Math 3350

Class Notes 1

1. We want to solve a 2^{nd} order equation

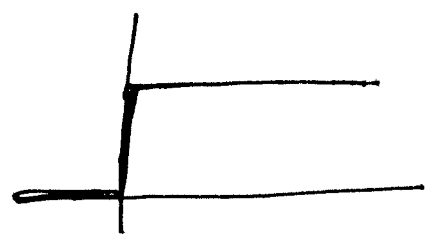
$$\ddot{y} + a\dot{y} + by = f(t)$$

where a and b are constants but not given $y(0) = 1/5$, $\dot{y}(0) = 15$. We also know that the impulse response

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2 + as + b}\right) = \frac{1}{2} e^{-t} \sin 2t.$$

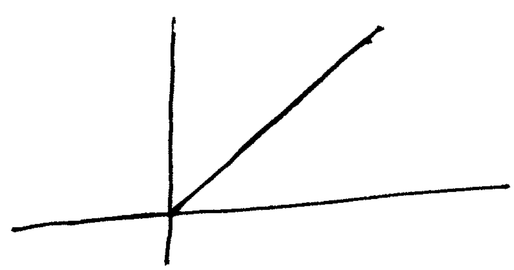
(a) Calculate ~~θ~~ $y(t)$ if $f(t)$ is an unit step input ie

$$f(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



(b) Calculate $y(t)$ if $f(t)$ is a ramp input ie

$$f(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$



1 Ans:

$$\mathcal{L}(\sin 2t) = \frac{2}{s^2+4}$$

$$\mathcal{L}\left(\frac{1}{2}e^{-t}\sin 2t\right) = \frac{2}{(s+1)^2+4}\,\frac{1}{2} = \frac{1}{s^2+2s+5}$$

$\therefore\ a=2,\ b=5$

$$Y(s) = \frac{s+a}{s^2+as+b}\,y(0) + \frac{1}{s^2+as+b}\,\dot{y}(0) + \frac{1}{s^2+as+b}\,F(s)$$

$$= y(0)\,\frac{s+2}{s^2+2s+5} + \dot{y}(0)\,\frac{1}{s^2+2s+5} + \frac{1}{s^2+2s+5}\,F(s)$$

— x —

$$\frac{s+2}{s^2+2s+5} = \frac{s+2}{(s+1)^2+2^2} = \frac{s+1}{(s+1)^2+2^2} + \frac{1}{2}\,\frac{2}{(s+1)^2+2^2}$$

$$\mathcal{L}^{-1}(\ \ ) = e^{-t}\left[\cos 2t + \frac{1}{2}\sin 2t\right]$$

$$y(t) = e^{-t}\left[\cos 2t + \frac{1}{2}\sin 2t\right] y(0)$$
$$+ \frac{1}{2} e^{-t} \sin 2t \; \dot{y}(0)$$
$$+ \mathcal{L}^{-1}\left(\frac{F(s)}{(s+1)^2 + 2^2}\right)$$

(a) $F(s) = \frac{1}{s}$

$$\frac{F(s)}{(s+1)^2+2^2} = \frac{1}{s\left[s^2+2s+5\right]} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$$

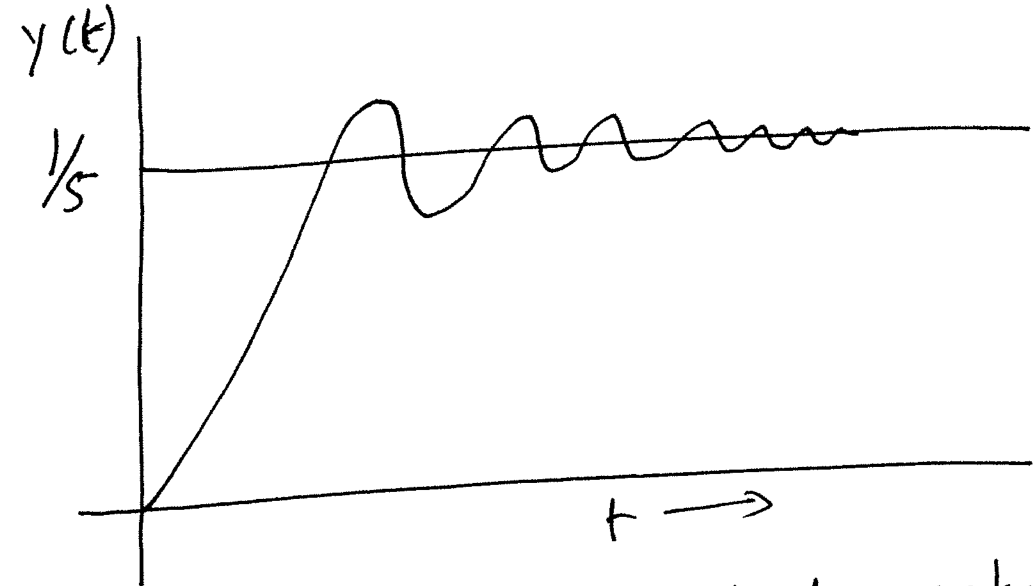
$$A = \frac{1}{5} \qquad B = -\frac{1}{5} \qquad 2A + C = 0$$
$$\Rightarrow C = -2A = -\frac{2}{5}$$

$$\therefore \frac{F(s)}{(s+1)^2+2^2} = \frac{1/5}{s} - \frac{1}{5}\left[\frac{s+2}{s^2+2s+5}\right]$$

$$\mathcal{L}^{-1}\left(\downarrow\right) = 1/5 - \frac{1}{5} e^{-t}\left[\cos 2t + \frac{1}{2}\sin 2t\right]$$

Hence

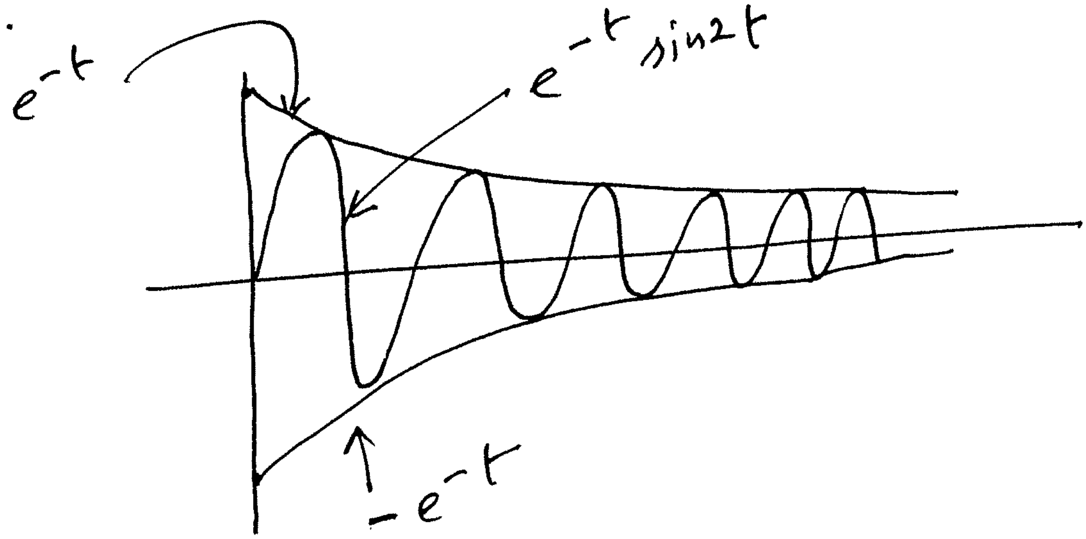
$$y(t) = \frac{1}{5} + e^{-t}\left[\cos 2t + \frac{1}{2}\sin 2t\right]\left[y(0) - \frac{1}{5}\right]$$
$$+ \dot{y}(0)\, \frac{1}{2}\, e^{-t} \sin 2t .$$



"underdamped response"

For $y(0) = 1/5$, we get a simplified expression
$\dot{y}(0) = 15$

$$y(t) = \frac{1}{5} + \frac{15}{2} e^{-t} \sin 2t$$



(b) $F(s) = \frac{1}{s^2}$

$$\frac{F(s)}{(s+1)^2+4} = \frac{1}{s^2[s^2+2s+5]} = \cancel{\frac{As+B}{s^2}}$$

$$= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+2s+5}$$

$$= \frac{As(s^2+2s+5) + B(s^2+2s+5) + (Cs+D) \times s^2}{s^2(s^2+2s+5)}$$

$\boxed{B = \frac{1}{5}}$

$$\therefore As^3 + 2As^2 + 5As + Bs^2 + 2Bs + 5B + Cs^3 + Ds^2 = 1$$

$$\Rightarrow (A+C)s^3 + (2A+B+D)s^2 + (5A+2B)s + 5B = 1$$

$$5A = -2B \Rightarrow A = -\frac{2}{5}\,\frac{1}{5} = -\frac{2}{25}$$

$$D = -2A - B = \frac{4}{25} - \frac{5}{25} = -\frac{1}{25}$$

$$C = -A = \frac{2}{25}$$

$$\therefore \frac{F(s)}{(s+1)^2+2^2} = -\frac{2}{25}\Big/s + \frac{5}{25}\Big/s^2$$

$$+ \frac{1}{25}\,\frac{2s-1}{s^2+2s+5}$$

$$\mathcal{L}^{-1}\frac{2s-1}{s^2+2s+5} = \mathcal{L}^{-1}\left(\frac{2(s+1)-3}{(s+1)^2+2^2}\right)$$

$$= 2\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] - \frac{3}{2}\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+2^2}\right]$$

$$= 2e^{-t}\cos 2t - \frac{3}{2}e^{-t}\sin 2t$$

$$\therefore \mathcal{L}^{-1}\left(\frac{F(s)}{s^2+2s+5}\right) = -\frac{2}{25} + \frac{5}{25}t +$$

$$\frac{2}{25}e^{-t}\cos 2t - \frac{3}{50}e^{-t}\sin 2t$$

$$y(t) = -\frac{2}{25} + \frac{5}{25}t +$$

$$e^{-t}\cos 2t\left[\frac{2}{25} + y(0)\right]$$

$$e^{-t}\sin 2t\left[-\frac{3}{50} + \frac{y(0)}{2} + \frac{\dot{y}(0)}{2}\right]$$

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when $y(0) = \frac{1}{5}$

$$y(0) + \frac{2}{25} = \frac{1}{5} + \frac{2}{25}$$

$$= \frac{7}{25}$$

when $\dot{y}(0) = 15$ we have

$$-\frac{3}{50} + \frac{y(0)}{2} + \frac{\dot{y}(0)}{2}$$

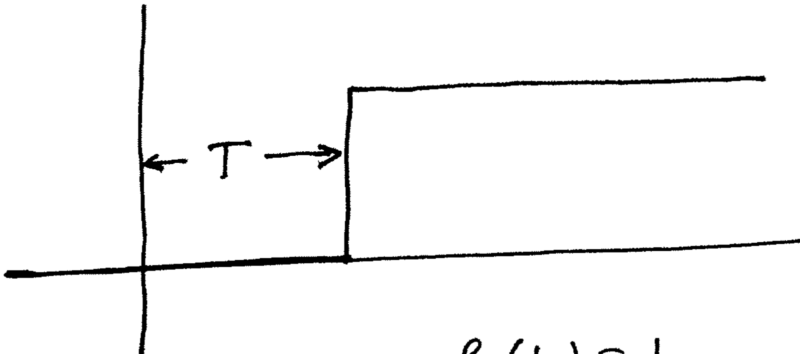
$$= -\frac{3}{50} + \frac{1}{10} + \frac{15}{2}$$

$$= \frac{-3+5+25.15}{50} = \frac{377}{50}$$

$$\therefore y(t) = -\frac{2}{25} + \frac{5}{25}t + \frac{7}{25}e^{-t}\cos 2t$$

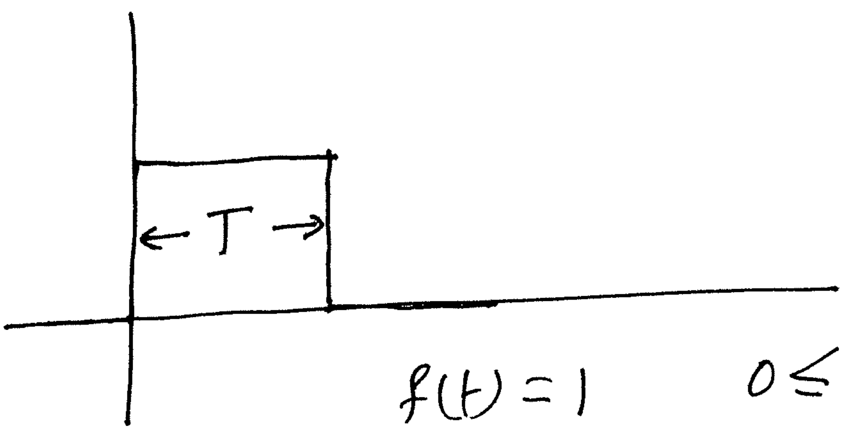
$$+ \frac{377}{50}e^{-t}\sin 2t .$$

2 Repeat problem 1 assuming $f(t)$ as follows:



$$f(t) = 1 \quad t \geq T$$
$$= 0 \quad t < \cancel{0}T$$

3 Repeat problem 1 assuming $f(t)$ as follows



$$f(t) = 1 \quad 0 \leq t \leq T$$
$$= 0 \quad \text{otherwise.}$$

② Ans: ⑨

We recover the solution from ①ⓐ.

Define a function

$$u(t-T) = 1 \quad t \geq T$$
$$= 0 \quad t < T$$

So $u(t)$ is the unit step function.

The solution for ①ⓐ is given as

$$y(t) = \left[e^{-t}\left[\cos 2t + \frac{1}{2}\sin 2t\right] y(0) + \frac{1}{2} e^{-t} \sin 2t \; \dot{y}(0)\right] u(t)$$
$$+ \left[\frac{1}{5} - \frac{1}{5} e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)\right] u(t)$$

Remark: $u(t)$ does not change anything because it is 1 for $t \geq 0$.

When the input is delayed by T units, the corresponding solution is also delayed by T units and we have for ②

Effect of initial condition is not delayed.

$$y(t) = \left[e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right) y(0) + \frac{1}{2} e^{-t} \sin 2t \; \dot{y}(0) \right] u(t)$$

$$+ \left[\frac{1}{5} - \frac{1}{5} e^{-(t-T)} \left(\cos 2(t-T) + \frac{1}{2} \sin 2(t-T) \right) \right] u(t-T)$$

Note that this part is obtained by delaying the answer obtained in 1(a) by T units of time

(3) Ans:

$$f(t) = u(t) - u(t-T)$$

Hence

$$y(t) =$$

$$\left[e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right) y(0) + \frac{1}{2} e^{-t} \sin 2t \, \dot{y}(0)\right] u(t)$$

$$+ \left[\frac{1}{5} - \frac{1}{5} e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)\right] u(t)$$

$$- \left[\frac{1}{5} - \frac{1}{5} e^{-(t-T)}\left(\cos 2(t-T) + \frac{1}{2}\sin 2(t-T)\right)\right] \cdot u(t-T)$$

$y(t) =$

$$e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right) y(0) \quad \text{for } 0 \leq t < T$$

$$+ \frac{1}{2} e^{-t} \sin 2t \, \dot{y}(0)$$

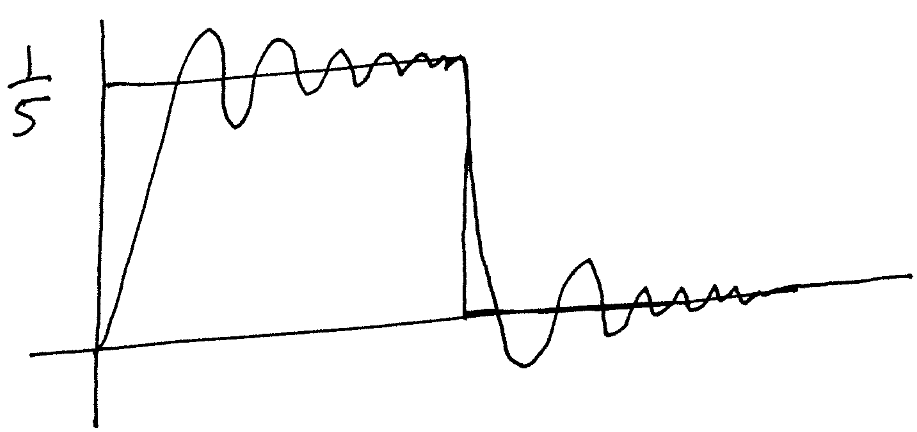
$$+ \frac{1}{5} - \frac{1}{5} e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

$$e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right) y(0)$$

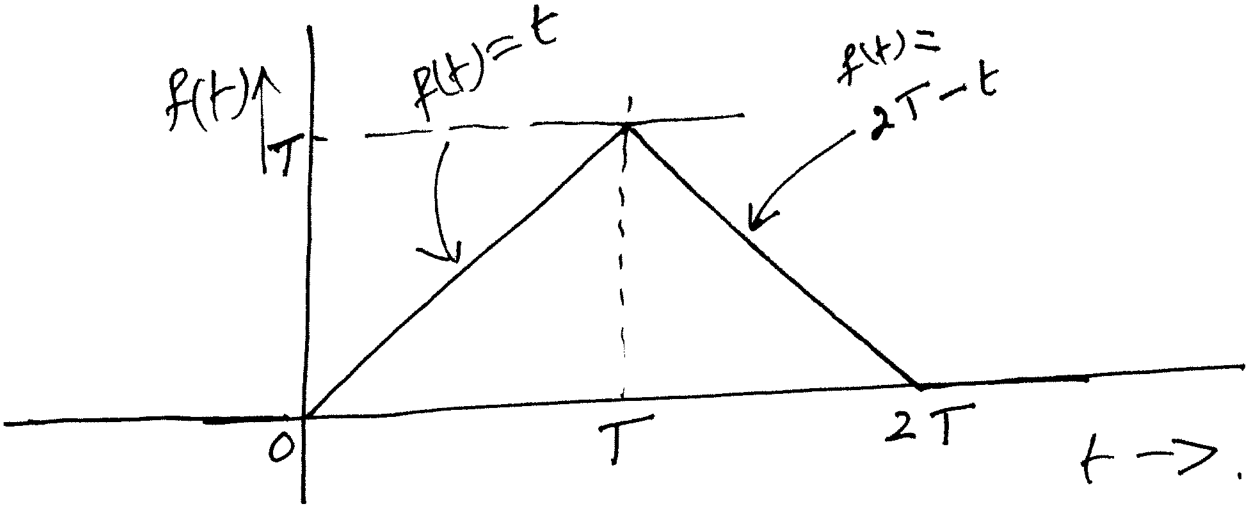
$$+ \frac{1}{2} e^{-t} \sin 2t \, \dot{y}(0)$$

$$+ \frac{1}{5}\left[e^{-(t-T)}\left(\cos 2(t-T) + \frac{1}{2}\sin 2(t-T)\right) - e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)\right]$$

$$\text{for } t \geq T$$



(4) Now try the problem for $f(t)$ sketched as follows



$$f(t) = t \qquad 0 \le t < T$$
$$= 2T - t \qquad T \le t < 2T$$
$$= 0 \qquad \text{otherwise}.$$