p(0) + y_h(0)$$

$$= \frac{1}{50} + \frac{-15}{49^2 + 15^2}$$

$$+ A + B = 0.$$

$$\Rightarrow \boxed{A + B = -\frac{1}{50} + \frac{15}{49^2 + 15^2}}$$

$$\dot{y}_p(0) = -\frac{15}{50} + \frac{49}{49^2+15^2}$$

$$\dot{y}_h(0) = -5A - 10B.$$

$$\dot{y}(0) = \dot{y}_p(0) + \dot{y}_h(0)$$

$$= -\frac{15}{50} + \frac{49}{49^2+15^2} - 5A - 10B = 0$$

$$\Rightarrow 5A + 10B = -\frac{15}{50} + \frac{49}{49^2+15^2}.$$

$$5A + 5B = -\frac{5}{50} + \frac{75}{49^2+15^2}$$

$$5B = -\frac{\cancel{10}}{\underset{5}{\cancel{50}}} - \frac{26}{49^2+15^2}.$$

$$B = \left[-\frac{1}{5} - \frac{26}{49^2+15^2}\right]\frac{1}{5}.$$

A can now be computed.

1 b

We will use convolution.

$$(s^2+15s+50)Y(s) = F(s).$$

$$Y(s) = \frac{1}{(s+5)(s+10)} \cdot F(s).$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s+5)(s+10)}\right) = Ae^{-5t} + Be^{-10t}$$

$$\frac{1}{(s+5)(s+10)} = \frac{A}{s+5} + \frac{B}{s+10}.$$

$$A = \left.\frac{1}{s+10}\right|_{s=-5} = \frac{1}{5}.$$

$$B = \left.\frac{1}{s+5}\right|_{s=-10} = -\frac{1}{5}.$$

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{(s+5)(s+10)}\right) = \frac{1}{5}\left[e^{-5t} - e^{-10t}\right]$$

$$y(t) = h(t) ** f(t).$$

$$\frac{1}{5}\int_0^t \left[e^{-5(t-\tau)} - e^{-10(t-\tau)}\right] f(\tau)\, d\tau.$$

$$= \frac{1}{5} e^{-5t} \underbrace{\int_0^t f(\tau) e^{5\tau}\, d\tau}_{= A(t)}$$

$$- \frac{1}{5} e^{-10t} \underbrace{\int_0^t f(\tau) e^{10\tau}\, d\tau}_{= B(t)}.$$

If

$$A(t) = \int_0^t f(\tau) e^{5\tau} \, d\tau.$$

$$B(t) = \int_0^t f(\tau) e^{10\tau} \, d\tau.$$

we have.

$$y(t) = \frac{1}{5} e^{-5t} A(t) - \frac{1}{5} e^{-10t} B(t).$$

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We calculate $A(t)$ & $B(t)$.

For $0 \leq t < T$

$$A(t) = \int_0^t 100 e^{5\tau} d\tau.$$

$$= \frac{\cancel{100}^{20} e^{5\tau}}{\cancel{5}} \Big|_0^t$$

$$= [20e^{5t} - 20] = 20[e^{5t} - 1]$$

$$B(t) = \int_0^t 100 e^{10\tau} d\tau.$$

$$= 10 e^{10\tau} \Big|_0^t = 10e^{10t} - 10.$$

$$= 10[e^{10t} - 1].$$

We calculate $A(t)$, $B(t)$

for $t > T$.

$$A(t) = \int_0^T 100 e^{5\tau}\, d\tau = A(T)$$

$$B(t) = \int_0^T 100 e^{10\tau}\, d\tau = B(T).$$

Plugging $A(t)$ & $B(t)$ back in $\circledast$

we have

$$y(t) = \frac{1}{5} e^{-5t} [20] [e^{5t} - 1]$$
$$- \frac{1}{5} e^{-10t} [10] [e^{10t} - 1]$$

for $0 \le t < T$

$$= 4 [1 - e^{-5t}]$$
$$- 2 [1 - e^{-10t}]$$

$$= 2 - 4e^{-5t} + 2e^{-10t}.$$

$$0 \le t < T$$

when $t > T$ we have

$$y(t) = 4e^{-5t}\left[e^{5T} - 1\right]$$

$$-2e^{-10t}\left[e^{10T} - 1\right]$$

$$= 4e^{-5(t-T)} - 2e^{-10(t-T)}.$$

$$-4e^{-5t} + 2e^{-10t}.$$

1 (c)

Let us solve the problem when

$f(t) = \sin t$.

We know from 1 (a) that

$$y_p(t) = \frac{49\sin t - 15\cos t}{49^2 + 15^2}.$$

$$y_h(t) = Ae^{-5t} + Be^{-10t}.$$

$$y(0) = y_p(0) + y_h(0) =$$

$$A + B - \frac{15}{49^2+15^2} = 0.$$

$$\Rightarrow \boxed{A + B = \frac{15}{49^2+15^2}.}$$

$$\dot{y}(0) = \dot{y}_p(0) + \dot{y}_h(0)$$

$$= \frac{49}{49^2+15^2} - 5A - 10B = 0.$$

$$\Rightarrow 5A + 10B = \frac{49}{49^2+15^2}.$$

$$5A + 5B = \frac{75}{49^2+15^2}.$$

$$5B = \frac{-26}{49^2+15^2}.$$

$$\boxed{B = \frac{-26/5}{49^2+15^2}.}$$

$$A = \frac{15 + \frac{26}{5}}{49^2+15^2} = \frac{1}{5}\,\frac{101}{49^2+15^2}$$

$$y(t) = \cancel{\frac{1}{5}\Big[}$$

$$\frac{1}{5}\,\frac{1}{49^2+15^2}\left[101e^{-5t}-26e^{-10t} + 245\sin t - 75\cos t\right]$$

when $f(t) = \sin(t-T) \quad t \geq T$

$\qquad\qquad\quad 0 \quad 0 \leq t < T$

$$y(t) = \frac{1}{5}\,\frac{1}{49^2+15^2}\left[101e^{-5(t-T)}-26e^{-10(t-T)} + 245\sin(t-T) - 75\cos(t-T)\right] \quad t \geq T.$$

$$0 \qquad 0 \leq t < T$$

2) Ans:

Char poly is λ^2+1.

Roots are at $\lambda = \pm i$

$y_h(t) = A\sin t + B\cos t$.

For $f(t) = \sin t$.

$y_p(t) = Ct\sin t + Dt\cos t$.

$= t(C\sin t + D\cos t)$.

$\dot{y}_p(t) = C\sin t + D\cos t$

$+ t(C\cos t - D\sin t)$.

$\ddot{y}_p = C\cos t - D\sin t$

$+ t(-C\sin t - D\cos t)$.

$+ (C\cos t - D\sin t)$.

$$\ddot{y}_p = 2C\cos t - 2D\sin t$$
$$- Ct\sin t - Dt\cos t.$$

$$\ddot{y}_p + y_p = 2C\cos t - 2D\sin t$$
$$= \sin t$$

$$\Rightarrow C = 0 \quad D = -\frac{1}{2}.$$

$$y_p(t) = t\left(-\frac{1}{2}\right)\cos t$$
$$= -\frac{t}{2}\cos t.$$

$$y(t) = A\sin t + B\cos t - \frac{t}{2}\cos t.$$

$$y(0) = 1 = B \Rightarrow B = 1.$$

$$\dot{y}(t) = A\cos t - B\sin t + \frac{t}{2}\sin t$$
$$- \frac{1}{2}\cos t.$$

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$\dot{y}(0) = A - \frac{1}{2} = \frac{1}{2}$.

$\boxed{A = 1}$.

$\therefore\ y(t) = \sin t + \cos t - \frac{t}{2}\cos t$.

$\boxed{y(t) = \sin t + \left(1 - \frac{t}{2}\right)\cos t}$.