Answers to Home Work Four

1. $A^2 + I = 0$

$\Rightarrow A^2 = -I$

$\Rightarrow \det(A^2) = \det(-I) = (-1)^n \det I$

$= (-1)^n$

More over

$\det(A^2) = (\det A)(\det A) =$

$(\det A)^2$

It follows that

$(\det A)^2 = (-1)^n$.

If n is odd, $(\det A)^2 = -1$ would not be possible since $\det A$ is a real number.

If n is even.

$(\det A)^2 = 1 \Rightarrow \det A = \pm 1$.

If n is even it would imply that $\det A$ is either 1 or -1.

for example

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ would suffice

Remark:

for a general 2×2 matrix

$$A = \begin{pmatrix} a & b \\ -c & d \end{pmatrix}$$

to satisfy $A^2 + \underline{I} = 0$, we can show

that $\boxed{a = -d}$ and $\boxed{bc = 1 + a^2}$

2 Ans

$$\det V = \det \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 \end{pmatrix}$$

$$= \det \begin{pmatrix} x_2 - x_1 & x_2^2 - x_1^2 \\ x_3 - x_1 & x_3^2 - x_1^2 \end{pmatrix}$$

$$= \det \begin{pmatrix} (x_2 - x_1) & (x_2 - x_1)(x_2 + x_1) \\ (x_3 - x_1) & (x_3 - x_1)(x_3 + x_1 \end{pmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1) \det \begin{pmatrix} 1 & (x_2 + x_1) \\ 1 & (x_3 + x_1) \end{pmatrix}$$

$$= (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$$

$$\det\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & x_2-x_1 & x_2^2-x_1^2 & x_2^3-x_1^3 \\ 0 & x_3-x_1 & x_3^2-x_1^2 & x_3^3-x_1^3 \\ 0 & x_4-x_1 & x_4^2-x_1^2 & x_4^3-x_1^3 \end{pmatrix}$$

$$= (x_2-x_1)(x_3-x_1)(x_4-x_1) \det\begin{pmatrix} 1 & x_2+x_1 & x_2^2+x_1x_2+x_1^2 \\ 1 & x_3+x_1 & x_3^2+x_1x_3+x_1^2 \\ 1 & x_4+x_1 & x_4^2+x_1x_4+x_1^2 \end{pmatrix}$$

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$$\det\begin{pmatrix} 1 & x_2+x_1 & x_2^2+x_1x_2+x_1^2 \\ 1 & x_3+x_1 & x_3^2+x_1x_3+x_1^2 \\ 1 & x_4+x_1 & x_4^2+x_1x_4+x_1^2 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & x_2+x_1 & x_2^2+x_1x_2+x_1^2 \\ 0 & x_3-x_2 & x_3^2-x_2^2+x_1x_3-x_1x_2 \\ 0 & x_4-x_2 & x_4^2-x_2^2+x_1x_4-x_1x_2 \end{pmatrix}$$

$$= \det\begin{pmatrix} (x_3-x_2) & (x_3^2-x_2^2)+x_1(x_3-x_2) \\ (x_4-x_2) & (x_4^2-x_2^2)+x_1(x_4-x_2) \end{pmatrix}$$

$$= \det\begin{pmatrix} (x_3-x_2) & (x_3-x_2)[x_1+x_2+x_3] \\ (x_4-x_2) & (x_4-x_2)[x_1+x_2+x_4] \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & x_1+x_2+x_3 \\ 1 & x_1+x_2+x_4 \end{pmatrix}$$

$(x_3-x_2)(x_4-x_2)$

$$= (x_3-x_2)(x_4-x_2)\left[\cancel{x_1}+\cancel{x_2}+x_4-\cancel{x_1}-\cancel{x_2}-x_3\right]$$

$$= (x_3-x_2)(x_4-x_2)(x_4-x_3).$$

$\therefore$

$$\boxed{\det V = (x_2-x_1)(x_3-x_1)(x_4-x_1)(x_3-x_2)(x_4-x_2)(x_4-x_3).}$$

Do you now see a pattern???

3) Ans:

If $A^T = -A$

$$\det(A^T) = \det(-A) = (-1)^n \det A.$$

However

$$\det(A^T) = \det A.$$

It follows that

$$\det A = (-1)^n \det A.$$

If n is odd we have

$$\det A = -\det A$$

$\Rightarrow \det A = 0 \Rightarrow A$ is singular.

(b)

$$\det \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}$$

$$= -a_{12} \det \begin{pmatrix} -a_{12} & a_{23} \\ -a_{13} & 0 \end{pmatrix}$$

$$+ a_{13} \det \begin{pmatrix} -a_{12} & 0 \\ -a_{13} & -a_{23} \end{pmatrix}$$

$$= -a_{12} \left[a_{13} a_{23} \right] + a_{13} \left[a_{12} a_{23} \right]$$

$$= 0.$$

④ (a) ⑨

$$\operatorname{cofac} A = \begin{pmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{23}a_{31} - a_{21}a_{33} & a_{21}a_{32} - a_{22}a_{31} \\ a_{13}a_{32} - a_{12}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{12}a_{31} - a_{11}a_{32} \\ a_{12}a_{23} - a_{13}a_{22} & a_{13}a_{21} - a_{11}a_{23} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}$$

adj A =

$$\begin{pmatrix} a_{22}a_{33}-a_{23}a_{32} & a_{13}a_{32}-a_{12}a_{33} & a_{12}a_{23}-a_{13}a_{22} \\ a_{23}a_{31}-a_{21}a_{33} & a_{11}a_{33}-a_{13}a_{31} & a_{13}a_{21}-a_{11}a_{23} \\ a_{21}a_{32}-a_{22}a_{31} & a_{12}a_{31}-a_{11}a_{32} & a_{11}a_{22}-a_{12}a_{21} \end{pmatrix}$$

(b) $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Let $A \cdot \operatorname{adj} A = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$

$c_{11} =$

$$a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right)$$

$$- a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right).$$

$$+ a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right).$$

$$= a_{11}\operatorname{cofac}(a_{11}) + a_{12}\operatorname{cofac}(a_{12})$$
$$+ a_{13}\operatorname{cofac}(a_{13}).$$

$$= \det A.$$

It can be verified that

$c_{ii} = \det A$ for $i = 1, 2, 3$.

$c_{ij} = 0$ for $i \neq j$.

(c) If $\det A = 0$ it follows that

$A \cdot \operatorname{adj} A = O$.

(5) Ans:

(a) $$R R^T = I$$

$$\Rightarrow \det(R R^T) = \det I = 1.$$

$$\Rightarrow (\det R)(\det R^T) = 1.$$

$$\Rightarrow (\det R)(\det R) = 1$$

$$\Rightarrow \det R = \pm 1.$$

(b) Let $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$R . R^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$= \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix}$$

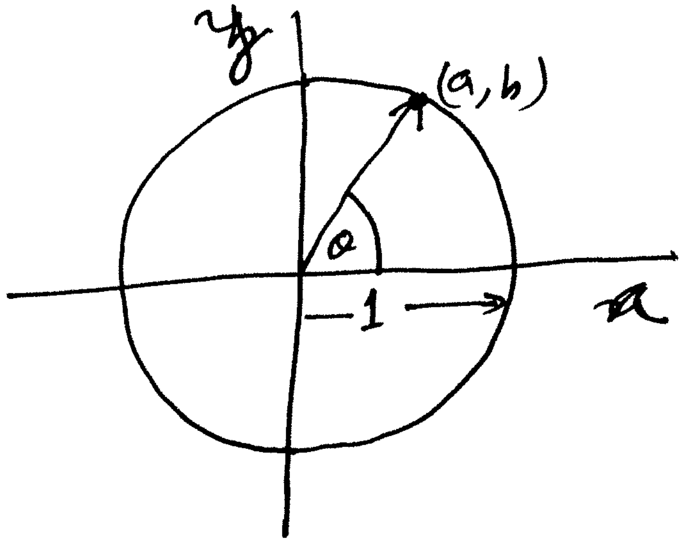
$$RR^T = I$$

$$\Rightarrow a^2+b^2 = 1$$

$$c^2+d^2 = 1.$$

$$ac+bd = 0.$$

$a^2+b^2=1 \Rightarrow$ the point (a,b) must be on the unit circle in the $(\cancel{x},\cancel{y})$ plane

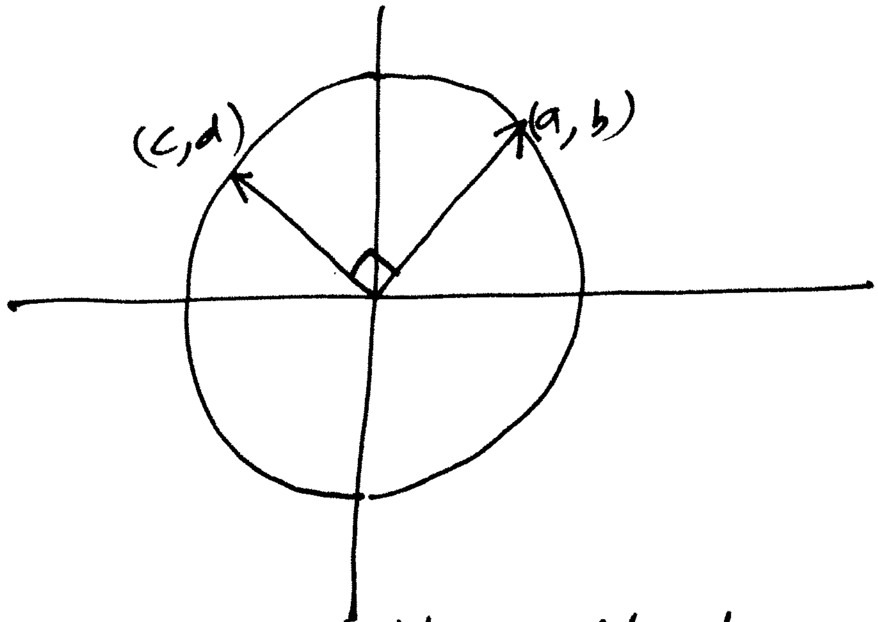


We can write

$$a = \cos\theta$$
$$b = \sin\theta.$$

$$ac+bd = 0 \Rightarrow ac = -bd \Rightarrow \frac{b}{a} = -\frac{c}{d}.$$

It follows that the vector (c,d) must be perp. to the vector (a,b).



It would follow that

$$c = -\alpha \sin\theta$$
$$d = +\alpha \cos\theta$$

Finally $c^2 + d^2 = 1 \Rightarrow \alpha = \pm 1$.

Thus if $a = \cos\theta$, $b = \sin\theta$

$$c = \mp \sin\theta \quad d = \pm \cos\theta$$

Remark: Note that $\det\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = 1$

$$\det\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = -1.$$