Math 3350

Home Work 1 (answers)

1 Ans:

$$v(t) = v_h(t) + v_p(t)$$

A Let us find the homogeneous solution $v_h(t)$

$$20 \frac{dv}{dt} + v = 0$$

$$\Rightarrow \int \frac{dv}{v} = -\int \frac{dt}{20}$$

$$\Rightarrow \ln v \Big|_{v_0}^{v} = -\frac{t}{20}\Big|_0^t$$

$$\Rightarrow \boxed{v = v_0 e^{-t/20}}$$ ← homogeneous sol^n

$$v_h(t) = v_0 e^{-t/20}$$

(B) Find particular solution for $f(t)=0$.

(2)

By inspection

$v_p(t) = 0$ would suffice

Hence

$$v(t) = v_0 e^{-t/20}$$

$\because v(0) = 100$ we have $v_0 = 100$

$$\therefore \boxed{v(t) = 100\, e^{-t/20}}$$

(C) Find particular sol^n for $f(t) = 10$

By inspection

$v_p(t) = 10$ would suffice

Hence

$$v(t) = 10 + v_0 e^{-t/20}$$

$100 = v(0) = 10 + v_0 \Rightarrow v_0 = 90$.

$$\therefore \boxed{v(t) = 10 + 90\, e^{-t/20}}$$

(D) Find particular solution for

$f(t) = 5\sin 30t$

Look at pages (7) & (8) for alternative solution

Let us guess that

$v_p(t) = A\sin 30t + B\cos 30t$. (*)

Substitute (*) in

$$20\frac{dv_p}{dt} + v_p = 5\sin 30t .$$

to obtain :→

$$\frac{dv_p}{dt} = 30A\cos 30t - 30B\sin 30t .$$

$$\therefore 20\frac{dv_p}{dt} = 600A\cos 30t - 600B\sin 30t .$$

Hence we obtain

$$(600A + B)\cos 30t + (A - 600B)\sin 30t = 5\sin 30t .$$

It follows that

$$B + 600A = 0 \Rightarrow B = -600A.$$

$$A - 600B = 5.$$

$$\Rightarrow A + (600)^2 A = 5.$$

$$\Rightarrow A = \frac{5}{1+(600)^2}$$

$$B = \frac{-600 \times 5}{1+(600)^2}.$$

$$\therefore v_p(t) = \frac{5}{1+(600)^2} \sin 30t$$

$$- \frac{3000}{1+(600)^2} \cos 30t.$$

$$= \frac{5}{\sqrt{1+(600)^2}} \left[\frac{1}{\sqrt{1+(600)^2}} \sin 30t - \frac{600}{\sqrt{1+(600)^2}} \cos 30t \right]$$

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Define an angle α:

$$\cos\alpha = \frac{1}{\sqrt{1+(600)^2}}$$

$$\sin\alpha = \frac{600}{\sqrt{1+(600)^2}}$$

we obtain.

$$v_p(t) = \frac{5}{\sqrt{1+(600)^2}} \sin(30t - \alpha)$$

It follows that

$$v(t) = v_p(t) + r\, v_h(t)$$

$$= \frac{5}{\sqrt{1+(600)^2}} \sin(30t-\alpha) + r e^{-t/20}$$

where r is arbitrary constant.

To obtain r we have

$$100 = v(0) = -\frac{5\sin\alpha}{\sqrt{1+(600)^2}} + r$$

$$= r - \frac{5 \cdot 600}{1+(600)^2}$$

$$\therefore r = 100 + \frac{5 \cdot 600}{1+(600)^2}$$

$$\approx 100 + \frac{5}{600} =$$

$$\approx 100 \cdot 01$$

Hence

$$v(t) \approx \frac{5}{600}\sin(30t-\alpha) + 100 \cdot 01\, e^{-t/20}$$

Oscillatory part

exponentially decaying part.

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Home Work 1 (answers)

① Ans:

$$20\frac{dv}{dt} + v = f(t)$$

$$\frac{dv}{dt} + \frac{1}{20}v(t) = \frac{1}{20}f(t)$$

This is a linear 1st order Eqn.

$$P(t) = \frac{1}{20}$$

$$Q(t) = \frac{1}{20}f(t)$$

$$\mu(t) = e^{\int \frac{1}{20}dt} = e^{t/20}$$

ⓒ $f(t) = 5\sin 30t$

$$y(t) = e^{-t/20}\left[\int e^{t/20}\frac{1}{\cancel{20}_{4}}\cancel{5}\sin 30t\,dt + C\right]$$

$$= e^{-t/20}\left[\frac{1}{4}\int e^{t/20}\sin 30t\,dt + C\right]$$

$$\int e^{t/20} \sin 30t \, dt$$

$$= \frac{e^{t/20}\left[\frac{1}{20}\sin 30t - 30\cos 30t\right]}{\frac{1}{400} + 900}$$

From the integration tables.

Thus

$$y(t) = c\, e^{-t/20} +$$

$$\frac{1}{4} \quad \frac{1}{900 + \frac{1}{400}} \left[\frac{1}{20}\sin 30t - 30\cos 30t\right]$$

2) Ans:

$$x(t) = x_h(t) + x_p(t).$$

A) Let us find the homogeneous sol^n. $x_h(t)$.

$$10\frac{d^2x}{dt^2} + 20x = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + 2x = 0 \quad (*)$$

Guess $x_h(t) = A\sin\sqrt{2}\,t + B\cos\sqrt{2}\,t$.

To verify we compute

$$\frac{d}{dt}x_h(t) = \sqrt{2}A\cos\sqrt{2}\,t - \sqrt{2}B\sin\sqrt{2}\,t.$$

$$\frac{d^2x_h(t)}{dt^2} = -2A\sin\sqrt{2}\,t - 2B\cos\sqrt{2}\,t.$$

$$= -2\left[A\sin\sqrt{2}\,t + B\cos\sqrt{2}\,t\right]$$

$$= -2x_h(t).$$

Hence $\dfrac{d^2x_h(t)}{dt^2} + 2x_h(t) = 0$.

$x_h(t) = A \sin\sqrt{2}\,t + B\cos\sqrt{2}\,t.$ 10 ~~8~~

B Find particular sol^n for $f(t)=0$

$x_p(t) = 0$ (By inspection).

Hence

$x(t) = A\sin\sqrt{2}\,t + B\cos\sqrt{2}\,t.$

$5 = x(0) = B \Rightarrow B = 5.$

$\Rightarrow x(t) = A\sin\sqrt{2}\,t + 5\cos\sqrt{2}\,t.$

$\Rightarrow \dfrac{dx(t)}{dt} = \sqrt{2}A\cos\sqrt{2}\,t - \sqrt{2}.5\sin\sqrt{2}\,t.$

$\Rightarrow \dfrac{dx(0)}{dt} = \sqrt{2}A = 15 \Rightarrow A = 15/\sqrt{2}.$

$\therefore \boxed{x(t) = \dfrac{15}{\sqrt{2}}\sin\sqrt{2}\,t + 5\cos\sqrt{2}\,t}$

(c) Find particular solution for $f(t)=10$

$x_p(t) = 1/2$ (By inspection)

Hence

$$x(t) = \frac{1}{2} + A\sin\sqrt{2}t + B\cos\sqrt{2}t.$$

$$5 = x(0) = \frac{1}{2} + B \Rightarrow B = 4\tfrac{1}{2} = 9/2.$$

$$\therefore x(t) = \frac{1}{2} + A\sin\sqrt{2}t + \frac{9}{2}\cos\sqrt{2}t.$$

$$\frac{dx(t)}{dt} = 0 + \sqrt{2}A\cos\sqrt{2}t - \frac{9}{2}\sqrt{2}\sin\sqrt{2}t.$$

$$\frac{dx(0)}{dt} = \sqrt{2}A = 15 \Rightarrow A = \frac{15}{\sqrt{2}}.$$

$$\boxed{\therefore x(t) = \frac{1}{2} + \frac{15}{\sqrt{2}}\sin\sqrt{2}t + \frac{9}{2}\cos\sqrt{2}t}$$

(D) Find a particular solution $x_p(t)$ to the equation

$$10 \frac{d^2x}{dt^2} + 20x = \sin 30t$$

guess

$$x_p(t) = \quad \sin 30t + \quad \cos 30t$$

$$x_p'(t) = 30 \quad \cos 30t - 30 \quad \sin 30t$$

$$x_p''(t) = -900 \quad \sin 30t - 900 \quad \cos 30t$$

$$10 x_p'' + 20 x_p =$$

$$-9000 \quad \sin 30t - 9000 \quad \cos 30t$$

$$+20 \quad \sin 30t + 20 \quad \cos 30t$$

$$= \sin 30t$$

$$\Rightarrow \quad (20 - 9000) \quad = 1 \quad \Rightarrow \quad = 0$$

$$(20 - 9000) \quad = 0 \quad \quad = \frac{-1}{8980}$$

$$\therefore \quad x_p(t) = -\frac{\sin 30t}{8980}$$

Hence

$x(t) =$

$$A \sin \sqrt{2}\, t + B \cos \sqrt{2}\, t - \frac{\sin 30t}{8980}.$$

To find A, B we use the initial conditions.

$$5 = x(0) = B \quad \Rightarrow B = 5$$

$$15 = x'(0) = \sqrt{2} A - \frac{30}{8980}$$

$$\Rightarrow \sqrt{2} A = 15 + \frac{3}{898}$$

$$A = \frac{1}{\sqrt{2}} \left[15 + \frac{3}{898} \right]$$

$\therefore x(t) =$

$$\frac{1}{\sqrt{2}} \left[15 + \frac{3}{898} \right] \sin \sqrt{2}\, t + 5 \cos \sqrt{2}\, t - \frac{\sin 30t}{8980}.$$