Surface Area and
Volume of an
Apple

In this problem, we want to calculate the surface area and volume of an apple. We start with a cardiod

$$r = 1 + \cos\theta \qquad (1)$$

described in polar co-ordinates.

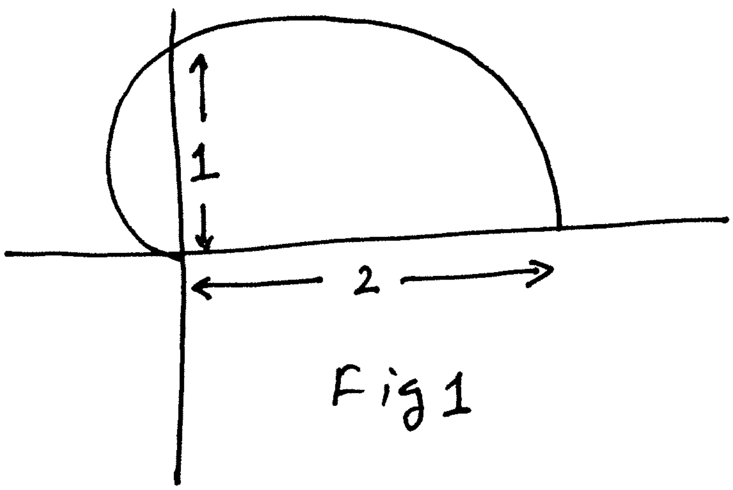

Fig 1

The graph of the cardiod for $\theta \in [0, \pi]$ is given by Fig 1. The full cardiod, for $\theta \in [0, 2\pi]$ is sketched in Fig 2. The full cardiod does not play any role in our problem.

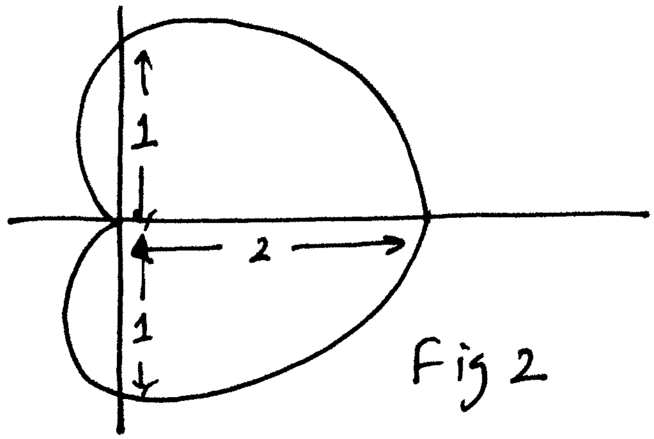

Fig 2

If we rotate the curve in Fig 1 about the x-axis we generate an apple.

Calculating the surface area of the apple

Surface area of any curve given by

$$y = f(x) \qquad (2)$$

when rotated about the x axis is given by

$$\int_{x=a}^{x=b} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad (3)$$

We can rewrite (3) as follows:

$$S = \int_{x=a}^{x=b} 2\pi y \sqrt{(dx)^2 + (dy)^2} \quad (4)$$

In our problem, the curve is not given as (2) but as (1) so we use the transformation

$$x = r\cos\theta \qquad (5)$$
$$y = r\sin\theta$$

from polar to cartesian co-ordinates.

Combining (1) and (5) we obtain

$$x = (1+\cos\theta)\cos\theta \qquad (6)$$
$$y = (1+\underset{\cos}{\cancel{\sin}}\,\theta)\sin\theta$$

from (6), we calculate dx and dy as follows:

$x = \cos\theta\,(1+\cos\theta)$

$dx = -\sin\theta\,[1+2\cos\theta]\,d\theta = -(\sin\theta + \sin 2\theta)\,d\theta$

(5)

$y = \sin\theta\,(1+\cos\theta)$

$dy = (\cos\theta + \cos 2\theta)\,d\theta$

It follows that

$(dx)^2 + (dy)^2 =$

$$\begin{aligned}
&[\sin^2\theta + \sin^2 2\theta + 2\sin\theta\sin 2\theta \\
&\quad + \cos^2\theta + \cos^2 2\theta + 2\cos\theta\cos 2\theta]\,(d\theta)^2 \\
&= 2\,[1 + \sin\theta\sin 2\theta + \cos\theta\cos 2\theta]\,(d\theta)^2 \\
&= 2\,[1 + \sin\theta\, 2\sin\theta\cos\theta \\
&\quad + \cos\theta\,(1 - 2\sin^2\theta)]\,(d\theta)^2 \\
&= 2\,(1+\cos\theta)\,(d\theta)^2
\end{aligned}$$

$$\therefore \sqrt{(dx)^2 + (dy)^2} = \sqrt{2}\,\sqrt{1+\cos\theta}\;d\theta$$

(6)

From (4), (5), and (6) we obtain

$$S = \int_{\theta=0}^{\theta=\pi} 2\pi \sin\theta (1+\cos\theta) \sqrt{2} \sqrt{1+\cos\theta} \, d\theta$$

$$= 2\sqrt{2}\,\pi \int_0^\pi \sin\theta \, (1+\cos\theta)^{3/2} \, d\theta \quad (7)$$

To calculate (7), note that

$$\frac{d}{d\theta} (1+\cos\theta)^{5/2} = \frac{5}{2} (1+\cos\theta)^{3/2} (-\sin\theta)$$

$$\Rightarrow \frac{d}{d\theta} \left[-\frac{2}{5} (1+\cos\theta)^{5/2} \right] = (1+\cos\theta)^{3/2} \sin\theta$$

It follows that

$$S = 2\sqrt{2}\,\pi \left[-\frac{2}{5} (1+\cos\theta)^{5/2} \right]_0^\pi$$

$$= -\frac{2\sqrt{2}\,\pi\, 2}{5} \left[\underbrace{(1+\cos\pi)^{5/2}}_{=0} - 2^{5/2} \right]$$

Hence

$$S = -\frac{4}{5}\sqrt{2}\,\pi \cancel{(-2)^{5/2}} \left[-(2)^{5/2}\right]$$

$$= \frac{4}{5}\sqrt{2}\,\pi \; 2\sqrt{2} = \frac{32}{5}\pi$$

$$= 20{\cdot}106$$

Volume of the apple

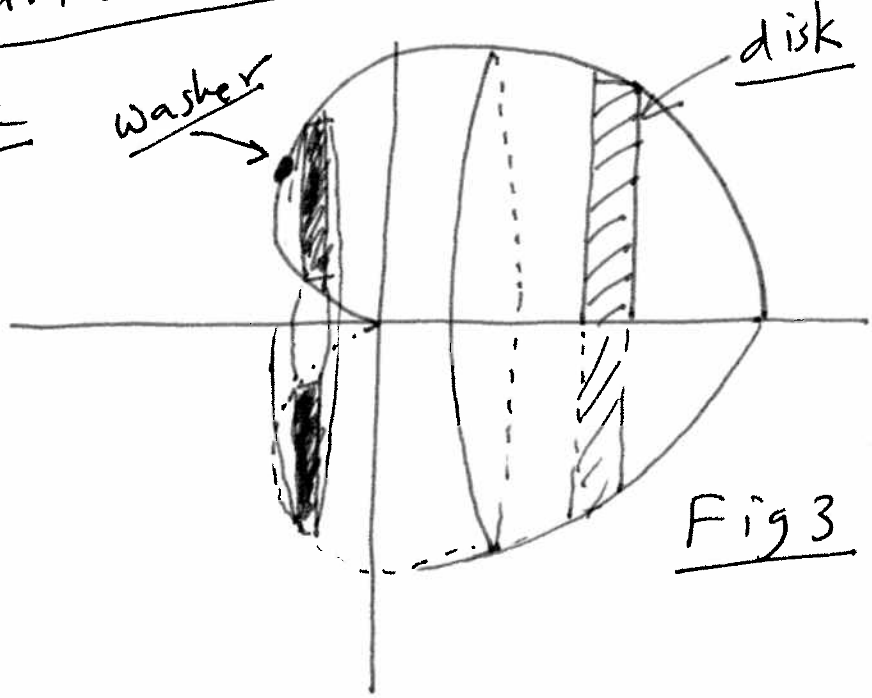

Fig 3

We now proceed to calculate the volume of the apple. As shown in figure 3, we would like to use the "disk" method which becomes complicated because for negative x, the disk becomes a "washer".

This happens because of the special shape of the apple.

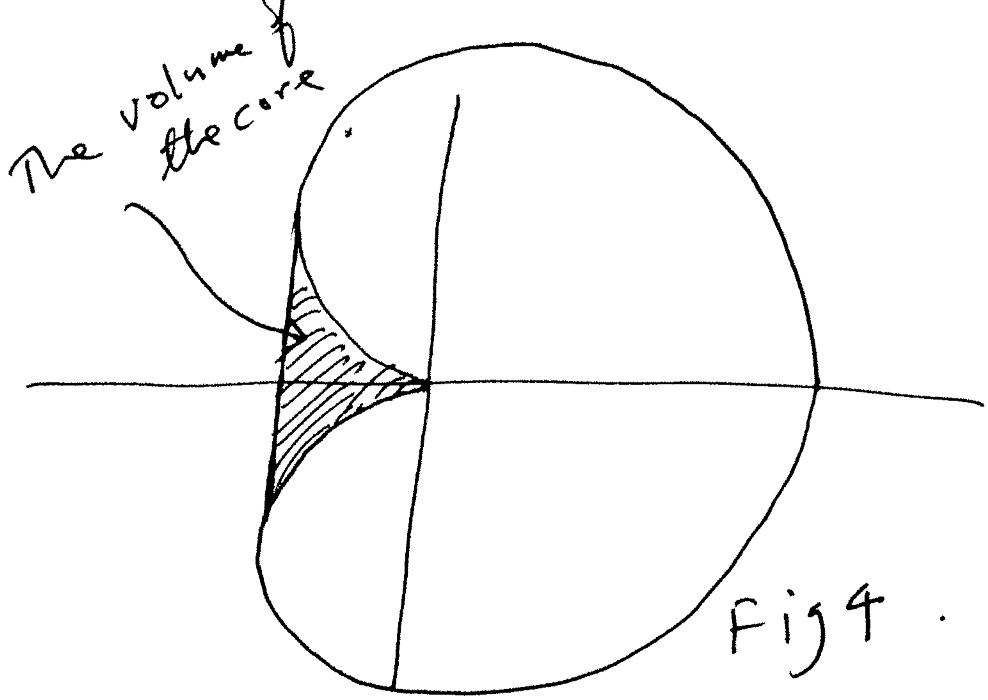

Fig 4.

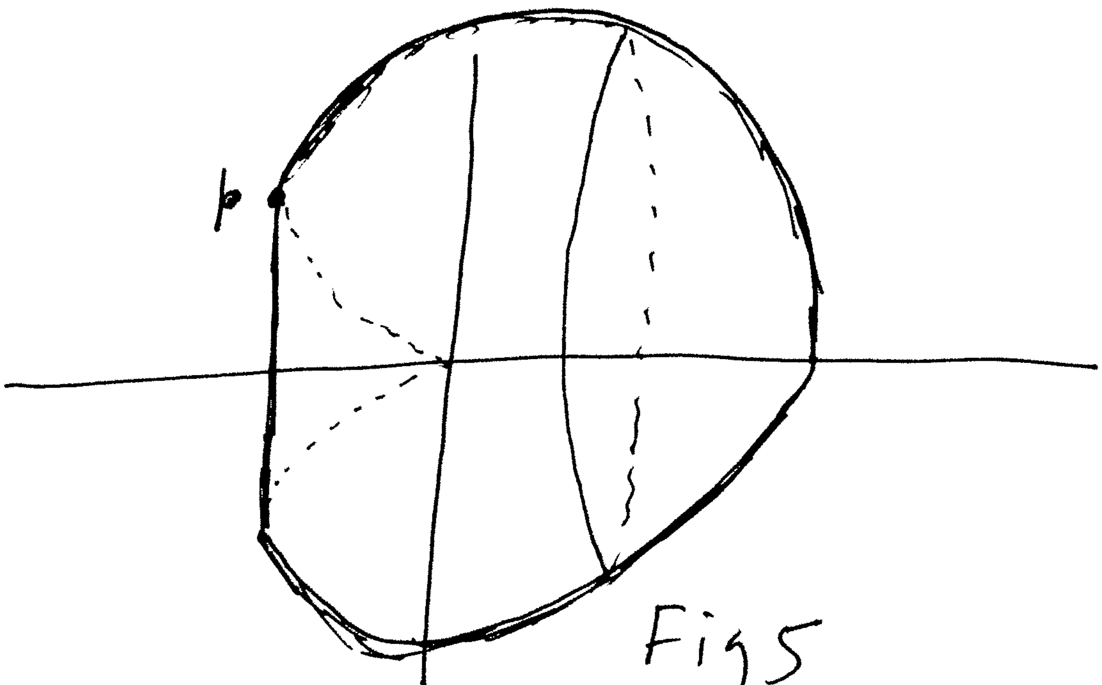

Fig 5

To circumvent the problem, we calculate the volume of the apple with core-filled (Fig 5) and subtract the volume of the core (Fig 4).

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Apple with core filled is generated by rotating a portion of the cardiod in Fig 1 sketched as follows:

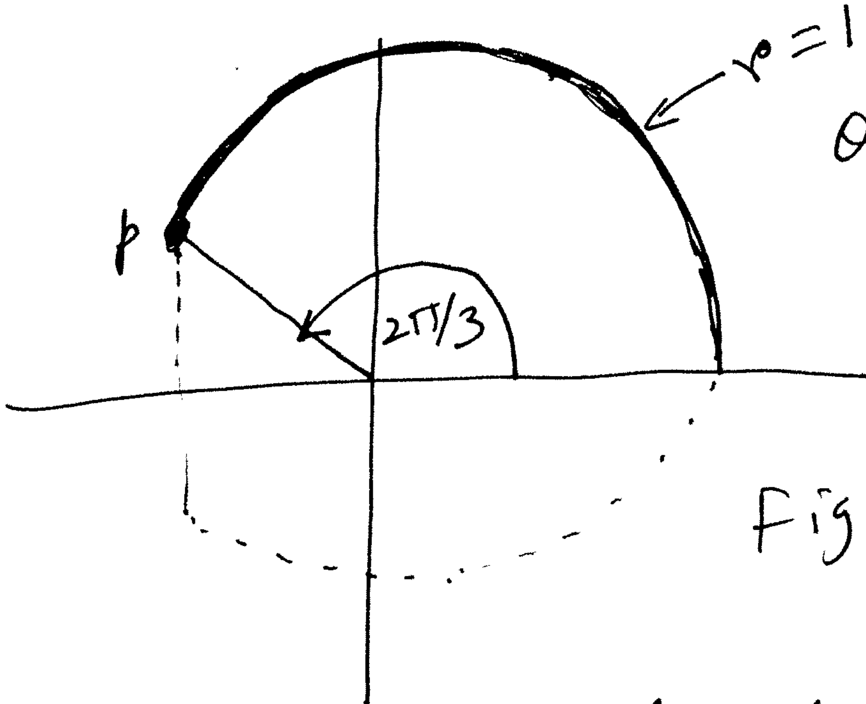


Fig 6.

Q: How to calculate the co-ordinates of p ??

Ans: Recall that

$$x = r\cos\theta = (1+\cos\theta)\cos\theta$$

θ is that angle at which the co-ordinate x attains a minimum.

To calculate that θ we compute

$$\frac{dx}{d\theta} = -\sin\theta\left[1 + 2\cos\theta\right]$$

and set $\frac{dx}{d\theta} = 0$

$$\Rightarrow \cos\theta = -\frac{1}{2}, \text{ or } \sin\theta = 0.$$

At $\sin\theta = 0$, $\theta = n\pi$, the co-ordinate x attains a local maxima.

At $\cos\theta = -\frac{1}{2}$, $\theta = \frac{2\pi}{3}$, the co-ordinate x attains a local minima.

Thus p is at an angle $2\pi/3$ as sketched in Fig 6. The x and y co-ordinates of p can be calculated as $\left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$. The polar co-ordinates are

$$r = \frac{1}{2}, \quad \theta = 2\pi/3.$$

We now calculate the volume of the apple with core filled, as shown in Fig 5. The volume formula using disk method (see Fig 3) is given by

$$V = \int_{x=-1/4}^{x=2} \pi y^2 \, dx .$$

Substituting y and dx as

$$y = (1+\cos\theta)\sin\theta$$

$$dx = -(\sin\theta + \sin 2\theta)\, d\theta = -\sin\theta[1+2\cos\theta]\, d\theta$$

we get

$$V = \int_{\theta=2\pi/3}^{0} \pi \sin^2\theta (1+\cos\theta)^2 \left[-(\sin\theta)(1+2\cos\theta)\right] d\theta$$

$$= -\pi \int_{\theta=2\pi/3}^{0} (1-\cos^2\theta)(1+\cos\theta)^2(1+2\cos\theta)\sin\theta \, d\theta .$$

To calculate $\textcircled{\star}$ we write

$$l = \cos\theta$$
$$dl = -\sin\theta \, d\theta$$

$$V = \pi \int_{-1/2}^{1} (1-l^2)(1+l)^2(1+2l)\, dl$$

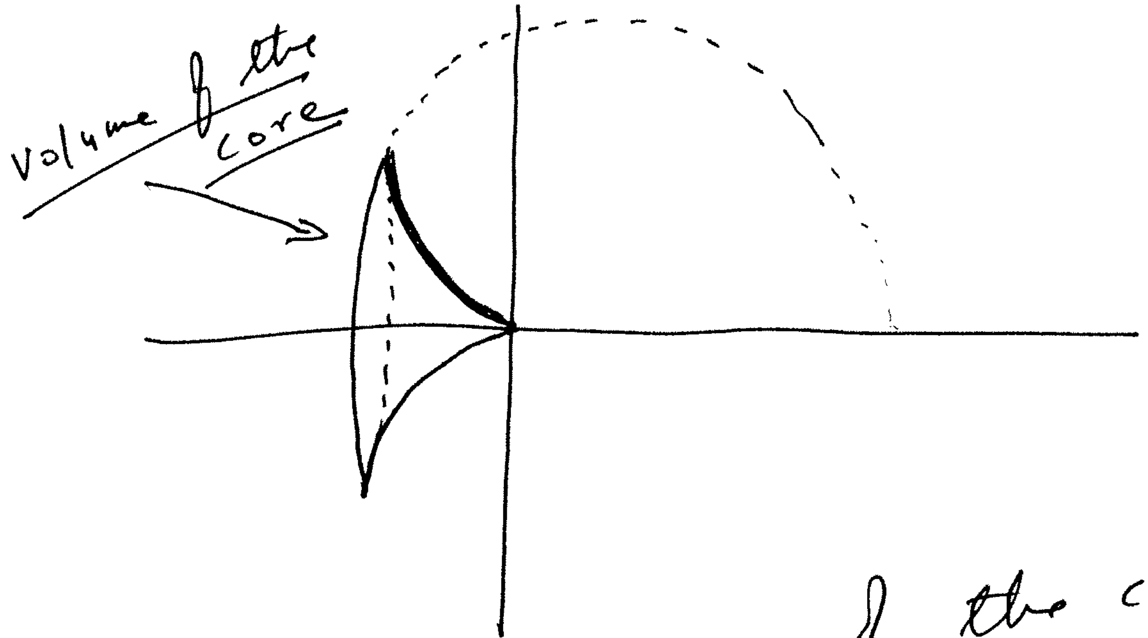
$$= \pi \int_{-1/2}^{1} \left[1 + 4l + 4l^2 - 2l^3 - 5l^4 - 2l^5\right] dl$$

$$V = \pi \left[l + 2l^2 + \frac{4}{3}l^3 - \frac{1}{2}l^4 - l^5 - \frac{1}{3}l^6 \right]_{-1/2}^{1}$$

$$= \pi \Big[\cos\theta + 2\cos^2\theta + \frac{4}{3}\cos^3\theta - \frac{1}{2}\cos^4\theta$$
$$- \cos^5\theta - \frac{1}{3}\cos^6\theta \Big]_{\theta = 2\pi/3}^{\theta = 0}.$$

$$= \pi (2 \cdot 671875)$$

Volume of the apple with core filled.



To calculate the volume of the core V_{core} we write

$$V_{core} = \int_{\theta=2\pi/3}^{\theta=\pi} \pi \sin^2\theta\,(1+\cos\theta)^2(-\sin\theta)(1+2\cos\theta)\,d\theta$$

analogous to ☆

$$= \pi\left[\cos\theta + 2\cos^2\theta + \frac{4}{3}\cos^3\theta - \frac{1}{2}\cos^4\theta - \cos^5\theta - \frac{1}{3}\cos^6\theta\right]_{\theta=2\pi/3}^{\theta=\pi}$$

$= .0052084\,\pi$ ← volume of the core.

The volume of the apple is given by.

$$(2{\cdot}671875 - {\cdot}0052084)\pi$$

$$V_{apple} = 2{\cdot}6666666\,\pi$$

Remark:

The volume of the core is not much. compared to the volume of the apple.

$${\cdot}195\,\%$$

Too bad the core gave us so much trouble.

Remark:

A spherical shaped apple with radius R having the same volume is given by.

$$\frac{4}{3}\pi R^3 = 2.666666\,\pi$$

$$R = 1.2599 = \sqrt[3]{2}.$$

If this apple was in the shape of an orange, "ie spherical" its radius would have been $\sqrt[3]{2}$ and its surface area $4\pi R^2$ would be 19.947853. which is slightly less than the surface area 20.106 of the apple computed before.