LINEAR ALGEBRA
HOMEWORK 3
PROBLEMS ON DETERMINANTS.

(1) $\det(A) = \begin{vmatrix} 4 & 6 \\ 3 & 5 \end{vmatrix} = 20 - 18 = 2$

(2) $\det(B) = \begin{vmatrix} 5 & 6 & 10 \\ 4 & 4 & 6 \\ -2 & 3 & 5 \end{vmatrix}$ Expand using the first column.

$$= 5\begin{vmatrix} 4 & 6 \\ 3 & 5 \end{vmatrix} - 4\begin{vmatrix} 6 & 10 \\ 3 & 5 \end{vmatrix} + (-2)\begin{vmatrix} 6 & 10 \\ 4 & 6 \end{vmatrix}$$

$$= 5 \times 2 - 0 - 2 \cdot 2 \begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix}$$ (factoring 2 from the first row)

$$= 10 + 4 \begin{vmatrix} 4 & 6 \\ 3 & 5 \end{vmatrix}$$ (Swapping the two rows)

$$= 10 + (4 \times 2)$$

$$= 18$$

(3) $$\begin{vmatrix} 3-\lambda & -4 \\ 1 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) + 4$$

$$= -6 + 2\lambda - 3\lambda + \lambda^2 + 4$$

$$= \lambda^2 - \lambda - 2$$

$$= (\lambda - 2)(\lambda + 1)$$

$\therefore$ $\lambda = 2$ or $\lambda = -1$ will make the determinat 0.

4 (a) $A = \begin{bmatrix} 4 & 6 \\ 3 & 5 \end{bmatrix}$

$\det(A) = 4\cdot 5 - 6\cdot 3 = 20 - 18 = 2.$

$\operatorname{adj}(A) = \begin{pmatrix} 5 & -6 \\ -3 & 4 \end{pmatrix}$

$\therefore A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)} = \begin{pmatrix} 5/2 & -3 \\ -3/2 & 2 \end{pmatrix}$

(b) $B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$\det(B) = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix}$ (Expand using first row)

$= 2\cdot(4-1) + 1(-2) = 4.$

$B_{11} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$; $B_{12} = -\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 2$; $B_{13} = \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1$

$B_{21} = \begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} = 2$; $B_{22} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$; $B_{23} = -\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} = 2$

$B_{31} = \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} = 1$; $B_{32} = -\begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} = 2$; $B_{33} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3.$

$\therefore \operatorname{adj}(B) = \begin{pmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \\ B_{13} & B_{23} & B_{33} \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

$B^{-1} = \dfrac{\operatorname{adj}(B)}{\det(B)} = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix}$

5 (a) $x+2y=9$

$2x+y=6$ $\Rightarrow$ $\underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 9 & 2 \\ 6 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}} = \frac{9-12}{-3} = \frac{-3}{-3} = 1$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 1 & 9 \\ 2 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}} = \frac{6-18}{-3} = \frac{-12}{-3} = 4$$

$\therefore x = 1 ; \quad y = 4.$

(b) $x+y+z=6$

$2x+2y+y=9$ $\Rightarrow$ $\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 11 \end{pmatrix}$

$3x+y+2z=11$

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{vmatrix} = -2$$

(see page 100-102)

$$\det(A_1) = \begin{vmatrix} 6 & 1 & 1 \\ 9 & 2 & 1 \\ 11 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 3 & 1 & 1 \\ -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & -1 & 2 \end{vmatrix} = -2$$

$$\det(A_2) = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 9 & 1 \\ 3 & 11 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & -4 & 2 \end{vmatrix} = -\begin{vmatrix} 0 & 0 & 1 \\ 0 & -4 & 2 \\ 1 & 0 & 2 \end{vmatrix} = -4$$

$$\det(A_3) = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 2 & 9 \\ 3 & 1 & 11 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 2 & -3 \\ 2 & 1 & -1 \end{vmatrix} = -\begin{vmatrix} 0 & 0 & 1 \\ 0 & -3 & 2 \\ 2 & -1 & 3 \end{vmatrix} = -6$$

$$\therefore x = \frac{\det(A_1)}{\det A} = 1 ; \quad y = \frac{\det(A_2)}{\det(A)} = 2 ; \quad z = \frac{\det(A_3)}{\det(A)} = 3 .$$

6 Since $C = D^T$,

$\det(C) = \det(D^T) = \det(D)$.

Since both matrices are $n \times n$,

$$\det(CD) = \det(C)\det(D) = \det(C)\det(C)$$
$$= [\det(C)]^2$$
$$= 1 \text{ (given)}$$

$\therefore \det(C) = \pm 1$.

* The key points are: $\det(D^T) = \det(D)$ and $\det(CD) = \det(C)\det(D)$. Both these were possible to say because both C and D are $n \times n$ (i.e. Both square and same size).
  If C was $n \times m$ and D was $m \times n$ and $m \neq n$, we cannot write $\det(CD) = \det(C)\det(D)$ because it would not make any sense as neither $\det(C)$ nor $\det(D)$ is defined.

7 (a) Since all determinants are non zero, all the matrices have well defined inverses.

(b) Since all matrices are square and of the same size multiplications and additions among them are well defined.

(i) $\det(PQRS) = \det(P)\det(Q)\det(R)\det(S)$

$= 1\cdot(-1)\cdot 2\cdot 4$
$= -8$

(ii) $\det(QR^T) = \det(Q)\det(R^T) = \det(Q)\det(R)$

$= -1\cdot 2$
$= -2$

(iii) $\det(2QR^{-1}) = ?$

The easiest way to think about this is to think of the factor 2 as $2I$. Then,

$$\det(2I) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$$

$$\det(2QR^{-1}) = \det((2I)QR^{-1})$$
$$= \det(2I)\det Q\det(R^{-1})$$
$$= \det(2I)\det(Q)\cdot\frac{1}{\det(R)}$$
$$= \frac{8\cdot(-1)}{2}$$
$$= -4$$

(iv) $\det(P + Q^{-1})$ It is not possible to compute this!

because $\det(A+B) \neq \det(A) + \det(B)$.

(v) $\det(R(PQ^{-1})^T)$

$= \det(R)\det((PQ^{-1})^T)$

$= \det(R)\det(PQ^{-1})$

$= \det(R)\det(P)\det(Q^{-1})$

$= \det(R)\det(P)\cdot\frac{1}{\det(Q)}$

$= \frac{4\cdot 1}{(-1)}$

$= -4.$