Final Exam
(Make Up)
Math 3350
Solutions

① Ans:

$$\frac{dy}{dt} + P(t)y = Q(t)$$

where $P(t) = a$

$$Q(t) = \frac{e^{-at}}{t+1}$$

$$I(t) = e^{\int P(t)\,dt} = e^{\int a\,dt} = e^{at}$$

$$y(t) = \frac{1}{I(t)}\left[\int I(t)\,Q(t)\,dt + c\right]$$

$$= e^{-at}\left[\int \frac{e^{at}e^{-at}}{t+1}\,dt + c\right]$$

$$= e^{-at}\left[\int \frac{dt}{t+1} + c\right]$$

$$= e^{-at}\left[\ln|(t+1)| + c\right]$$

To find c we compute

$$5 = y(0) = [\ln 1 + c] = c$$

$$\therefore c = 5$$

$$y(t) = e^{-at}\left[\ln|(t+1)| + 5\right]$$

(2) Ans:

We will use Laplace Transform.

$$\mathcal{L}(y) = Y(s)$$

$$\mathcal{L}(\dot{y}) = sY(s) - y(0) = sY(s)$$

$$\because y(0) = 0$$

$$\mathcal{L}(\ddot{y}) = s\mathcal{L}(\dot{y}) - \dot{y}(0)$$

$$= s^2 Y(s) - 1 \quad \because \dot{y}(0) = 1$$

$$\mathcal{L}(\ddot{y}) + 3\mathcal{L}(\dot{y}) + 2\mathcal{L}(y) = \mathcal{L}(\sin 5t)$$

$$s^2 \cdot Y(s) - 1 + 3sY(s) + 2Y(s) = \frac{5}{s^2 + 5^2}$$

$$(s^2 + 3s + 2)\,Y(s) = 1 + \frac{5}{s^2+25} = \frac{s^2+30}{s^2+25}$$

It follows that

$$Y(s) = \frac{s^2+30}{(s^2+3s+2)(s^2+25)}$$

$$= \frac{s^2+30}{(s+1)(s+2)(s^2+25)}.$$

We now use partial fraction:

$$\frac{s^2+30}{(s+1)(s+2)(s^2+25)} = \frac{As+B}{s^2+25} + \frac{C}{s+1} + \frac{D}{s+2}$$

$$C = \left.\frac{s^2+30}{(s+2)(s^2+25)}\right|_{s=-1} = \frac{31}{26}. = 1.1923$$

$$D = \left.\frac{s^2+30}{(s+1)(s^2+25)}\right|_{s=-2} = -\frac{34}{29}$$

$$= -1.1724$$

We also have

$$s^2+30 = (As+B)(s+1)(s+2) + C(s^2+25)(s+2) + D(s^2+25)(s+1)$$

$$2B + 50C + 25D = 30$$ ← comparing the constants.

$$B = \frac{30 - 50\cdot\frac{31}{26} + 25\cdot\frac{34}{29}}{2}$$

$$= \left(15 - \frac{25\cdot 31}{26} + \frac{25\cdot 17}{29}\right) = -.1525$$

$$A + C + D = 0$$ ← comparing the co-eff of s^3

$$A = -\frac{31}{26} + \frac{34}{29} = -.02$$

$$\frac{s^2+30}{(s+1)(s+2)(s^2+25)} \approx \frac{1.1923}{s+1} - \frac{1.1724}{s+2} - \frac{.02s+.1525}{s^2+25}$$

$$\frac{.02s+.1525}{s^2+25} = \frac{1}{50}\,\frac{s+7.625}{s^2+25}$$

$$= \frac{1}{50}\left[\frac{s}{s^2+25} + \frac{7.625}{5}\,\frac{5}{s^2+25}\right]$$

$$= \frac{1}{50}\left[\frac{s}{s^2+25} + 1.525\,\frac{5}{s^2+25}\right]$$

$$y(t) \approx 1.1923e^{-t} - 1.1724e^{-2t} - \frac{1}{50}\left[\cos 5t + 1.525\sin 5t\right]$$

③ Ans:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx.$$

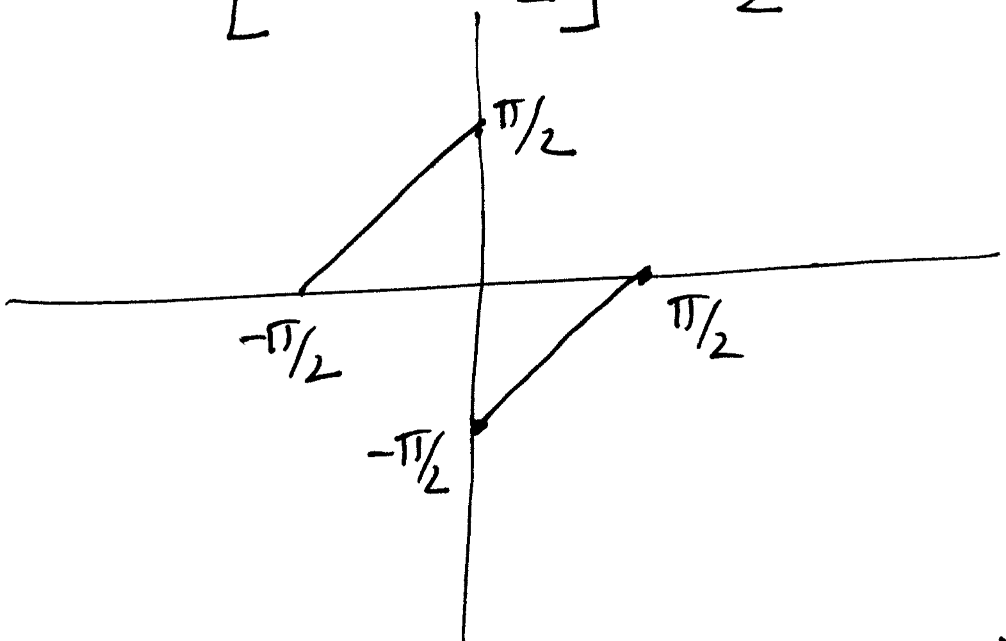
where $L = \pi/2$

The problem becomes easy if we define

$$g(x) = f(x) - \pi/2$$

It follows that

$$g(x) = \begin{cases} \left[x - \pi/2\right] & 0 < x < \frac{\pi}{2} \\ \left[x + \pi/2\right] & -\frac{\pi}{2} < x < 0 \end{cases}$$



$g(x)$ is of period π and is an odd function.

$$g(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$L = \pi/2$$

$$= \sum_{n=1}^{\infty} b_n \sin 2nx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} g(x) \sin 2nx \, dx$$

$$= \frac{2}{L} \int_{0}^{L} g(x) \sin 2nx \, dx$$

because $g(x)$ is odd fn.

$$= \frac{4}{\pi} \int_{0}^{\pi/2} \left(x - \pi/2\right) \sin 2nx \, dx$$

$$\int_{0}^{\pi/2} x \sin 2nx \, dx = \frac{\pi}{4n} (-1)^{n+1}$$

$$\int_0^{\pi/2} \sin 2nx \, dx$$

$$= -\left.\frac{\cos 2nx}{2n}\right|_0^{\pi/2}$$

$$= -\frac{1}{2n}\left[\cos \pi n - 1\right]$$

$$= \frac{1}{2n}\left[1-(-1)^n\right]$$

$$\therefore\ b_n = \frac{4}{\pi}\left[\frac{\pi}{4n}(-1)^{n+1} - \frac{\pi}{2}\frac{1}{2n}\left(1-(-1)^n\right)\right]$$

$$= \frac{\cancel{4}}{\cancel{\pi}}\frac{\cancel{\pi}}{\cancel{4}n}\left[(-1)^{n+1} - 1 + (-1)^n\right]$$

$$= \frac{1}{n}(-1) = -\frac{1}{n}$$

$$f(0) = \frac{\pi}{2}$$

$$\therefore\ g(x) = -\sum_{n=1}^{\infty} \frac{\sin 2nx}{n}$$

$$\therefore\ f(x) = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{\sin 2nx}{n}$$