# Mid Term III
# Solutions.

1

$$\int_0^a x e^{-3x}\,dx$$

$$= \left. \frac{x e^{-3x}}{-3} \right|_0^a - \int_0^a \frac{1\, e^{-3x}}{-3}\,dx$$

$$= \left. -\frac{1}{3} x e^{-3x} \right|_0^a + \frac{1}{3} \int_0^a e^{-3x}\,dx$$

$$= \left. -\frac{1}{3} x e^{-3x} + \frac{1}{3} \frac{e^{-3x}}{-3} \right|_0^a$$

$$= \left. -\frac{1}{3} x e^{-3x} - \frac{1}{9} e^{-3x} \right|_0^a$$

$$= \left[ -\frac{1}{3} a e^{-3a} - \frac{1}{9} e^{-3a} \right] - \left[ -\frac{1}{9} \right]$$

$$= \frac{1}{9} - \frac{1}{9} e^{-3a} - \frac{1}{3} a e^{-3a}.$$

2

$$\int_0^\infty x e^{-3x}\,dx$$

$$= \frac{1}{9} - \frac{1}{9}\,\underset{a\to\infty}{Lt}\left[e^{-3a}\right] - \frac{1}{3}\,\underset{a\to\infty}{Lt}\left[a e^{-3a}\right]$$

$$\underset{a\to\infty}{Lt}\, e^{-3a} = 0$$

$$\underset{a\to\infty}{Lt}\,\frac{a}{e^{3a}} = \underset{a\to\infty}{Lt}\,\frac{1}{3e^{3a}} = 0$$

$$\boxed{\therefore \int_0^\infty x e^{-3x}\,dx = \frac{1}{9}}$$

2 Ans

$$\int \frac{1}{x^2}\, dx$$

$$= \int x^{-2}\, dx$$

$$= \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x}$$

$$\int_1^a \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_1^a$$

$$= -\frac{1}{a} + 1 = 1 - \frac{1}{a}$$

$$\int_1^\infty \frac{1}{x^2}\, dx = \underset{a\to\infty}{Lt} \left(1 - \frac{1}{a}\right) = \underline{1}$$

$$V = \pi \int_1^\infty \frac{1}{x^2}\, dx = \pi$$

3 Ans:

$$\int_1^4 \frac{1}{x-2}\,dx$$

$$= \underset{\substack{a\to 2\\ a<2}}{Lt} \int_1^a \frac{1}{x-2}\,dx + \underset{\substack{a\to 2\\ a>2}}{Lt} \int_a^4 \frac{1}{x-2}\,dx$$

$$= \underset{\substack{a\to 2\\ a<2}}{Lt} \ln|(x-2)|\Big|_1^a + \underset{\substack{a\to 2\\ a>2}}{Lt} \ln|(x-2)|\Big|_a^4$$

$$= \underset{\substack{a\to 2\\ a<2}}{Lt} \left[\ln|(a-2)| - \ln 1\right] + \underset{\substack{a\to 2\\ a>2}}{Lt} \left[\ln 2 - \ln|(a-2)|\right]$$

$$= \underset{\substack{a\to 2\\ a<2}}{Lt} \ln|(a-2)| - \underset{\substack{a\to 2\\ a>2}}{Lt} \ln(a-2) + \ln 2$$

The first two do not exist. Hence the integral diverges.

5

4

Let

$$S = \frac{1}{4} + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^7 + \left(\frac{1}{4}\right)^{10} + \cdots$$

$$\left(\frac{1}{4}\right)^3 S = \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^7 + \left(\frac{1}{4}\right)^{10} + \left(\frac{1}{4}\right)^{13} + \cdots$$

$$S - \left(\frac{1}{4}\right)^3 S = \frac{1}{4}$$

$$\Rightarrow S = \frac{1/4}{1 - \left(\frac{1}{4}\right)^3} = \cdot 2539682$$

Hence the sequence converges and it converges to

$\cdot 2539682$.

5 $u_n = \dfrac{1}{n(n+1)}$

$$u_{n+1} = \frac{1}{(n+1)(n+2)}$$

$$\frac{u_{n+1}}{u_n} = \frac{1}{\cancel{(n+1)}(n+2)} \; n\cancel{(n+1)}.$$

$$= \frac{n}{n+2}$$

$$\underset{n\to\infty}{\mathrm{Lt}} \; \frac{u_{n+1}}{u_n} = 1 \leftarrow \text{ratio test fails.}$$

——— x ———

Let us look at

$$\int_2^\infty \frac{1}{x(x+1)}\, dx .$$

7

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$= \frac{(A+B)x + A}{x(x+1)}.$$

$A + B = 0 \quad A = 1$

$\Rightarrow B = -1$

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

$$\int_2^a \frac{1}{x(x+1)}\,dx = \Big[\ln x - \ln(x+1)\Big]_2^a$$

$$= \Big[\ln a - \ln(a+1)\Big] - \Big[\ln 2 - \ln 3\Big]$$

$$= \ln \frac{a}{a+1} + \ln 3 - \ln 2$$

8

$$\int_2^{\infty} \frac{1}{x(x+1)}\,dx.$$

$$= \underset{a\to\infty}{Lt}\ \ln\frac{a}{a+1} + \ln\left(\frac{3}{2}\right)$$

$$= \ln 1 + \ln 3/2$$

$$= \ln 3/2 \doteq .4054651$$

Hence the series converges.

———— x ————

Actually ~~note~~ note that

$$u_2 = \frac{1}{2} - \frac{1}{3}$$

$$u_2 + u_3 = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = \frac{1}{2} - \frac{1}{4}$$

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$$u_2 + u_3 + \cdots + u_n = \frac{1}{2} - \frac{1}{n+1}$$

$$\text{Sum} = \frac{1}{2} - \underset{n\to\infty}{Lt}\frac{1}{n+1} = \frac{1}{2}.$$

Hence the series converges.

(6) We compute

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \int_2^{\infty} \frac{du}{u^2}$$

$$\boxed{\text{If } u = \ln x \quad du = \frac{1}{x}\,dx}$$

$$\int_2^{a} \frac{du}{u^2} = \frac{u^{-1}}{-1} = -\frac{1}{u}$$

$$= -\frac{1}{\ln x}\Bigg|_2^{a}$$

$$= -\frac{1}{\ln a} + \frac{1}{\ln 2}$$

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = -\lim_{a\to\infty} \frac{1}{\ln a} + \frac{1}{\ln 2}.$$

$$= \frac{1}{\ln 2} = 1{\cdot}442695.$$

Hence by integral test,

the series converges.

—— x ——

7 $$u_n = \frac{3^n}{n^2}$$

$$u_{n+1} = \frac{3^{n+1}}{(n+1)^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{3^{n+1}}{(n+1)^2} \frac{n^2}{3^n}$$

$$= 3\left(\frac{n}{n+1}\right)^2$$

$$\rho = \underset{n \to \infty}{Lt} \frac{u_{n+1}}{u_n} = 3$$

Hence the series diverges.