H. W. 2 (solutions)

1

(i) $M = x + xy^2$

$N = e^{x^2} y$

$\frac{\partial M}{\partial y} = x\,2y = 2xy$

$\Rightarrow$ Not exact.

$\frac{\partial N}{\partial x} = 2xe^{x^2} y$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{N} = \frac{2x\cancel{y}(e^{x^2} - 1)}{e^{x^2}\cancel{y}} = 2x(1 - e^{-x^2})$$

$\therefore \frac{\partial M/\partial y - \partial N/\partial x}{N}$ is a function of x

$$\mu(x) = e^{\int (e^{-x^2} - 1)\,2x\,dx}$$

$$= e^{\left[-e^{-x^2} - x^2\right]}$$

$\mu M\,dx + \mu N\,dy$ is exact where

$$\mu M = x(1+y^2)\, e^{-[x^2 + e^{-x^2}]}$$

$$\mu N = y\, \cancel{e^{x^2}}\, e^{-e^{-x^2}}\, \cancel{e^{-x^2}}$$

$$= y\, e^{-e^{-x^2}}$$

To find $F(x,y)$ we proceed as follows:

$$\frac{\partial F}{\partial y} = \mu N$$

$$\Rightarrow F = e^{-e^{-x^2}} \frac{y^2}{2} + g(x).$$

$$\frac{\partial F}{\partial x} = \frac{1}{\cancel{2}} y^2\, e^{-e^{-x^2}} (+1)\, e^{-x^2} (+\cancel{2}x) + g'(x)$$

$$= x y^2 e^{-x^2} e^{-e^{-x^2}} + g'(x) = \mu M$$

$$= x\, e^{-[x^2 + e^{-x^2}]} + x y^2 e^{-x^2} e^{-e^{-x^2}}$$

$$g'(x) = x e^{-[x^2 + e^{-x^2}]}$$

$$\cancel{\frac{d}{dx} e^{-[x^2+e^{-x^2}]} = e^{-[x^2+e^{-x^2}]}(-1)\left[2x + e^{-x^2}(-2x)\right]}$$

$$\cancel{\frac{d}{dx} e^{-x^2} = e^{-x^2}(-1)(2x) = -2x e^{-x^2}}$$

$$\cancel{\frac{d}{dx}\left(-\frac{1}{2} e^{-x^2}\right) = x e^{-x^2}}$$

$$\frac{d}{dx} e^{-e^{-x^2}} = e^{-e^{-x^2}} (-1) e^{-x^2} (-2x)$$

$$= 2x e^{-x^2} e^{-e^{-x^2}}$$

$$\frac{d}{dx}\left(\frac{1}{2} e^{-e^{-x^2}}\right) = x e^{-[x^2+e^{-x^2}]}$$

(4)

$$g(x) = \frac{1}{2} e^{-e^{-x^2}} + C$$

$$\therefore F = \frac{1}{2} y^2 e^{-e^{-x^2}} + \frac{1}{2} e^{-e^{-x^2}} + C$$

$$= \frac{1}{2} e^{-e^{-x^2}} \left[y^2 + 1 \right] + C.$$

Sol^{n}

$$\frac{1}{2} e^{-e^{-x^2}} (1+y^2) = \text{const.}$$

$$\Rightarrow e^{e^{-x^2}} = (1+y^2)\,\text{const.}$$

$$\Rightarrow e^{-x^2} = \ln(1+y^2) + C_1$$

a constant.

(ii)

$$\frac{dx}{dt} = x - x^2$$

$$\Rightarrow \int \frac{dx}{x - x^2} = \int dt$$

I/D

Note that

$$\frac{1}{x - x^2} = \frac{1}{x} + \frac{1}{1-x}$$

$$\therefore \int \frac{1}{x-x^2}\,dx = \ln|x| + \ln|1-x| + C$$

$$= \ln\left|\frac{x}{1-x}\right| + C$$

$$\therefore \ln\left|\frac{x}{1-x}\right| = t + C$$

$$\Rightarrow \left|\frac{x}{1-x}\right| = e^C e^t \Rightarrow \frac{x}{1-x} = \pm K e^t$$

K is a constant.

Solving for x we obtain

$$x(1 \pm ke^t) = \pm ke^t$$

$$\Rightarrow x = \frac{\pm ke^t}{1 \pm ke^t}$$

$$x(t) = \frac{ke^t}{1+ke^t} \quad \text{or} \quad \frac{-ke^t}{1-ke^t}.$$

1 (iii)

$$\sqrt{y}\,dx = -(1+x)\,dy$$

$$\Rightarrow \int \frac{dy}{\sqrt{y}} = -\int \frac{dx}{1+x}.$$

$$\Rightarrow 2\sqrt{y} = -\ln|1+x| + c.$$

$$\Rightarrow \ln|1+x| = -2\sqrt{y} + c.$$

$$\Rightarrow |1+x| = e^{-2\sqrt{y}}\, e^{c}$$

$$\Rightarrow 1+x = \pm K e^{-2\sqrt{y}}.$$

$$\Rightarrow x = -1 \pm K e^{-2\sqrt{y}}$$

$\because$ $y(0)=1$ we have

$0 = -1 \pm K e^{-2}$ $\Rightarrow K = e^{2}$ and $\pm$ is replaced by $+$.

$$\therefore x = -1 + e^{2(1-\sqrt{y})}$$

(2)

(i) $y\frac{dx}{dy} + 2x = 5y^3$

$$\Rightarrow \frac{dx}{dy} + \frac{2}{y}x = 5y^2.$$

$\leftarrow$ Linear Eqn in x

$P(y) = \frac{2}{y} \qquad Q(y) = 5y^2.$

$$\mu(y) = e^{\int \frac{2}{y} dy} = e^{2\ln|y|} c$$

$$= y^2 c$$

$$\mu(y) = y^2$$

$$x(y) = \frac{1}{y^2}\left[\int y^2 5y^2 \, dy + c\right]$$

$$= \frac{1}{y^2}\left[\int 5y^4 \, dy + c\right]$$

$$= \frac{1}{y^2}\left[y^5 + c\right] = y^3 + cy^{-2}$$

$x = y^3 + c/y^2$ $\leftarrow$ Soln.

(2)(ii) $\frac{dy}{dx}(1+x^2) = x(1-y)$.

9

$$\int \frac{dy}{1-y} = \int \frac{x}{1+x^2}\,dx$$

$$\int \frac{x}{1+x^2}\,dx \qquad \begin{aligned} u &= 1+x^2 \\ du &= 2x\,dx \\ \tfrac{1}{2}du &= x\,dx \end{aligned}$$

$$= \int \frac{1}{2}\frac{du}{u}$$

$$= \frac{1}{2}\ln|u| = \frac{1}{2}\ln(1+x^2) = \ln(1+x^2)^{1/2}$$

$$\int \frac{dy}{1-y} = -\ln|(1-y)|$$

$$\therefore \ln|(1-y)| = -\ln(1+x^2)^{1/2} + C.$$

$$= \ln\left[(1+x^2)^{-1/2}\right] + C$$

$$(1-y) = \pm e^{C}(1+x^2)^{-1/2} = \frac{K}{\sqrt{1+x^2}}$$

$$y = 1 - \frac{K}{\sqrt{1+x^2}}$$

K is any constant.

2 iii

$$\frac{dy}{dx} + 4y = x^2 e^{-4x}$$

$$P = 4 \quad Q = x^2 e^{-4x}.$$

$$\mu = e^{4x}$$

$$\mu Q = x^2.$$

$$\therefore y(x) = e^{-4x}\left[\int x^2 dx + C\right]$$

$$= e^{-4x}\left[\frac{x^3}{3} + C\right]$$

$$y(x) = \frac{1}{3}x^3 e^{-4x} + C e^{-4x}.$$

3 Ans

$$\frac{dy}{dx} + 2y = \frac{x}{y^2}$$

$$V = y^3$$

$$\frac{dV}{dx} = 3y^2 \frac{dy}{dx} = 3y^2\left[-2y + \frac{x}{y^2}\right]$$

$$= -6V + 3x$$

$$\frac{dV}{dx} + 6V = 3x$$

Linear

$P = 6 \qquad Q = 3x$

$$\mu(x) = e^{6x}$$

$$\mu Q = 3xe^{6x}$$

$$V(x) = e^{-6x}\left[\int 3xe^{6x}\,dx + C\right]$$

$$\int x e^{6x}\,dx$$

$$= \frac{x e^{6x}}{6} - \int \frac{e^{6x}}{6}\,dx$$

$$= \frac{1}{6} x e^{6x} - \frac{1}{36} e^{6x}$$

$$\therefore V(x) = e^{-6x}\left[\frac{1}{2} x e^{6x} - \frac{1}{12} e^{6x} + c\right]$$

$$= \frac{1}{2} x - \frac{1}{12} + c e^{-6x}$$

$$y^3 = \frac{1}{2} x - \frac{1}{12} + c e^{-6x}$$

Sol^{n}

4 Ans:

a $M = 5x^2y + 6x^3y^2 + 4xy^2$.

$N = 2x^3 + 3x^4y + 3x^2y$.

$$\frac{\partial M}{\partial y} = 5x^2 + 6x^3 2y + 4x\,2y.$$

$$= 5x^2 + 12x^3y + 8xy$$

$$\frac{\partial N}{\partial x} = 6x^2 + 3y\,4x^3 + 3y\,2x.$$

$$= 6x^2 + 12x^3y + 6xy.$$

Not equal.

Hence not exact

b The new M & N are as follows.

$$M = 5x^{n+2}y^{m+1} + 6x^{n+3}y^{m+2} + 4x^{n+1}y^{m+2}.$$

$$\frac{\partial M}{\partial y} = 5x^{n+2}(m+1)y^{m} + 6x^{n+3}(m+2)y^{m+1} + 4x^{n+1}(m+2)y^{m+1}.$$

$$N = (2x^3 + 3x^4 y + 3x^2 y) . x^n y^m$$

$$= 2x^{n+3} y^m + 3x^{n+4} y^{m+1} + 3x^{n+2} y^{m+1}$$

$$\frac{\partial N}{\partial x} = 2y^m (n+3) x^{n+2} + 3y^{m+1} (n+4) x^{n+3}$$
$$+ 3y^{m+1} (n+2) x^{n+1}.$$

——— × ———

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$$

$$5(m+1) = 2(n+3)$$
$$6(m+2) = 3(n+4) \quad \Rightarrow m = 1$$
$$n = 2$$
$$4(m+2) = 3(n+2)$$

——— × ———

$$M = 5x^4 y^2 + 6x^5 y^3 + 4x^3 y^3$$
$$N = 2x^5 y + 3x^6 y^2 + 3x^4 y^2.$$

$$\frac{\partial F}{\partial x} = M \qquad \frac{\partial F}{\partial y} = N.$$

$$F = \cancel{5} y^2 \frac{x^5}{\cancel{5}} + \cancel{6} y^3 \frac{x^6}{\cancel{6}} + \cancel{4} y^3 \frac{x^4}{\cancel{4}} + g(y)$$

$$= x^5 y^2 + x^6 y^3 + x^4 y^3 + g(y).$$

$$\frac{\partial F}{\partial y} = x^5 2y + x^6 3y^2 + x^4 3y^2 + g'(y)$$

$$= N.$$

$$\Rightarrow g'(y) = 0.$$

$$\Rightarrow g(y) = C$$

$$\therefore F = x^5 y^2 + x^6 y^3 + x^4 y^3 + C$$

Sol$^{\underline{n}}$: $x^5 y^2 + x^6 y^3 + x^4 y^3 = \text{const}$

5 Ans:

$$xM + yN \equiv 0$$

$$\Rightarrow xM = -yN$$

$$\Rightarrow \frac{M}{N} = -\frac{y}{x}$$

$$M\,dx + N\,dy = 0$$

$$\Rightarrow \frac{M}{N}dx + 1\,dy = 0$$

$$\Rightarrow -\frac{y}{x}dx + dy = 0$$

$$\Rightarrow -y\,dx + x\,dy = 0$$

$$\Rightarrow y\,dx = x\,dy$$

$$\Rightarrow \int\frac{dx}{x} = \int\frac{dy}{y} \Rightarrow \ln x = \ln y + C$$

$$\Rightarrow \boxed{x = Ky}$$