Final Exam.
Math 3350
Solutions.

① Ans:

$f(x)$ is an odd $f^{\underline{n}}$ of period $2\pi$.
we have only the sine terms.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$\int_0^{\pi} x\sin nx\,dx = -\frac{1}{n}x\cos nx\Big|_0^{\pi} + \int_0^{\pi}\frac{1}{n}\cos nx\,dx$$

$$= -\frac{1}{n}\pi\cos n\pi + \overset{0}{\cancel{\frac{1}{n^2}\sin nx\Big|_0^{\pi}}}$$

$$= -\frac{1}{n}\pi(-1)^n = (-1)^{n+1}\frac{\pi}{n}$$

Thus

$$f(x) = \pi \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

$f(\pi/2) = \pi/2$ and we have

$$\frac{\cancel{\pi}}{2} = \cancel{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \frac{1}{2} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots\cdots$$

② Ans:

We will use Laplace's Transform:

$$\mathcal{L}(y) = Y(s).$$

$$\mathcal{L}(\dot{y}) = sY(s) - y(0) = sY(s)$$

$$\because y(0) = 0$$

$$\mathcal{L}(\ddot{y}) = s\mathcal{L}(\dot{y}) - \dot{y}(0)$$

$$= s^2 Y(s) - \underline{1} \quad \because \dot{y}(0) = 1.$$

$$\therefore \mathcal{L}(\ddot{y}) + 2\mathcal{L}(y) = \mathcal{L}(\sin 5t)$$

$$= \frac{5}{s^2+25}$$

$$(s^2+2)Y(s) = 1 + \frac{5}{s^2+25} = \frac{s^2+30}{s^2+25}$$

It follows that

$$Y(s) = \frac{s^2+30}{(s^2+2)(s^2+25)}.$$

We now use partial fraction:

$$\frac{s^2+30}{(s^2+2)(s^2+25)} = \frac{As+B}{s^2+2} + \frac{Cs+D}{s^2+25}.$$

$$= \frac{As^3+25As+Bs^2+25B + Cs^3+2Cs+Ds^2+2D}{(s^2+2)(s^2+25)}.$$

$$= \frac{(A+C)s^3+(B+D)s^2+(25A+2C)s + (25B+2D)}{(s^2+2)(s^2+25)}$$

$$A + C = 0 \Rightarrow A = -C$$

$$B + D = 1 \Rightarrow B = 1 - D$$

$$\Rightarrow A = C = 0$$

$$25A + 2C = 0 \Rightarrow C = -\frac{25}{2}A$$

$$25B + 2D = 30$$

we have

$$25(1-D) + 2D = 30$$

$$\Rightarrow 25 - 23D = 30$$

$$\Rightarrow -23D = 5 \Rightarrow D = -5/23.$$

$$B = 1 + 5/23 = \frac{28}{23}$$

$$\therefore \frac{s^2+30}{(s^2+2)(s^2+25)} = \frac{28}{23}\,\frac{1}{s^2+2} - \frac{5}{23}\,\frac{1}{s^2+25}$$

$$= \frac{28}{23\sqrt{2}}\,\frac{\sqrt{2}}{s^2+(\sqrt{2})^2} - \frac{1}{23}\,\frac{5}{s^2+5^2}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{s^2+30}{(s^2+2)(s^2+25)}\right)$$

$$= \frac{28}{23\sqrt{2}}\sin(\sqrt{2}\,t) - \frac{1}{23}\sin 5t .$$

③ Ans:—

$$P(x,y) = \sin y - x^2 e^{-x}.$$

$$Q(x,y) = \cos y - y^2 e^{-x}$$

$$\frac{\partial P}{\partial y} = \cos y \; ; \; \frac{\partial Q}{\partial x} = +y^2 e^{-x}$$

$\Longrightarrow$ Not exact.

To make things exact try multiplying by a $f^n$ of $x$, say $\phi(x)$.

$$\phi(x)\left[\sin y - x^2 e^{-x}\right] dx + \phi(x)\left[\cos y - y^2 e^{-x}\right] dy = 0$$

$$P(x) = \phi(x)\left[\sin y - x^2 e^{-x}\right]$$

$$Q(x) = \phi(x)\left[\cos y - y^2 e^{-x}\right]$$

$$\frac{\partial P}{\partial y} = \phi(x)\cos y$$

$$\frac{\partial Q}{\partial x} = \phi'(x)\left[\cos y - y^2 e^{-x}\right] + \phi(x)\left[+y^2 e^{-x}\right]$$

For exactness we need

$$\phi'(x)\left[\cos y - y^2 e^{-x}\right] + \phi(x)\, y^2 e^{-x} = \phi(x)\cos y$$

$$\Rightarrow \frac{\phi'(x)}{\phi(x)} = \frac{\cos y - y^2 e^{-x}}{\cos y - y^2 e^{-x}} = 1$$

$$\Rightarrow \boxed{\phi(x) = e^{x}}$$

We have

$$e^{x}\left[\sin y - x^2 e^{-x}\right]dx + e^{x}\left[\cos y - y^2 e^{-x}\right]dy = 0$$

$$\Rightarrow \left[e^{x}\sin y - x^2\right]dx + \left[e^{x}\cos y - y^2\right]dy = 0.$$

We want a $f^n$ $F(x,y)$:

$$\frac{\partial F}{\partial x} = e^{x}\sin y - x^2$$

$$\frac{\partial F}{\partial y} = e^{x}\cos y - y^2$$

$$F = e^{x}\sin y - \frac{x^3}{3} + h(y)$$

$$\frac{\partial F}{\partial y} = e^{x}\cos y + h'(y) = e^{x}\cos y - y^2$$

$$h'(y) = -y^2$$

$$h(y) = -\frac{y^3}{3} + c$$

$$\therefore F(x,y) = e^x \sin y - \left(\frac{x^3}{3} + \frac{y^3}{3}\right) + c$$

choose $c = 0$

we have

$$d\left[e^x \sin y - \frac{1}{3}(x^3 + y^3)\right] = 0$$

$$\Rightarrow \boxed{e^x \sin y - \frac{1}{3}(x^3 + y^3) = c \quad \text{a constant.}}$$

Required solution