Math 1352

Midterm III

Answers

1 Ans:

$$\frac{dy}{dx} + \frac{1}{x} y = \frac{1}{x}$$

$$P(x) = \frac{1}{x} \;;\quad Q(x) = \frac{1}{x}$$

$$F(x) = e^{\int P(x)\,dx} = e^{\int \frac{1}{x} dx} = e^{\ln|x|}$$

$$= |x|$$

For $x > 0$, $F(x) = x$

$$y(x) = \frac{1}{F(x)} \left[\int Q(x) F(x)\, dx + C \right]$$

$$= \frac{1}{x} \left[\int \frac{1}{\cancel{x}} \cancel{x}\, dx + C \right]$$

$$= \frac{1}{x} \left[x + C \right] = 1 + \frac{C}{x}$$

$$\boxed{y(x) = 1 + \frac{C}{x}}$$

2 Ans:

$$xy\,dx = (x-4)(x-5)\,dy$$

$$\Rightarrow \int \frac{x}{(x-4)(x-5)}\,dx = \int \frac{dy}{y}$$

$$\Rightarrow \ln|y| = \int \frac{x}{(x-4)(x-5)}\,dx$$

$$\frac{x}{(x-4)(x-5)} = \frac{A}{x-4} + \frac{B}{x-5}.$$

Solving for A & B we get.

$$(A+B)x - (5A+4B) = x$$

$$\Rightarrow \left.\begin{aligned} A+B&=1 \\ 5A+4B&=0 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} A &= -4 \\ B &= 5 \end{aligned}}$$

(3)

$$\int \frac{x}{(x-4)(x-5)}\,dx = \int \frac{5}{x-5}\,dx - \int \frac{4}{x-4}\,dx$$

$$= 5\ln|x-5| - 4\ln|x-4| + C$$

$$\therefore \ln|y| = 5\ln|x-5| - 4\ln|x-4| + C$$

$$= \ln|x-5|^5 - \ln|x-4|^4 + C$$

$$= \ln\left[\frac{|x-5|^5}{|x-4|^4}\right] + C$$

$$\therefore |y| = \frac{|x-5|^5}{|x-4|^4} e^C$$

writing $e^C = k$, a constant, we have

$$|y| = k\frac{|x-5|^5}{|x-4|^4}$$

Removing the $|\cdot|$ and absorbing the sign inside k we have

$$y = k\,\frac{(x-5)^5}{(x-4)^4}$$

where k is any constant.

3 Ans:

1 $$\int_0^3 \frac{1}{(x-1)^{5/6}}\,dx$$

~~∴ the integrand is dis discontinuous~~

~~at x = 1 a~~

∴ the integrand is not defined for $x < 1$, it would follow that the integral does not exist.

— × —

Alternative problem

$$\int_0^3 \frac{1}{(x-1)^{5/7}}\,dx$$

Here the integrand is defined for all values in the interval $[0, 3]$ with the exception of $x = 1$.

We write

$$\int_0^3 \frac{dx}{(x-1)^{5/7}} = \int_0^1 \frac{dx}{(x-1)^{5/7}} + \int_1^3 \frac{dx}{(x-1)^{5/7}}.$$

$$= \underset{b \to 1^-}{Lt} \int_0^b \frac{dx}{(x-1)^{5/7}}$$

$$+ \underset{b \to 1^+}{Lt} \int_b^3 \frac{dx}{(x-1)^{5/7}}.$$

$$\int \frac{dx}{(x-1)^{5/7}} = \int (x-1)^{-5/7}\, dx$$

$$= \frac{(x-1)^{-5/7+1}}{-5/7+1} = \frac{7}{2}(x-1)^{2/7}$$

$$\int_0^b \frac{dx}{(x-1)^{5/7}} = \frac{7}{2}(x-1)^{2/7}\Big|_0^b = \frac{7}{2}(b-1)^{2/7} - \frac{7}{2}$$

$$= \frac{7}{2}(b-1)^{2/7} - \frac{7}{2}(-1)^{2/7}.$$

7

$$\underset{b\to 1^-}{Lt} \int_0^b \frac{dx}{(x-1)^{5/7}} =$$

$$\underset{b\to 1^-}{Lt} \frac{7}{2}(b-1)^{2/7} - \frac{7}{2}$$

$$= -7/2$$

$$\int_b^3 \frac{dx}{(x-1)^{5/7}} = \frac{7}{2}(x-1)^{2/7}\Big|_b^3$$

$$= \frac{7}{2}2^{2/7} - \frac{7}{2}(b-1)^{2/7}$$

$$\underset{b\to 1^+}{Lt} \int_b^3 \frac{dx}{(x-1)^{5/7}} = \underset{b\to 1^+}{Lt}\left[-\frac{7}{2}(b-1)^{2/7}\right]$$

$$+ \frac{7}{2}2^{2/7}$$

$$= \frac{7}{2}2^{2/7}$$

It would follow that

$$\int_0^3 \frac{dx}{(x-1)^{5/7}} = -\frac{7}{2} + \frac{7}{2}\, 2^{2/7}$$

$$= \frac{7}{2}\left[2^{2/7} - 1\right]$$

3 Ans

(ii) $$\int_{-1}^{1} \frac{1}{x}\,dx = \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx$$

$$= \lim_{b \to 0^-} \int_{-1}^{b} \frac{1}{x}\,dx$$

$$+ \lim_{b \to 0^+} \int_{b}^{1} \frac{1}{x}\,dx$$

$$= \lim_{b \to 0^-} \ln|x| \Big|_{-1}^{b} + \lim_{b \to 0^+} \ln|x| \Big|_{b}^{1}$$

$$= \lim_{b \to 0^-} \Big[\ln|b| - \ln|-1| \Big] + \lim_{b \to 0^+} \Big[\ln|1| - \ln|b| \Big]$$

$$= \underbrace{\underset{b \to 0^-}{Lt} \ln|b|}_{} - \underbrace{\underset{b \to 0^+}{Lt} \ln|b|}_{}$$

These two limits do not exist.

Hence $\int_{-1}^{1} \frac{1}{x} \, dx$ does not exist.

4

a $S_n = 2 + 4 + 6 + \cdots + 2n$

$S_n = 2n + (2n-2) + \cdots + 2$

Adding, we have

$2S_n = (2+2n) + (4+(2n-2)) + \cdots + (2n+2)$

$= (2+2n)n = 2(n+1)n$.

$\therefore \boxed{S_n = n(n+1)}$

4 b

$$S_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-1}}.$$

$$\frac{1}{3} S_n = \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-1}} + \frac{1}{3^n}$$

subbracting, we have

$$S_n - \frac{1}{3} S_n = 1 - \frac{1}{3^n}.$$

$$\frac{2}{3} S_n = 1 - \frac{1}{3^n}.$$

$$\boxed{S_n = \frac{3}{2}\left[1 - \frac{1}{3^n}\right]}$$

(13)

(5) Ans:

The integral test says that

$$\sum_{k=1}^{\infty} k e^{-k^2}$$

and

$$\int_1^{\infty} x e^{-x^2} dx$$

both converge or diverge.

compute $\int_1^{\infty} x e^{-x^2} dx$ as follows:

Let $u = x^2$

$du = 2x \, dx$.

$$\int_1^N x e^{-x^2} dx$$

$$= \int_1^N \frac{1}{2} du\, e^{-u}$$

$$= \frac{1}{2} \int_1^N e^{-u} du$$

$$= -\frac{1}{2} e^{-u} \Big|_1^N$$

$$= -\frac{1}{2} \left[e^{-N} - e^{-1} \right]$$

$$= \frac{1}{2} \left[e^{-1} - e^{-N} \right]$$

$$\int_1^\infty x e^{-x^2} dx = \lim_{N \to \infty} \int_1^N x e^{-x^2} dx$$

$$= \frac{1}{2}\left[e^{-1} - \operatorname*{Lt}_{N\to\infty} e^{-N}\right]$$

$$= \frac{1}{2} e^{-1}$$

The integral converges and ~~hens~~ hence the series converges as well