$\boxed{\text{Review}}$

$\boxed{\text{Math 3350}}$

PRACTICE PROBLEMS FOR THE FINALS & THEIR SOLUTIONS.

1 We have the following 2nd order differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 50y = f(t)$$

a) calculate $y(t)$ assuming

$f(t) = \sin 7t \qquad t \geq 0$

b) calculate $y(t)$ assuming

$$f(t) = \begin{cases} \sin 7t & 0 < t < \frac{2\pi}{7} \\ 0 & t > 2\pi/7 \end{cases}$$

Assume $y(0) = 0$, $y'(0) = 0$

1 Ans:

Taking Laplace's Transform we have

$$(s^2+2s+50)\,Y(s) = \mathcal{L}[f(t)]$$

a $f(t) = \sin 7t\,,\quad t \geq 0$

$$\mathcal{L}(f(t)) = \frac{7}{s^2+49}$$

$$\therefore\ Y(s) = \frac{7}{(s^2+2s+50)(s^2+49)}$$

We need to write

$$\frac{7}{(s^2+2s+50)(s^2+49)} = \frac{As+B}{s^2+49} + \frac{Cs+D}{s^2+2s+50}$$

$$\therefore (As+B)(s^2+2s+50)+(Cs+D)(s^2+49) = 7$$

we have by comparing the co-efficients

$A=-C \qquad 2A+B+D=0$

$A=-2B \qquad D=3B$

$B=\frac{7}{197}, \quad A=\frac{-14}{197}, \quad D=\frac{21}{197}$

$C=\frac{14}{197}$

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$$\frac{7}{(s^2+2s+50)(s^2+49)} =$$

$$\frac{1}{197}\left[\frac{-14s+7}{s^2+49} + \frac{14s+21}{s^2+2s+50}\right]$$

$$\boxed{\begin{array}{l} \dfrac{-14s+7}{s^2+49} = -14\dfrac{s}{s^2+49} + \dfrac{7}{s^2+7^2} \\ \quad\quad\downarrow \\ \mathcal{L}^{-1}\left(\quad\right) = -14\cos 7t + \sin 7t \end{array}}$$

$$\frac{14s+21}{s^2+2s+50} = \frac{14s+21}{(s+1)^2+7^2} = \frac{14(s+1)+7}{(s+1)^2+7^2}$$

$$= 14\frac{s+1}{(s+1)^2+7^2} + \frac{7}{(s+1)^2+7^2}$$

$$\mathcal{L}^{-1}\left(\frac{14s+21}{s^2+2s+50}\right)$$

$$= 14e^{-t}\cos 7t + e^{-t}\sin 7t$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{7}{(s^2+2s+50)(s^2+49)}\right]$$

$$= \frac{1}{197}\left[14\cos 7t\left(e^{-t}-1\right) + \sin 7t\left(e^{-t}+1\right)\right]$$

b ~~If u(t) is define~~

If $u(t-a)$ is defind as follows

$$u(t-a) = \begin{cases} 0, & 0 < t < a \\ 1, & t \geq a \end{cases}$$

we have

$$\xcancel{f(t) = (\sin 7t)\left[u(t) - u\left(t - \frac{2\pi}{7}\right)\right]}$$

$$f(t) = \sin 7t \; u(t) - \sin 7\left(t - \frac{2\pi}{7}\right) u\left(t - \frac{2\pi}{7}\right)$$

$$\mathcal{L}\left[f(t)\right] = \left[\frac{7}{s^2+7^2} - \frac{7}{s^2+7^2} e^{-\frac{2\pi}{7}s}\right]$$

For this case

$y(t) =$

$$\frac{1}{197}\left[14\cos 7t\,(e^{-t}-1) + \sin 7t\,(e^{-t}+1)\right]u(t)$$

$$-\left\{\frac{1}{197}\left[14\cos\left[7\left(t-\frac{2\pi}{7}\right)\right]\left[e^{-\left(t-\frac{2\pi}{7}\right)}-1\right] + \sin\left[7\left(t-\frac{2\pi}{7}\right)\right]\left[e^{-\left(t-\frac{2\pi}{7}\right)}+1\right]\right]\right\} u\left(t-\frac{2\pi}{7}\right)$$

Remark: We delay the sol^{n} by $2\pi/7$ and subtract.

$\therefore \cos\left[7\left(t-\frac{2\pi}{7}\right)\right] = \cos 7t$

$\sin\left[7\left(t-\frac{2\pi}{7}\right)\right] = \sin 7t$

we have

$$y(t) = \frac{1}{197}\Bigg[14\cos 7t\left\{\left(e^{-t}-1\right)-\left(e^{-\left(t-\frac{2\pi}{7}\right)}-1\right)\cdot u\left(t-\frac{2\pi}{7}\right)\right\}$$

$$+\sin 7t\left\{\left(e^{-t}+1\right)-\left(e^{-\left(t-\frac{2\pi}{7}\right)}+1\right)\cdot u\left(t-\frac{2\pi}{7}\right)\right\}\Bigg]$$

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y for $0 < t < \frac{2\pi}{7}$

$$y(t) = \frac{1}{197}\left[14\cos 7t\left(e^{-t}+1\right) + \sin 7t\left(e^{-t}+1\right)\right]$$

———— x ————

for $t > 2\pi/7$

$$y(t) = \frac{1}{197}\left[14\cos 7t\left(e^{-t} - e^{-\left(t-\frac{2\pi}{7}\right)}\right) + \sin 7t\left(e^{-t} - e^{-\left(t-\frac{2\pi}{7}\right)}\right)\right]$$

$$= \frac{1}{197}\left[14\cos 7t + \sin 7t\right]\left(e^{-t} - e^{-\left(t-\frac{2\pi}{7}\right)}\right)$$

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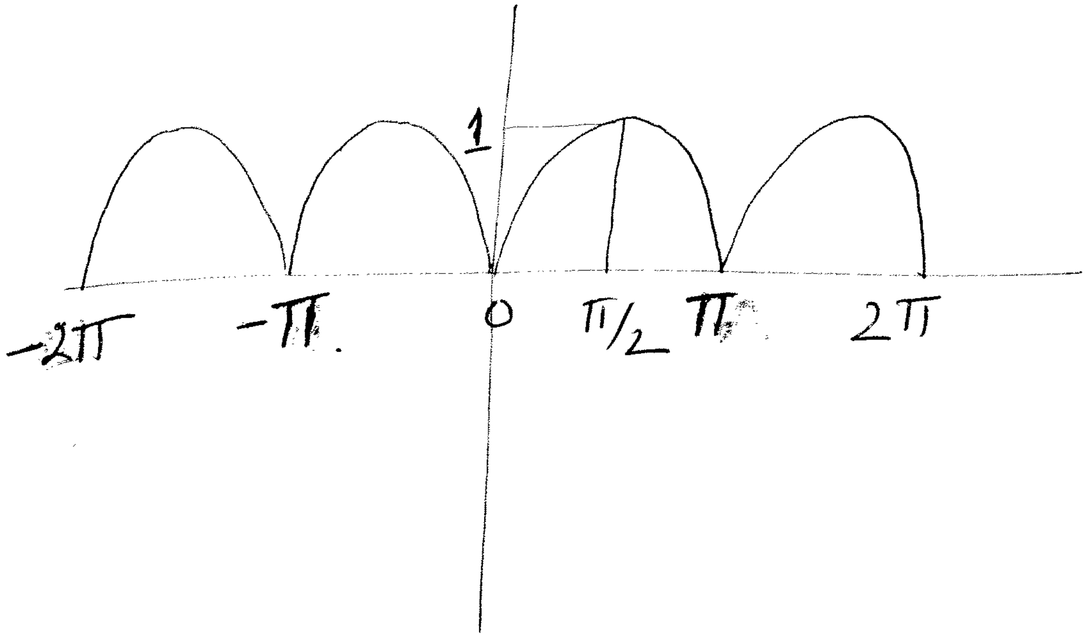
$$= \frac{1}{197}\left[14\cos 7t + \sin 7t\right] e^{-t}$$

$$\left(1 - e^{2\pi/7}\right)$$



2 Let $f(x)$ be the ~~ro~~ rectified sine wave i.e.

$$f(x) = |\sin x|$$

Calculate the Fourier cosine series expansion of $f(x)$ and show that

~~Recall we have shown~~

1. $$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}.$$

2. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}.$$

3. $$\sum_{n=0}^{\infty} \frac{1}{16n^2+16n+3} = \frac{\pi}{8}$$

2 Ans:

$f(x)$ is an even function of period π (not 2π)

$$p = \pi \qquad L = p/2 = \pi/2$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2nx$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \qquad a_0 = \frac{2}{\pi}$$

$$= \frac{1}{L} \int_0^L f(x)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \sin x\, dx = \frac{-2}{\pi} \cos x \Big|_0^{\pi/2}$$

$$= +2/\pi$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos 2nx\,dx$$

$$= \frac{2}{L}\int_{0}^{L} f(x)\cos 2nx\,dx$$

$$\cancel{= \frac{2}{L}\int}$$

$$= \frac{4}{\pi}\int_{0}^{\pi/2} \sin x\cos 2nx\,dx$$

We use trignometric formula

$$\sin x\cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$

to obtain

$$\sin x\cos 2nx = \frac{1}{2}\left[\sin(2n+1)x + \sin(1-2n)x\right]$$

Thus we have

$$a_n = \frac{2}{\pi}\int_0^{\pi/2} \sin\left[(1+2n)x\right] + \sin\left[(1-2n)x\right] \, dx$$

$$= -\frac{2}{\pi}\left[\frac{\cos\left[(1+2n)x\right]}{1+2n} + \frac{\cos\left[(1-2n)x\right]}{1-2n}\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[\frac{\cos\left[(2n+1)x\right]}{2n+1} - \frac{\cos\left[(2n-1)x\right]}{2n-1}\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[\cancelto{0}{\frac{\cos\left[(2n+1)\frac{\pi}{2}\right]}{2n+1}} - \cancelto{0}{\frac{\cos\left[(2n-1)\frac{\pi}{2}\right]}{2n-1}}\right]$$

$$+ \frac{2}{\pi}\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right]$$

$$= \frac{2}{\pi} \; \frac{\cancel{2n}-1-\cancel{2n}-1}{4n^2-1} = \frac{-4}{\pi(4n^2-1)}$$

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$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos 2nx$$

At $x=0$ $f(0) = 0$

$\cos 2nx = 1$

$$\therefore \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{2}{\pi}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$$

$$\Rightarrow \boxed{\frac{1}{3} + \frac{1}{15} + \frac{1}{33} + \cdots = \frac{1}{2}}$$

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At $x = \pi/2$

$f(x) = 1$

$$1 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos n\pi$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \left(1 - \frac{2}{\pi}\right)\frac{\pi}{4}(-1)$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}\right)(-1)$$

$$= \frac{1}{2} - \frac{\pi}{4}$$

$$\boxed{-\frac{1}{3} + \frac{1}{15} - \frac{1}{33} + \frac{1}{63} - \cdots = \frac{1}{2} - \frac{\pi}{4}}$$

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Subtracting (**) from (*) we get (18)

$$2 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{4n^2-1} = \frac{\pi}{4}.$$

$$\Rightarrow \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{4n^2-1} = \frac{\pi}{8}.$$

$n = 2m+1$.

$$\Rightarrow \sum_{m=0}^{\infty} \frac{1}{4(2m+1)^2-1} = \frac{\pi}{8}$$

$$4(2m+1)^2 = (4m^2+4m+1)4$$
$$= 16m^2+16m+4$$

$$\boxed{\sum_{m=0}^{\infty} \frac{1}{16m^2+16m+3} = \pi/8}$$

3) Solve the following 1^{st} order Equation.

$$x\,dy - y\,dx = xy^2\,dx$$

assume $y(1) = \frac{2}{5}$ ie y at $x = 1$ is $2/5$.

Ans:

$$\underbrace{(xy^2 + y)}_{M}dx - \underbrace{x}_{N}dy = 0 \quad *$$

$M = xy^2 + y \qquad \frac{\partial M}{\partial y} = 2xy + 1$

$N = -x \qquad \frac{\partial N}{\partial x} = -1$

$\boxed{\text{Not exact}}$.

Multiplying * by $F(y)$ we get

$$\underbrace{FM}\,dx + \underbrace{FN}\,dy = 0.$$

$$\frac{\partial}{\partial y}(FM) = F_y M + F(2xy+1).$$

$$\frac{\partial}{\partial x} FN = F\frac{\partial N}{\partial x} = -F.$$

For exactness we want

$$F_y M + F(2xy+1) = -F.$$

$$\frac{F_y}{F} = \frac{-2xy-2}{M} = \frac{-2\cancel{(1+xy)}}{y\cancel{(1+xy)}} = -\frac{2}{y}.$$

$$\int \frac{dF}{F} = \int -\frac{2}{y}\,dy.$$

$$\ln F = -2\ln y + c = \ln\left(\frac{1}{y^2}\right) + c$$

$$F = \frac{1}{y^2} e^c = k \frac{1}{y^2}$$

We choose $k = 1$ & $F(y) = \frac{1}{y^2}$.

(*) is written as

multiplying (*) by $\frac{1}{y^2}$

$$\frac{\cancel{y}(1+xy)}{y^{\cancel{2}}} dx - \frac{x}{y^2} dy = 0$$

which is exact.

Let $d\,u(x,y) = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

we write

$$\frac{\partial u}{\partial x} = \frac{1+xy}{y}, \quad \frac{\partial u}{\partial y} = -\frac{x}{y^2}$$

$$= \frac{1}{y} + x,$$

$$\frac{\partial u}{\partial x} = \frac{1}{y} + x$$

$$u(x,y) = \frac{x}{y} + \frac{x^2}{2} + h(y).$$

$$\frac{\partial u}{\partial y} = x\frac{(-1)}{y^2} + h'(y) = -\frac{x}{y^2}$$

$$\Rightarrow h'(y) = 0$$

$$\Rightarrow h(y) = \text{const}.$$

$$\therefore u(x,y) = \frac{x}{y} + \frac{x^2}{2} + c$$

(*) is written as.

$$du = 0$$

$$\Rightarrow \frac{x}{y} + \frac{x^2}{2} = \text{const} = c_1$$

$$\frac{1}{y} + \frac{x}{2} = \frac{c_1}{x}.$$

$$\Rightarrow \frac{1}{y} = \frac{c_1}{x} - \frac{x}{2} = \frac{2c_1 - x^2}{2x}$$

$$\Rightarrow \boxed{y = \frac{2x}{2c_1 - x^2}}.$$

$x = 1$ $\quad y(1) = \dfrac{2}{2c_1 - 1} = \dfrac{2}{5}$ (say)

$$c_1 = 3$$

$$y(1) = \frac{2}{5}$$

$$y = \frac{2x}{6 - x^2}$$