Volume of an Equilateral pyramid.

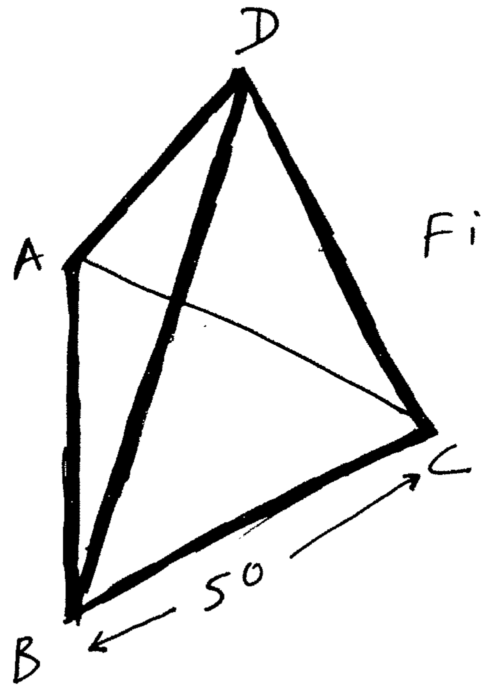


Fig 1

An equitaleral pyramid, sketched in Fig 1 is a pyramid with a equilateral triangular base, where each of the four side surfaces are also equilateral triangles. We assume that the length of every edge is 50 ft. The problem is to calculate the volume of this pyramid.

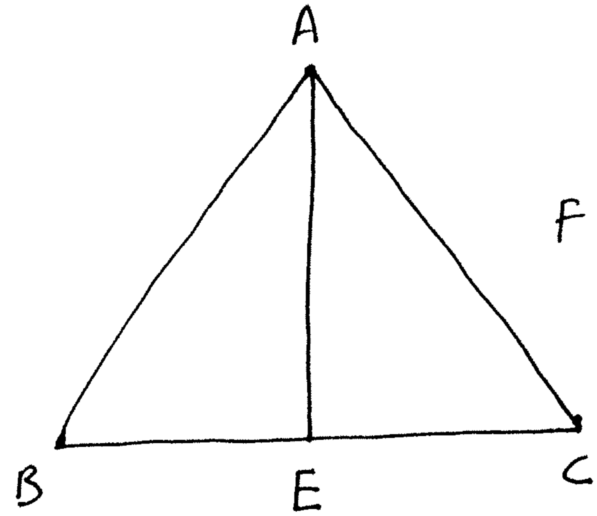


Fig 2

In Fig 2 we show the base of the pyramid. The height AE is calculated as follows.

$$AE^2 + BE^2 = AB^2$$

$$\Rightarrow AE = \sqrt{AB^2 - BE^2}$$

We know that $AB = 50$, $BE = 25$

It follows that

$$AE = 43 \cdot 30127.$$

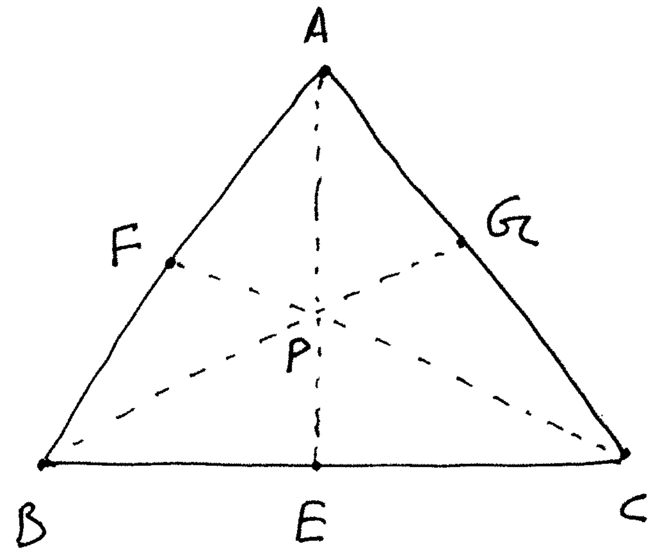


Fig3

Let E, F, G, be the mid points of segments BC, AB and AC respectively.

We know from geometry that the segments AE, CF and BG intersect at a point P (see Fig3)

Since the angle $\angle PBE = 30°$ it follows that

$$\frac{PE}{BE} = \tan 30 \Rightarrow PE = BE \tan 30$$

$$= 25 \tan 30$$

$$= 14.433757$$

Hence AP = AE − PE = 28.867513

We want to calculate the height DP of the pyramid. (see Fig 4).

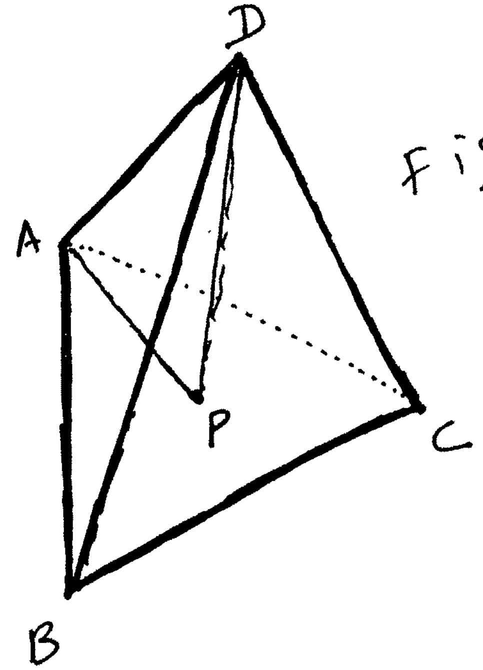
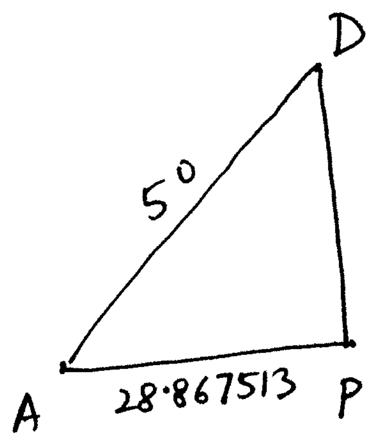


Fig 4



Note that APD is a right triangle

Hence $DP = \sqrt{50^2 - 28{\cdot}867513^2}$

$= 40{\cdot}824829$

The equilateral pyramid has a base area of

$$BE . AE = 25 . 43 \cdot 30127$$

$$= 1082 \cdot 5318 \text{ sq ft.}$$

and a height of $DP = 40 \cdot 824829$.

If we believe in the formula that

A volume of a pyramid =

$$\frac{1}{3} \text{ area of the base . height}$$

$$= \frac{1}{3} 1082 \cdot 5318 . 40 \cdot 824829 .$$

$$= 14731 \cdot 392 \text{ cu ft.}$$

Since we are doing calculus II, we dont believe in this formula. So we calculate the volume by taking a cross-section "A'B'C'" h ft above the base (see Fig 5).

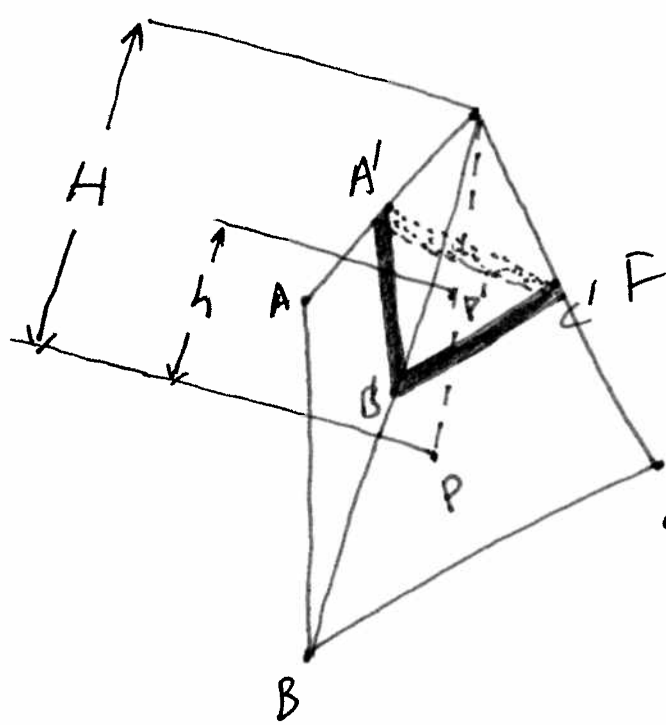


Fig 5

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We would first like to calculate the length l' of the equilateral triangle $A'B'C'$. Using ratio and proportion we write

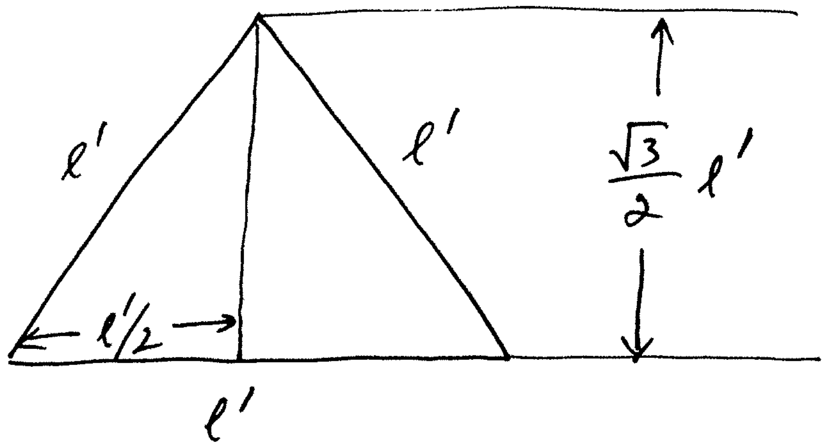
$$\frac{l'}{50} = \frac{H-h}{H}.$$

follows from the fact that the length of the cross-section scales in proportion to the height of the pyramid above the cross-section.

$$\therefore\ l' = \left(1 - \frac{h}{H}\right) 50 .$$

$$l' = \left(1 - \frac{h}{40.824829}\right) 50 .$$

Next we calculate the area of the cross section of length l'



The height of the cross section is given by $\sqrt{l'^2 - \left(\frac{l'}{2}\right)^2}$

$1082.5\left(1-\frac{h}{312}\right)^2$

$$= \frac{\sqrt{3}}{2} l'$$

Area of the cross section =

$$\frac{l'}{2}\,\frac{\sqrt{3}}{2} l' = \frac{\sqrt{3}}{4} l'^2 \text{ sq ft}$$

$$= \frac{\sqrt{3}}{4}\left(1 - \frac{h}{H}\right)^2 (50)^2 \text{ sq ft}.$$

If the cross section is thickness Δh, the cross sectional volume is given by

$$\Delta V = \overset{312}{1082.5}\left(1 - \frac{h}{H}\right)^2 \Delta h .$$

Hence

$$V = \int_0^H \underbrace{1082.5}_{\frac{\sqrt{3}}{4}50^2} \left(1 - \frac{h}{H}\right)^2 dH .$$

$$= 1082.5 \int_0^H \left(1 - \frac{h}{H}\right)^2 dH .$$

$$\int_0^H \left(1-\frac{h}{H}\right)^2 dH$$

$$= -\frac{H}{3}\left(1-\frac{h}{H}\right)^3 \Bigg|_0^H$$

$$= \frac{H}{3}.$$

height.

$$\therefore \text{Volume} = \frac{1}{3}\underbrace{1082.5312}_{\text{Area of the base}}\ H$$

We know that $H = 40.824829$.

$$\therefore \text{Volume} = \frac{1}{3}\, 1082.5312 \cdot 40.824829.$$

$$= 14731.392 \ \text{cu ft}$$

Remark:

We did not quite show this but the volume of an equilateral pyramid of side s can be shown to be

$$\boxed{Vol = .1178511\, s^3}$$

For $s = 50$ we get $Vol = 14731.4$.

which matches our earlier calculation.