Midterm II

Answers.

(1) (a)

- $s^2+2s+2=(s+1)^2+1$
- $3s+4 = 3(s+1)+1$

$$\frac{3s+4}{s^2+2s+2} = \frac{3(s+1)+1}{(s+1)^2+1}$$

$$= 3\frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}.$$

- $\mathcal{L}^{-1}\frac{s}{s^2+1} = \cos t \Rightarrow \mathcal{L}^{-1}\frac{s+1}{(s+1)^2+1} = e^{-t}\cos t$
- $\mathcal{L}^{-1}\frac{1}{s^2+1} = \sin t \Rightarrow$

$$\Rightarrow \mathcal{L}^{-1}\frac{1}{(s+1)^2+1} = e^{-t}\sin t$$

$$\therefore \mathcal{L}^{-1}\frac{3s+4}{s^2+2s+2} = 3e^{-t}\cos t + e^{-t}\sin t.$$

$$= e^{-t}(3\cos t + \sin t).$$

(b) $$F(s) = \frac{3s+4}{s^2+2s+2}.$$

$$F'(s) = \frac{(s^2+2s+2)3 - (3s+4)(2s+2)}{(s^2+2s+2)^2}$$

$$= \frac{(3s^2+6s+6) - (6s^2+14s+8)}{(s^2+2s+2)^2}$$

$$= \frac{-3s^2-8s-2}{(s^2+2s+2)^2}$$

$$-F'(s) = \frac{3s^2+8s+2}{(s^2+2s+2)^2}.$$

If $\mathcal{L}^{-1} F(s) = f(t)$

then $\mathcal{L}^{-1}(-F'(s)) = t f(t)$

Thus we have the result.

(2) Laplace's Method.

$$\ddot{y} + 5\dot{y} + 6y = 6 + 2e^{-t}.$$

$$\mathcal{L}\left[\ddot{y} + 5\dot{y} + 6y\right] = \frac{6}{s} + \frac{2}{s+1}.$$

$$= \frac{6s + 6 + 2s}{s(s+1)}.$$

$$= \frac{8s + 6}{s(s+1)}.$$

$$(s^2 + 5s + 6)\,Y(s) = \frac{8s+6}{s(s+1)}.$$

$$Y(s) = \frac{8s+6}{s(s+1)(s+2)(s+3)}.$$

$$y(t) = \mathcal{L}^{-1}\,Y(s).$$

$$Y(s) = \frac{8s+6}{s(s+1)(s+2)(s+3)}.$$

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} + \frac{D}{s+3}.$$

- $$A = \left.\frac{8s+6.}{(s+1)(s+2)(s+3)}\right|_{s=0} = \frac{6}{6} = 1.$$
- $$B = \left.\frac{8s+6}{s(s+2)(s+3)}\right|_{s=-1.} = \frac{-2}{-2} = 1.$$
- $$C = \left.\frac{8s+6}{s(s+1)(s+3).}\right|_{s=-2} = \frac{-10}{2} = -5$$
- $$D = \left.\frac{8s+6.}{s(s+1)(s+2)}\right|_{s=-3} = \frac{-18}{-6} = +3.$$

$$y(t) = A + Be^{-t} + Ce^{-2t} + De^{-3t}$$

$$= 1 + e^{-t} - 5e^{-2t} + 3e^{-3t}$$

$\blacksquare$

Char poly is $\lambda^2 + 5\lambda + 6$

$$= (\lambda+2)(\lambda+3).$$

$$y_h(t) = Ae^{-2t} + Be^{-3t}.$$

$$y_p(t) = C + De^{-t}$$ (from table in the book)

$$\dot{y}_p = -De^{-t}.$$

$$\ddot{y}_p = De^{-t}.$$

Thus we have

$$De^{-t} - 5De^{-t} + 6C + 6De^{-t} = 6 + 2e^{-t}$$

$$\Rightarrow 2De^{-t} + 6C = 6 + 2e^{-t}$$

$\Rightarrow$

$C = 1$ & $D = 1$.

$\therefore y_p(t) = 1 + e^{-t}$.

- $\therefore y(t) = Ae^{-2t} + Be^{-3t} + 1 + e^{-t}$.

- To find A, B use the initial conditions as follows.

$y(0) = A + B + 1 + 1 = 0 \Rightarrow A + B = -2$.

$\dot{y}(0) = \left(-2Ae^{-2t} - 3Be^{-3t} - e^{-t}\right)\Big|_{t=0}$.

- $= -2A - 3B - 1 = 0$

- $\Rightarrow 2A + 3B = -1$.

Solving $A + B = -2$, $2A + 3B = -1$. $\Rightarrow A = -5$, $B = +3$.

Hence

$y(t) = -5e^{-2t} + 3e^{-3t} + 1 + e^{-t}$.

(3) Ans:

(a) For $f(t) = e^{-2t}$

$$y_p(t) = A e^{-2t}$$

- $\Rightarrow y_h(t) = 2e^{-t} + e^{-5t}$.

- $\therefore$ char. poly must have roots at

$$\lambda = -1 \text{ and } \lambda = -5$$

$$\Rightarrow \text{char poly} = (\lambda+1)(\lambda+5)$$
$$= \lambda^2 + 6\lambda + 5.$$

- $\therefore$ $a = 6$ & $b = 5$.

- The eqn is.

$$\ddot{y} + 6\dot{y} + 5y = f(t).$$

To find $y_p(t)$ calculate.

$$\dot{y}_p(t) = -2Ae^{-2t}.$$

$$\ddot{y}_p(t) = 4Ae^{-2t}.$$

- Thus.
- $$\ddot{y}_p + 6\dot{y}_p + 5y_p =$$

$$(4A - 12A + 5A)e^{-2t}$$

$$= -3Ae^{-2t}$$

- Thus $$\ddot{y}_p + 6\dot{y}_p + 5y_p = e^{-2t}$$
- $$\Rightarrow -3Ae^{-2t} = e^{-2t}$$

$$A = -\frac{1}{3}.$$

$$\therefore y_p(t) = -\frac{1}{3}e^{-2t}.$$

Thus for $f(t) = e^{-2t}$,

$y(t)$ should have been recorded as

$$y(t) = -\frac{1}{3}e^{-2t} + 2e^{-t} + e^{-5t}.$$

(sorry that was a typo in the question paper)

Hence

$$y(0) = -\frac{1}{3} + 2 + 1 = 3 - \frac{1}{3}$$

$$= \frac{8}{3}$$

$$\dot{y}(t) = -\frac{1}{3}(-2)e^{-2t} - 2e^{-t} - 5e^{-5t}$$

$$\dot{y}(0) = \frac{2}{3} - 2 - 5 = \frac{2}{3} - 7 = \frac{2-21}{3}$$

$$= -\frac{19}{3}$$

$$\therefore\ y_0 = \frac{8}{3},\ v_0 = -\frac{19}{3}$$

3(b)

We need to solve

$$\ddot{y}(t) + 6\dot{y}(t) + 5y(t) = \sin t$$

- $y(0) = \frac{8}{3}, \quad \dot{y}(0) = -\frac{19}{3}.$

- $y_p(t) = A\sin t + B\cos t.$

$\dot{y}_p(t) = -B\sin t + A\cos t.$

$\ddot{y}_p(t) = -A\sin t - B\cos t$

- $\ddot{y}_p + 6\dot{y}_p + 5y_p = \sin t$

- $\Rightarrow (-A - 6B + 5A)\sin t \cdot (-B + 6A + 5B)\cos t = \sin t$

$\Rightarrow (4A - 6B)\sin t + (4B + 6A)\cos t = \sin t.$

$$4B + 6A = 0 \Rightarrow B = -\frac{\cancel{6}^{3}}{\cancel{4}_{2}} A$$

$$\boxed{B = -\frac{3}{2} A}$$

- $4A - 6B = 1$
- $\Rightarrow 4A - 6\left(-\frac{3}{2}\right) A = 1.$

$$\Rightarrow 4A + 9A = 1$$

$$A = \frac{1}{13} = \frac{2}{26}.$$

- $B = -\frac{3}{2} \frac{1}{13} = -\frac{3}{26}$
- $\boxed{\therefore y_p(t) = \frac{1}{13} \sin t - \frac{3}{26} \cos t}.$

$$y(t) = Ce^{-t} + De^{-5t} +$$
$$\frac{1}{13}\sin t - \frac{3}{26}\cos t.$$

- $y(0) = C + D - \frac{3}{26} = \frac{8}{3}.$
- $\therefore \boxed{C + D = \frac{3}{26} + \frac{8}{3}}.$

$$\dot{y}(t) = -Ce^{-t} - 5De^{-5t}$$
$$+ \frac{1}{13}\cos t + \frac{3}{26}\sin t.$$

-
$$\dot{y}(0) = -C - 5D + \frac{1}{13} = -\frac{19}{3}.$$
-

$$C + 5D = \frac{2}{26} + \frac{19}{3}$$
$$C + D = \frac{3}{26} + \frac{8}{3}$$

$$4D = -\frac{1}{26} + \frac{11}{3}$$

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$$D = \frac{1}{4}\left[\frac{11}{3} - \frac{1}{26}\right]$$

$$C = \frac{3}{26} + \frac{8}{3} - \frac{1}{4}\left[\frac{11}{3} - \frac{1}{26}\right]$$

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