Practise questions and answers for Midterm II (make up)

For your own benefit, please try the questions before you look ~~through~~ through the answers.

① Calculate inverse Laplace Transform of the following function

ⓐ $$\frac{3s^2+5s+9}{(s+1)^2(s+2)}$$

ⓑ $$\frac{5s+4}{s^2+6s+8}$$

ⓒ $$\frac{5s+4}{s^2+6s+10}$$

② We are solving the equation

$$\dddot{y}+a\ddot{y}+b\dot{y}+cy=f(t)$$

The roots of the characteristic polynomial are given as

$$\lambda_1=-5,\ \lambda_2=-3+2i,\ \lambda_3=-3-2i$$

Let $f(t)=e^{-6t}$, and $y(0)=\dot{y}(0)=\ddot{y}(0)=0$

ⓐ write down a, b, c.

ⓑ write down $y_p(t)$, the particular sol$^{\underline{n}}$

ⓒ write down $y_h(t)$, the homogeneous sol$^{\underline{n}}$

ⓓ Calculate $y(t)$.

(3) $\ddot{y} + 6\dot{y} + 8y = \sin 3t$

$y(0) = \dot{y}(0) = 0$

(a) Calculate $y(t)$ using particular/homogeneous solutions.

(b) " " " Laplace Transforms.

(4) $\ddot{y} + 6\dot{y} + 8y = f(t)$

$$f(t) = t \qquad 0 < t < \pi$$
$$= 0 \qquad \text{otherwise}.$$

$y(0) = \dot{y}(0) = 0.$

Calculate $y(t)$ using convolution.

① **Ans:**

ⓐ $$\frac{3s^2+5s+9}{(s+1)^2(s+2)} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$A = \left.\frac{3s^2+5s+9}{(s+1)^2}\right|_{s=-2}$$

$$\boxed{Ae^{-2t} + Be^{-t} + Ce^{-t}t}$$

$$C = \left.\frac{3s^2+5s+9}{s+2}\right|_{s=-1}$$

To find B, we write

$$RHS = \frac{A(s+1)^2 + B(s+1)(s+2) + C(s+2)}{(s+1)^2(s+2)}$$

comparing the co-efficients of s^2 we get

$$A + B = 3$$

$$\therefore B = 3 - A$$

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(b) $\dfrac{5s+4}{s^2+6s+8}$

$$= \frac{5s+4}{(s+4)(s+2)} = \frac{A}{s+4} + \frac{B}{s+2}$$

$$A = \left.\frac{5s+4}{s+2}\right|_{s=-4} \qquad \boxed{Ae^{-4t} + Be^{-2t}}$$

$$B = \left.\frac{5s+4}{s+4}\right|_{s=-2}$$

(c) $s^2 + 6s + 10$

$$= (s+3)^2 + 1^2 \qquad \boxed{e^{-3t}\left[5\,\cancel{\sin t}\ \cos t - 11 \sin t\right]}$$

$$5s+4 = 5(s+3) - 11$$

$$\therefore \frac{5s+4}{s^2+6s+10} = \frac{5(s+3)}{(s+3)^2+1^2} - 11\,\frac{1}{(s+3)^2+1^2}$$

$$= 5\,\frac{s+3}{(s+3)^2+1^2} - 11\,\frac{1}{(s+3)^2+1^2}$$

(2)

Char polynomial is

$$(\lambda+5)(\lambda+3-2i)(\lambda+3+2i)$$

$$=(\lambda+5)\left[(\lambda+3)^2+2^2\right]$$

$$=(\lambda+5)\left[\lambda^2+6\lambda+9+4\right]$$

$$= \lambda^2+6\lambda+13$$

$$\lambda+5$$

$$\lambda^3+6\lambda^2+13\lambda$$

$$5\lambda^2+30\lambda+65$$

$$\lambda^3+11\lambda^2+43\lambda+65$$

(a) $\boxed{a=11,\ b=43,\ c=65}$

(b) $y_p(t)= Ae^{-6t}$

$$y_p = Ae^{-6t}$$

$$\dot{y}_p = -6Ae^{-6t}$$

$$\ddot{y}_p = 36Ae^{-6t}$$

$$\dddot{y}_p = -216Ae^{-6t}$$

$$y_p = -\frac{1}{13}e^{-6t}$$

$$\dddot{y}_p + 11\ddot{y}_p + 43\dot{y}_p + 65y_p = e^{-6t}$$

$$-216A + 11 \times 36A - 43 \times 6A + 65A = 1$$

$$\begin{array}{r} 36 \\ 11 \\ \hline 396 \end{array} \quad \begin{array}{r} 43 \\ 6 \\ \hline 258 \end{array} \qquad \begin{array}{r} -216 \\ -258 \\ \hline -474 \end{array} \quad \begin{array}{r} 396 \\ 65 \\ \hline 461 \end{array}$$

$$-13$$

$$-13A = 1 \qquad A = -\frac{1}{13}$$

(b)

$$y_h(t) = \alpha e^{-5t} + e^{-3t}\left[\beta \sin 2t + \gamma \cos 2t\right]$$

$$y(t) = \alpha e^{-5t} + e^{-3t}\left[\beta \sin 2t + \gamma \cos 2t\right] - \frac{1}{13} e^{-6t}.$$

α, β, γ can be calculated from the initial conditions.

3 Ans:

a $\lambda^2+6\lambda+8$

$=(\lambda+4)(\lambda+2)$

$y_h(t)=\alpha e^{-4t}+\beta e^{-2t}$.

$y_p(t) = A\sin 3t + B\cos 3t$.

$\dot{y}_p(t) = -3B\sin 3t + 3A\cos 3t$

$\ddot{y}_p(t) = -9A\sin 3t - 9B\cos 3t$

$-8A-3B=1$

$-8B+3A=0$

$$\begin{pmatrix} -8 & -3 \\ 3 & -8 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$A = \frac{-8}{64+9} = -\frac{8}{73}$

$B = \frac{-3}{73}$

$$\boxed{y_p(t) = -\frac{1}{73}\left[8\sin 3t + 3\cos 3t\right]}$$

$$Y(t) = \alpha e^{-4t} + \beta e^{-2t} - \frac{1}{73}\left[8\sin 3t + 3\cos 3t\right]$$

Calculate α, β from i.c.

— x —

(b) $\mathcal{L}(Y) = Y(s)$

$\mathcal{L}(\dot{y}) = s\,Y(s)$

$\mathcal{L}(\ddot{y}) = s^2\,Y(s)$

$$\therefore\ Y(s) = \frac{1}{s^2+6s+8}\ \frac{3}{s^2+9}.$$

$$= \frac{3}{(s+4)(s+2)(s^2+9)} = \frac{A}{s+4} + \frac{B}{s+2} + \frac{Cs+D}{s^2+9}$$

$$A = \left.\frac{3}{(s+2)(s^2+9)}\right|_{s=-4} = \frac{3}{(-2)(25)}$$

$$= -\frac{3}{50}$$

$$B = \left. \frac{3}{(s+4)(s^2+9)} \right|_{s=-2}$$

$$= \frac{3}{(2)(13)} = \frac{3}{26}$$

$$y(t) = -\frac{3}{50} e^{-4t} + \frac{3}{26} e^{-2t} - \frac{1}{73}\left[8 \sin 3t + 3 \cos 3t\right]$$

You need to find C, D and verify these numbers.

4 $(s^2+6s+8)Y(s) = \mathcal{L}(f(t)) = F(s).$

$$Y(s) = \frac{1}{(s+4)(s+2)} F(s).$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s+4)(s+2)}\right) = Ae^{-4t} + Be^{-2t}.$$

$$\frac{1}{(s+4)(s+2)} = \frac{A}{s+4} + \frac{B}{s+2}$$

$$A = \left.\frac{1}{s+2}\right|_{s=-4} = \frac{1}{2-4} = -\frac{1}{2}$$

$$B = \left.\frac{1}{s+4}\right|_{s=-2} = \frac{1}{4-2} = \frac{1}{2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s+4)(s+2)}\right) = \frac{1}{2}\left[e^{-2t} - e^{-4t}\right]$$

$$y(t) = \frac{1}{2}\left[\left(e^{-2t} - e^{-4t}\right) \ast\ast\, f(t)\right]$$

$$= \frac{1}{2}\int_0^t \left[e^{-2(t-\tau)} - e^{-4(t-\tau)}\right] f(\tau)\, d\tau.$$

$$= \frac{1}{2} e^{-2t} \underbrace{\int_0^t e^{2\tau} f(\tau)\, d\tau.}_{=A(t)}$$

$$- \frac{1}{2} e^{-4t} \underbrace{\int_0^t e^{4\tau} f(\tau)\, d\tau}_{=B(t).}$$

Define $A(t) = \int_0^t e^{2\tau} f(\tau)\, d\tau.$

$$B(t) = \int_0^t e^{4\tau} f(\tau)\, d\tau.$$

$$y(t) = \frac{1}{2} e^{-2t} A(t) - \frac{1}{2} e^{-4t} B(t)$$

Calculating $A(t)$.

For $t < \pi$

$$A(t) = \int_0^t \tau e^{2\tau} d\tau$$

$$\int \tau e^{2\tau} d\tau = \frac{\tau e^{2\tau}}{2} - \int \frac{e^{2\tau}}{2} d\tau.$$

$$= \frac{1}{2}\tau e^{2\tau} - \frac{1}{4}e^{2\tau}.$$

$$\int_0^t \tau e^{2\tau} d\tau = \left(\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t}\right) - \left(0 - \frac{1}{4}\right)$$

$$A(t) = \frac{1}{4} + \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t}$$

$t < \pi$

For $t > \pi$

$$A(t) = \int_0^\pi \tau e^{2\tau} d\tau = A(\pi).$$

you can calculate $B(t)$ like wise.

For $t > \pi$ we would have

$$y(t) = \frac{A(\pi)}{2} e^{-2t} - \frac{B(\pi)}{2} e^{-4t}$$

For $t < \pi$

$$y(t) = \frac{A(t)}{2} e^{-2t} - \frac{B(t)}{2} e^{-4t}.$$