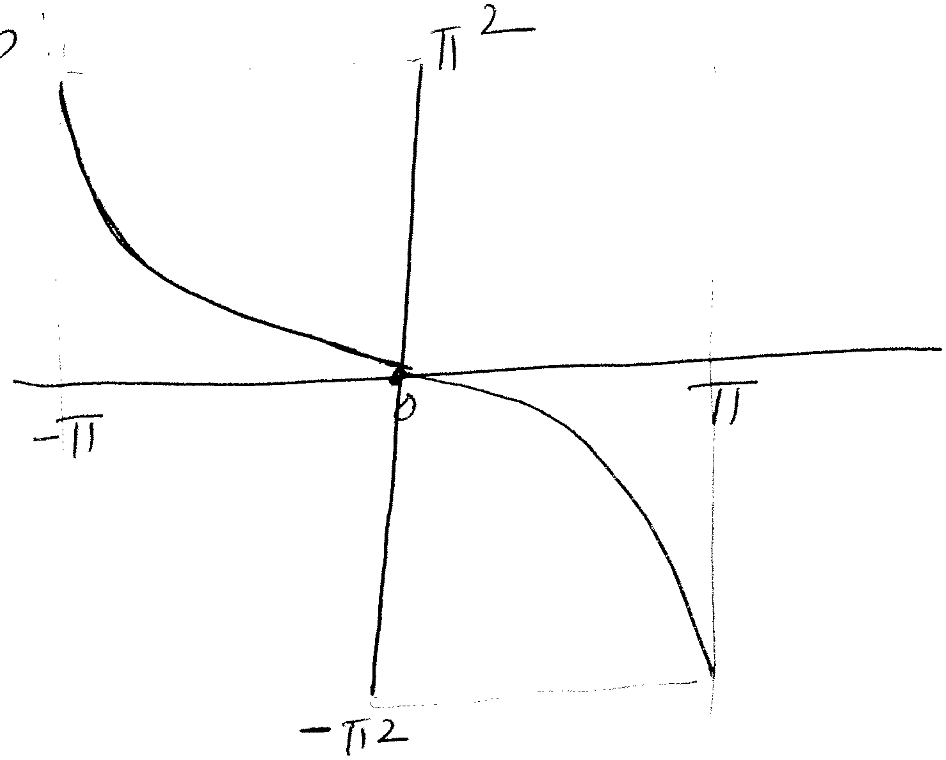
Math 3350

Home Work 7

Answers.

1 Ans:



The function $f(x)$ is an odd fn

Hence

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

$$b_n = -\frac{2}{\pi}\int_0^{\pi} x^2 \sin nx \, dx.$$

I $\int x^2 \sin nx \, dx$

$$= -x^2 \frac{\cos nx}{n} + \int 2x \frac{\cos nx}{n} dx$$

$$= -\frac{1}{n} x^2 \cos nx + \frac{2}{n}\int x \cos nx \, dx.$$

II $\int x \cos nx \, dx =$

$$x \frac{\sin nx}{n} - \int \frac{\sin nx}{n} dx$$

$$= \frac{1}{n}\left[x \sin nx\right] - \frac{1}{n}\int \sin nx \, dx.$$

III $\int \sin nx \, dx$

$$= -\frac{\cos nx}{n}$$

II & III $\Rightarrow$

$\int x \cos nx \, dx =$

$$\frac{1}{n}[x \sin nx] + \frac{1}{n^2} \cos nx$$

I, II, III $\Rightarrow$

$\int x^2 \sin nx \, dx =$

$$-\frac{1}{n} x^2 \cos nx + \frac{2}{n^2} x \sin nx$$

$$+ \frac{2}{n^3} \cos nx$$

④

$$\int_0^{\pi} x^2 \sin nx \, dx =$$

$$\left[-\frac{1}{n}\pi^2\cos n\pi + \frac{2}{n^2}\pi \underbrace{\sin n\pi}_{0} + \frac{2}{n^3}\cos n\pi\right.$$

$$\left.-\left[0+0+\frac{2}{n^3}\right]\right]$$

$$= -\frac{1}{n}\pi^2(-1)^n + \frac{2}{n^3}(-1)^n - \frac{2}{n^3}$$

$$\therefore\ b_n = -\frac{2}{\pi}\left[-\frac{\pi^2}{n}(-1)^n - \frac{2}{n^3} + \frac{2}{n^3}(-1)^n\right]$$

$$= \frac{2\pi}{n}(-1)^n + \frac{4}{\pi n^3}\left[1-(-1)^n\right]$$

2) Ans:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$p = 2L$ p is the period.

In our problem $p = \pi \Rightarrow L = \pi/2$.

$$\therefore f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2nx + \sum_{n=1}^{\infty} b_n \sin 2nx$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_0 = \frac{1}{2}$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi/2} dx = \frac{1}{\pi}\,\pi/2 = \frac{1}{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos 2nx\,dx$$

$$= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\cos 2nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi/2}\cos 2nx\,dx$$

$$a_n = \frac{\cancel{2}}{\pi} \frac{\sin 2nx}{\cancel{2}n} \Big|_0^{\pi/2}$$

$$= \frac{1}{\pi n} \left[\sin\left(\cancel{2}n \frac{\pi}{\cancel{2}}\right) - 0 \right] = 0$$

—— x ——

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin 2nx \, dx$$

$$= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(x) \sin 2nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \sin 2nx \, dx$$

$$= -\frac{2}{\pi} \frac{\cos 2nx}{2n} \Big|_0^{\pi/2}$$

$$b_n = -\frac{1}{\pi n}\left[\cos\left(2n\frac{\pi}{2}\right) - 1\right]$$

$$= \frac{1}{\pi n}\left[1 - (-1)^n\right]$$

$b_n = 0$ $\quad n$ even

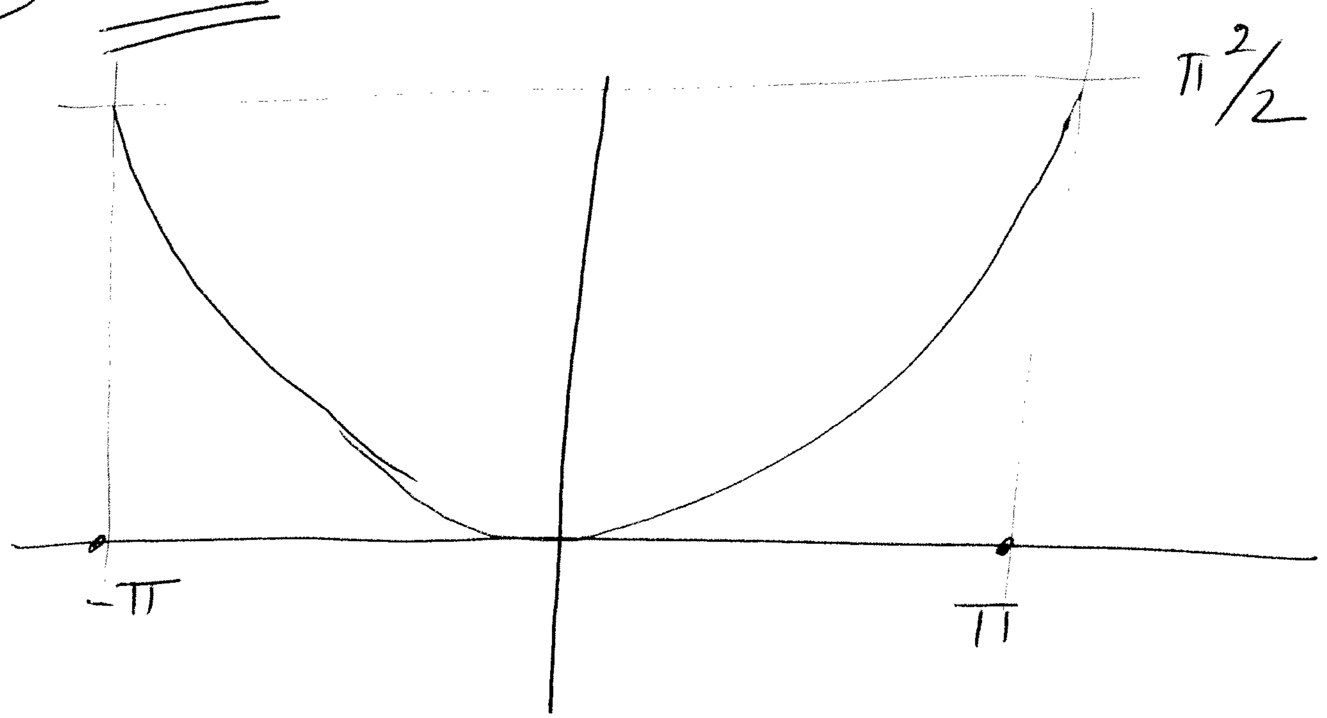
$= \dfrac{2}{\pi n}$ $\quad n$ odd

$n = 2m+1$

$$b_{2m+1} = \frac{2}{\pi(2m+1)} \qquad m = 0, 1, 2, \ldots\ldots$$

$$\therefore f(x) = \frac{1}{2} + \sum_{m=0}^{\infty} \frac{2}{\pi(2m+1)} \sin\left[2(2m+1)x\right]$$

3 Ans:



f(x) is an even $f^{\underline{n}}$ of period 2π.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\cancel{\cos nx}\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} f(x)\,\cancel{\cos nx}\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \frac{x^2}{2}\,dx = \frac{1}{2\pi}\,\frac{x^3}{3}\Big|_0^{\pi} = \frac{\pi^3}{6\pi} = \pi^2/6$$

$$\boxed{a_0 = \frac{\pi^2}{6}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx.$$

$$= \frac{\cancel{2}}{\pi}\int_{0}^{\pi} \frac{x^2}{\cancel{4}}\cos nx\,dx.$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx.$$

$$\int_0^{\pi} x^2 \cos nx \, dx$$

$$= x^2 \frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi} 2x \frac{\sin nx}{n} \, dx$$

$$= \frac{1}{n} x^2 \sin nx \Big|_0^{\pi} - \frac{2}{n} \int_0^{\pi} x \sin nx \, dx$$

$$= -\frac{2}{n} \int_0^{\pi} x \sin nx \, dx$$

$$a_n = -\frac{2}{n\pi} \int_0^{\pi} x \sin nx \, dx$$

$$\int_0^{\pi} x \sin nx \, dx =$$

$$- x \frac{\cos nx}{n}\Big|_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} \, dx$$

$$= -\frac{1}{n} x\cos nx + \frac{1}{n}\frac{\sin nx}{n}\Bigg|_0^{\pi}$$

$$= -\frac{1}{n}\pi\cos n\pi$$

$$a_n = \left(-\frac{2}{n\pi}\right)\left(-\frac{\pi}{n}\right)(-1)^n$$

$$= \frac{2\cancel{\pi}}{n^2\cancel{\pi}}(-1)^n$$

$$= \frac{2}{n^2}(-1)^n$$

$\therefore$

$f(x) =$

$$\frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n \cos nx .$$

$\circledast$

Evaluting $\circledast$ at $x = 0$ we have

$$0 = f(0) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n$$

$$= \frac{\pi^2}{6} + 2\left[(-1) + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

$$\Rightarrow \boxed{1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} = \frac{\pi^2}{12}}$$

Evaluating (*) at $x = \pi$ we have

$$\frac{\pi^2}{2} = f(\pi) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2} (-1)^n \cos n\pi$$

$$\because (-1)^n \cos n\pi = 1 \quad \forall n$$

we have

$$\sum_{n=1}^{\infty} \frac{2}{n^2} = \frac{\pi^2}{2} - \frac{\pi^2}{6} = \frac{\cancel{2}\pi^2}{\cancel{6}\,3}$$

$$= \frac{\pi^2}{3}.$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.}$$

← This is the well known p-series for $p = 2$.

(4)

$$f(x) = k \quad -\pi/2 < x \leq \pi/2$$

$$0 \quad \left\{ \begin{array}{l} \pi > x > \pi/2 \\ -\pi < x < -\pi/2 \end{array} \right.$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi/2} k\, dx = \frac{k}{\cancel{\pi}} \frac{\cancel{\pi}}{2} = \frac{k}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx.$$

$$= \frac{2}{\pi}\int_{0}^{\pi/2} k\cos nx\, dx.$$

$$= \frac{2k}{\pi}\int_{0}^{\pi/2} \cos nx\, dx$$

$$= \frac{2k}{\pi} \left. \frac{\sin nx}{n} \right|_{0}^{\pi/2}$$

$$= \frac{2k}{n\pi} \sin \frac{n\pi}{2}$$

$a_n = 0$ n even.

$a_n = \frac{2k}{n\pi}$ $n = 1, 5, 9, \ldots$

$\quad\;\; -\frac{2k}{n\pi}$ $n = 3, 7, 11, \ldots$

writing

$n = 2m+1$

$$a_{2m+1} = \frac{2k}{(2m+1)\pi}(-1)^m$$

$$\boxed{f(x) = \frac{k}{2} + \sum_{m=0}^{\infty} \frac{2k}{(2m+1)\pi}(-1)^m \cos(2m+1)x}$$

Note that at $x = \pi$

$f(\pi) = 0$

and

$\cos[(2m+1)\pi] = -1 \quad \forall m = 0, 1, \ldots$

It follows that

$$\sum_{m=0}^{\infty} \frac{2k}{(2m+1)\pi}(-1)^m = \frac{k}{2}$$

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$$\Rightarrow \sum_{m=0}^{\infty} (-1)^m \frac{1}{2m+1} = \frac{\pi}{4}.$$

$$\Rightarrow \boxed{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots\cdots = \frac{\pi}{4}.}$$